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Existence of periodic solutions with prescribed minimal period of a 2nth-order discrete system

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Abstract: In this paper, we concern with a 2nth-order discrete system. Using the critical point theory, we establish various sets of sufficient conditions for the existence of periodic solutions with prescribed minimal period. To the best of our knowledge, this is the first time to discuss the periodic solutions with prescribed minimal period for a 2nth-order discrete system.

Keywords: minimal period, 2nth-order, nonlinear, discrete system, critical point

MSC: 39A23, 34C25, 58E05, 37J45

1 Introduction

Below $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ denote the sets of all natural numbers, integers and real numbers, respectively. * denotes the transpose of a vector. $[\cdot]$ is denoted by the greatest-integer function. Let $a, b \in \mathbb{Z}$, we define $\mathbb{Z}(a) = \{a, a+1, \cdots\}$ and when $a < b$, $\mathbb{Z}(a, b) = \{a, a+1, \cdots, b\}$.

In this paper, we shall study the 2nth-order discrete system

$$\Delta^{2n} u_{k-n} = (-1)^n f(k, u_k),\ n \in \mathbb{Z}(3),\ k \in \mathbb{Z}, \tag{1.1}$$

where Δ is the forward difference operator $\Delta u_k = u_{k+1} - u_k$, $\Delta^i u_k = \Delta(\Delta^{i-1} u_k)$ for $i \geq 2$, $f \in C^1(\mathbb{R}^2, \mathbb{R})$, $f(k+T, u) = f(k, u)$ for a given integer $T \geq 3$.

He and Chen [1] in 2008 concerned with the existence of a periodic solution for the following second order discrete convex systems involving the p-Laplacian:

$$\Delta\left[\phi_p\left(u_{n-1}\right)\right] + \nabla F(k, u_k) = 0,\ k \in \mathbb{Z}.$$

Some existence theorems are obtained by using the dual least action principle.

In 2007, Cai and Yu [2] considered the 2nth-order difference equation

$$\Delta^n\left(r_{k-n} \Delta^n u_{k-n}\right) + f(k, u_k) = 0,\ n \in \mathbb{Z}(3),\ k \in \mathbb{Z}.$$

By the Linking Theorem, some new criteria are obtained for the existence and multiplicity of periodic solutions of the above equation.

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By establishing a proper variational framework and using the critical point theory, Chen and Tang [3] established some new existence criteria to guarantee the $2n$th-order nonlinear difference equation containing both many advances and retardations

$$\Delta^n\left(r_{k-n}\Delta^n u_{k-n}\right)+q_n u_k=f(k,u_{k+n},\cdots,u_k,\cdots,u_{k-n}),\ n\in\mathbb{Z}(3),\ k\in\mathbb{Z},$$

has at least one or infinitely many homoclinic orbits. Their conditions on the potential are rather relaxed and some existing results in the literature are improved.

Leng in 2016 considered the $2n$th-order difference equation with ϕ_c-Laplacian

$$\Delta^n\left(r_{k-n}\phi_c\left(\Delta^n u_{k-n}\right)\right)=(-1)^n f(k,u_{k+1},u_k,u_{k-1}),\ k\in\mathbb{Z},$$

where n is a fixed positive integer, Δ is the forward difference operator $\Delta u_k=u_{k+1}-u_k$, $\Delta^n u_k=\Delta(\Delta^{n-1}u_k)$, r_k is real valued for each $k\in\mathbb{Z}$, ϕ_c is a special ϕ-Laplacian operator defined by $\phi_c(s)=\frac{s}{\sqrt{1+s^2}}$, $f\in C(\mathbb{Z}\times\mathbb{R}^3,\mathbb{R})$, r_k and $f(k,v_1,v_2,v_3)$ are T-periodic in k for a given positive integer T. By using the critical point theory, some new criteria for the existence and multiplicity of periodic and subharmonic solutions are established.

In the aforementioned references, most of the results are periodic solutions or homoclinic orbits of difference equations. Yu, Long and Guo [4] established some existence criteria to periodic solutions with prescribed minimal period of second-order difference equation

$$\Delta^2 u_{k-1}+A\sin u_k=f(k),\ k\in\mathbb{Z},$$

by making use of the variational methods.

