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Research Article

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A quantitative obstruction to collapsing surfaces

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Abstract: We provide a quantitative obstruction to collapsing surfaces of genus at least 2 under a lower curvature bound and an upper diameter bound.

Keywords: curvature, diameter, volume, filling radius, systole, Gromov-Hausdorff distance

1 Introduction

S. Alesker posed the following question at MathOverflow [1]. Let (M_i) be a sequence of 2-dimensional orientable closed surfaces of genus $g \geq 2$ endowed with smooth Riemannian metrics of Gaussian curvature at least -1 and diameter at most D. By the Gromov compactness theorem, one can choose a subsequence converging in the Gromov-Hausdorff (GH) sense to a compact Alexandrov space with curvature at least -1 and Hausdorff dimension 0, 1, or 2. Let us assume that the limit space has dimension 1. Then it is either a circle or a segment. Can these possibilities (circle and segment) be obtained in the limit M of (M_i)? We show that these possibilities cannot occur, and quantify this statement by providing an explicit lower bound for the filling radius of M. For related results see [2].

2 Impossibility of collapse

We prove the impossibility of collapse in dimension 2, in the following sense.

Theorem 2.1. *The distance between a strongly isometric map from a closed orientable surface M of genus $g \geq 2$ of Gaussian curvature $K \geq -1$ and diameter at most D to a metric space Z, and a map from M to a graph in Z, is at least* $\frac{\pi(g-1)}{3\sinh D}$.

Thus we obtain a quantitative lower bound rather than merely the nonexistence of Shioya-Yamaguchi-type collapse to spaces of positive codimension (see [3, 4]).

Corollary 2.2. *Let $D > 0$. GH limits of metrics on a closed orientable surface of genus $g \geq 2$ with Gaussian curvature at least -1 and diameter at most D are necessarily 2-dimensional.*

Recall that the *systole* of a Riemannian manifold M is the least length of a noncontractible loop of M. For an overview of systolic geometry see [5].

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The *filling radius* FillRad M of a closed n-dimensional manifold M is defined as the infimum of all $\epsilon > 0$ such that the inclusion of M in its ϵ-neighborhood in any strongly isometric embedding of M in a Banach space sends the fundamental homology class $[M]$ of M to the zero class, by means of the induced homomorphism on $H_n(M)$. Here the embedding can be taken to be into the space of bounded functions on M which sends a point $p \in M$ to the distance function from p. This embedding is strongly isometric (ambient distance restricted to M coincides with intrinsic distance on M) if the function space is equipped with the sup-norm.

Lemma 2.3 (Gromov's lemma). *The systole of an aspherical manifold M is at most six times the filling radius of M.*

Proof. Consider a strongly isometric embedding of the surface M into a Banach space B. The space B can be assumed finite-dimensional if the metric condition is relaxed to a requirement of being bilipschitz with to a bilipschitz factor arbitrarily close to 1; see [6]. Suppose M is "filled" (in the homological sense) by a chain C (in the sense that M is the boundary of C). Then the induced homomorphism $H_n(M) \to H_n(C)$ sends $[M]$ to the zero class. Consider a triangulation of C into infinitesimal simplices (here the term "infinitesimal" is used informally in its meaning "sufficiently small" though this could be rendered rigorous as in [7]).

We argue by contradiction. Let $R > 0$ be strictly smaller than a sixth of the systole. Suppose the chain C is contained in an open R-neighborhood of M in B. We will retract C back to M, while fixing the subset $M \subseteq C$, contradicting the fact that the nonvanishing fundamental class $[M]$ is sent to a zero class in C.

For each vertex of the triangulation of C, we choose a nearest point of M. To extend the retraction to the 1-skeleton of C, we map each edge (of a triangle of the triangulation) to a minimizing path joining the images of the two vertices in M. The length of such a minimizing path is less than $2R$ (plus the infinitesimal sidelength of the triangle) by the triangle inequality. Hence the boundary of each 2-cell of the triangulation is sent to a loop of length at most $6R$ (plus an infinitesimal). Since this length is less than the systole of M, the map can now be extended to the 2-skeleton of C.

To extend the map to the 3 skeleton, note that the universal cover of M is contractible and hence $\pi_2(M) = 0$, and similarly for the higher homotopy groups. Therefore the skeletal retraction extends to all of C inductively. The contradiction completes the proof of the lemma. □

Proof of Theorem 2.1. We exploit Gromov's notion of the filling radius of a manifold [8]. The argument relies only on basic Jacobi field estimates and basic homotopy theory. We seek a suitable lower bound so as to rule out positive-codimension collapse. Choose a noncontractible closed geodesic $\gamma \subseteq M$ of length equal to the systole sys(M). Consider the normal exponential map along γ. Using the lower curvature bound, we obtain an upper bound on the total area of M as $2\,\mathrm{sys}(M)\sinh(D)$, where D is the diameter. The bound follows by applying Rauch bounds on Jacobi fields (this is an ingredient in the proof of Toponogov's theorem); see e.g., Cheeger-Ebin [9, Theorem 5.8, pp. 97–98]. The bound results from comparison with the area of a hyperbolic collar of width D around a closed geodesic of the same length as γ. Therefore, the systole is bounded below as follows:

$$\mathrm{sys}(M) \geq \frac{\mathrm{area(M)}}{2\sinh D}. \tag{2.1}$$

Meanwhile the area is bounded below by the Gauss-Bonnet theorem:

$$\mathrm{area}(M) \geq -\int_M K = 2\pi(2g-2),$$

where g is the genus. Furthermore the filling radius of M is bounded below by a sixth of the systole by Gromov's Lemma 2.3. Therefore the bound (2.1) implies

$$\mathrm{FillRad}(M) \geq \frac{1}{6}\mathrm{sys}(M) \geq \frac{\mathrm{area}(M)}{12\sinh D} \geq \frac{\pi(g-1)}{3\sinh D}. \tag{2.2}$$

The theorem now follows from the fact the distance between a strongly isometric map from M to a metric space Z and a map from M to a graph in Z is bounded below by the filling radius; see e.g., [8, p. 127, Example]. This proves that aspherical surfaces of curvature bounded below by -1 with diameter bounded above by D cannot collapse, so that a GH limit is necessarily 2-dimensional as follows. □

To prove Corollary 2.2, note that if a metric on M is sufficiently close to a finite graph Γ in the sense of the GH distance, then the construction of the proof of Lemma 2.3 produces a map from M to Γ which is close to the embedding of M in Z, contradicting the lower bound (2.2).

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