Existence of solutions of higher-order nonlinear differential equations has been the subject of many investigations [5-15]. Difference equations, the discrete analogs of differential equations, occur widely in numerous settings and forms, not only in mathematics itself but also in several economical and population problems. However, to the best of our knowledge, this is the first time to discuss the periodic solutions with prescribed minimal period for a $2n$th-order discrete system [1-4, 16-22]. The difficulty lies in the fact that there are very scarce techniques to study the existence of periodic solutions with prescribed minimal period. The purpose of this paper is to establish various sets of sufficient conditions for the existence of periodic solutions with prescribed minimal period to a $2n$th-order nonlinear discrete system. The main approach used in our paper is a variational technique. The motivation for the present work stems from the recent papers [18, 20].

Set

$$\omega=\frac{2\pi}{T}.$$

The rest of this paper is organized as follows. Firstly, in Section 2, we shall establish the variational framework associated with (1.1) and reduce the existence of periodic solutions of (1.1) to seeking the existence of critical points of the corresponding functional. Then, in Section 3, we shall give auxiliary results which will be of fundamental importance in proving our main results. Finally, in Section 4, we shall prove the existence results by using the variational methods.

To conclude the Introduction, the reader is referred to [23-25] for the general background on difference equations and [26] for the basic knowledge of variational methods.

2 Variational framework

The purpose of this section is to establish the variational framework associated with (1.1) and to state some basic notations for the coming discussion.

Let U be a pT-dimensional Euclidean space consisting of functions $\mathbb{Z}\to\mathbb{R}$ and for any $k\in\mathbb{Z}$,

$$U=\{u|u_{k+pT}=u_k\}.$$

U is equipped with a norm

$$\|u\| = \left(\sum_{i=1}^{pT} u_i^2\right)^{\frac{1}{2}},$$

and an inner product

$$(u, v) = \sum_{i=1}^{pT} u_i v_i.$$

Let us consider a functional J defined on U as follows

$$J(u) = \frac{1}{2}\sum_{k=1}^{pT} \left(\Delta^n u_{k-1}\right)^2 - \sum_{k=1}^{pT} F(k, u_k). \tag{2.2}$$

It is obvious that $J \in C^1(U, \mathbb{R})$ and for any $u \in U$, we can calculate

$$\frac{\partial J}{\partial u_k} = (-1)^n \Delta^{2n} u_{k-n} - f(k, u_k),\ \forall k \in \mathbb{Z}(1, pT).$$

As a consequence, u is a critical point of $J(u)$ on U if and only if

$$\Delta^{2n} u_{k-n} = (-1)^n f(k, u_k),\ \forall k \in \mathbb{Z}(1, pT).$$

Set

$$M = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ -1 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}_{pT\times pT}.$$

It is easy to know that the eigenvalues of M are

$$\lambda_k = 4\sin^2\frac{k\pi}{pT}, k = 0, 1, 2, \cdots, pT-1,$$

0 is an eigenvalue of M and

$$\min\{\lambda_1, \lambda_2, \cdots, \lambda_{pT-1}\} = 4\sin^2\frac{\pi}{pT},\ \max\{\lambda_1, \lambda_2, \cdots, \lambda_{pT-1}\} \le 4.$$

Let

$$K = \ker M = \{u \in U | Mu = 0\},$$

and the eigenvectors of M corresponding to λ_k by

$$\mu_i = \left(\cos\frac{2i\pi}{pT}, \cos\frac{2i\pi\cdot 2}{pT}, \cdots, \cos\frac{2i\pi\cdot pT}{pT}\right)^*, i = 1, 2, \cdots, \left[\frac{pT-1}{2}\right],$$

and

$$\nu_i = \left(\sin\frac{2i\pi}{pT}, \sin\frac{2i\pi\cdot 2}{pT}, \cdots, \sin\frac{2i\pi\cdot pT}{pT}\right)^*, i = 1, 2, \cdots, \left[\frac{pT-1}{2}\right].$$

We define

$$A = \operatorname{span}\{\mu_i\}, i = 1, 2, \cdots, \left[\tfrac{pT-1}{2}\right],$$

and

$$B = \operatorname{span}\{\nu_i\}, i = 1, 2, \cdots, \left[\tfrac{pT-1}{2}\right].$$

If pT is odd, then $U = K \oplus A \oplus B$. For any $u \in U$ and $k \in \mathbb{Z}$,

$$u_k = a + (-1)^k b + \sum_{i=1}^{\left[\frac{pT-1}{2}\right]} \left(a_i\cos\frac{\omega i}{p}k + b_i\sin\frac{\omega i}{p}k\right),$$

where a, b, a_i and b_i are constants.

If pT is even, then 4 is the eigenvalue of M. Let ξ be the eigenvector corresponding to 4, and D = span $\{\xi\}$. We have $U = K \oplus A \oplus B \oplus D$. For any $u \in U$ and $k \in \mathbb{Z}$,

$$u_k = a + \sum_{i=1}^{\left[\frac{pT-1}{2}\right]} \left(a_i \cos \frac{\omega i}{p} k + b_i \sin \frac{\omega i}{p} k \right),$$

where a, a_i and b_i are constants.

Now, we give the existence results of at least one periodic solution with minimal period pT as follows.

Theorem 2.1. *Assume that the following conditions are satisfied:*
(F_1) *there is a function $F(s,u) \in C^2(\mathbb{R}^2, \mathbb{R})$ with $F(s+T,u) = F(s,u)$, $F(-s,-u) = F(s,u)$, $F(s,u) \geq 0$ and it satisfies*

$$\frac{\partial F(s,u)}{\partial u} = f(s,u), \forall s \in \mathbb{R};$$

(F_2) $\lim\limits_{|u| \to +\infty} \frac{F(s,u)}{u^2} = 0$ *uniformly for $s \in \mathbb{R}$;*

(F_3) *there exist three constants $\eta > 0$ and $\epsilon > \varepsilon > 0$ such that*

$$\left(\frac{\partial^2 F(s,u)}{\partial u^2} \theta, \theta \right) \leq \epsilon \theta^2, \forall (s,u) \in \mathbb{R}^2, \theta \in \mathbb{R}$$

and

$$\left(\frac{\partial^2 F(s,u)}{\partial u^2} \theta, \theta \right) \geq \varepsilon \theta^2, \forall |u| \leq \eta, s \in \mathbb{R}, \theta \in \mathbb{R};$$

(F_4) *if u is a solution of (1.1) with a minimal period ϕT, ϕ is a rational number, and $f(s,u)$ also has a minimal period ϕT, then ϕ must be an integer;*

(F_5) *let $p > 1$ be a given positive integer and l_p denote the least prime factor of p,*

$$\left(4 \sin^2 \frac{\omega l_p}{2p} \right)^n > \epsilon, \ \left(4 \sin^2 \frac{\omega}{2p} \right)^n < \varepsilon$$

and

$$\sum_{k=1}^{pT} f^2(k,0) < \frac{4\pi\eta^2 \left[\left(4 \sin^2 \frac{\omega l_p}{2p} \right)^n - \epsilon \right] \left[\varepsilon - \left(4 \sin^2 \frac{\omega}{2p} \right)^n \right]}{\omega}.$$

Then (1.1) has at least one periodic solution with minimal period pT.

Remark 2.1. The assumption (F_2) implies that
($\tilde{F}_2$) there is a constant $C > 0$ such that

$$F(s,u) \leq C, \ \forall (s,u) \in \mathbb{R}^2.$$

Corollary 2.1. *Suppose that (F_1) – (F_4) are satisfied and*

$$\left(4 \sin^2 \frac{\omega}{2} \right)^n > \epsilon, \ \left(4 \sin^2 \frac{\omega}{2p} \right)^n < \varepsilon.$$

If

$$\sum_{k=1}^{pT} f^2(k,0) < \frac{4\pi\eta^2 \left[\left(4 \sin^2 \frac{\omega}{2} \right)^n - \epsilon \right] \varepsilon}{\omega},$$

then there is $P > 0$ such that for any prime integer $p > P$, (1.1) has at least one periodic solution with minimal period pT.

Theorem 2.2. *Assume that the assumptions* $(F_1)-(F_4)$ *hold. If*

$$f(k,0)=0,\forall k\in\mathbb{Z},$$

and

$$\left(4\sin^2\frac{\omega l_p}{2p}\right)^n>\epsilon,\ \left(4\sin^2\frac{\omega}{2p}\right)^n<\varepsilon,$$

then (1.1) has at least one periodic solution with minimal period pT.

Corollary 2.2. *Suppose that* $(F_1)-(F_4)$ *are satisfied. If*

$$f(k,0)=0,\forall k\in\mathbb{Z},$$

and

$$\left(4\sin^2\frac{\omega}{2}\right)^n>\epsilon,\ \left(4\sin^2\frac{\omega}{2p}\right)^n<\varepsilon,$$

then there is $P>0$ *such that for any prime integer* $p>P$*, (1.1) has at least one periodic solution with minimal period* pT.

3 Auxiliary results

In this section, we shall give auxiliary results which will be of fundamental importance in proving our main results.

Let B_r denote the open ball in U about 0 of radius r and ∂B_r denote its boundary.

Lemma 3.1. [26] *Let* U *be a finite dimensional Hilbert space,* $J(u)\in C^1(U,\mathbb{R})$ *is coercive, i.e.,* $J(u)\to+\infty$*, as* $\|u\|\to+\infty$. *Then* $J(u)$ *attains its minimal at some* $\tilde{u}$ *on* U.

Set

$$\tilde{U}=\{u\in U|u_{-k}=u_k,\forall k\in\mathbb{Z}\}.$$

We have $\tilde{U}=B$, then

$$u_k=\sum_{i=1}^{\left[\frac{pT-1}{2}\right]}b_i\sin\frac{\omega i}{p}k,\forall k\in\mathbb{Z}.$$

Lemma 3.2. *Assume that the assumptions* $(F_1)-(F_5)$ *hold. Then* $J(u)$ *attains its minimal at some* $\tilde{u}$ *on* $\tilde{U}$.

Proof. From $(\tilde{F}_2)$, for any $u\in\bar{U}$,

$$\begin{aligned}J(u)&=\frac{1}{2}\sum_{k=1}^{pT}(\Delta^n u_{k-1},\Delta^n u_{k-1})-\sum_{k=1}^{pT}F(k,u_k)\\&=\frac{1}{2}\sum_{k=1}^{pT}(\Delta^n u_k,\Delta^n u_k)-\sum_{k=1}^{pT}F(k,u_k)\\&=\frac{1}{2}x^\star Mx-\sum_{k=1}^{pT}F(k,u_k)\\&\geq\frac{1}{2}\times4\sin^2\frac{\pi}{pT}\|x\|^2-pTC\\&=2\sin^2\frac{\pi}{pT}\|x\|^2-pTC,\end{aligned}$$

where $x=(\Delta^{n-1}u_1,\Delta^{n-1}u_2,\cdots,\Delta^{n-1}u_{pT})^\star$. Since

$$\|x\|^2=\sum_{k=1}^{pT}\left(\Delta^{n-2}u_{k+1}-\Delta^{n-2}u_k\right)^2\geq4\sin^2\frac{\pi}{pT}\sum_{k=1}^{pT}\left(\Delta^{n-2}u_k\right)^2\geq\cdots\geq\left(4\sin^2\frac{\pi}{pT}\right)^{n-1}\|u\|^2,$$

we have

$$J(u) \geq \frac{1}{2}\left(4\sin^2\frac{\pi}{pT}\right)^n \|u\|^2 - pTC \to +\infty,$$

as $\|u\| \to +\infty$. By Lemma 3.1, the conclusion of Lemma 3.2 is true. □

Lemma 3.3. *Assume that u is a critical point of $J(u)$ on $\tilde{U}$. Then u is a critical point of $J(u)$ on U.*

The proof of Lemma 3.3 is similar to that of Lemma 2.2 in [4]. For the simplicity, we omit its proof.

Let

$$\Psi_\phi = -\frac{p}{2\left[\left(4\sin^2\frac{\omega\phi}{2p}\right)^n - \epsilon\right]}\sum_{k=1}^{pT} f^2(k,0).$$

Lemma 3.4. *Assume that the assumptions $(F_1)-(F_5)$ hold and $J(u) < \Psi_{l_p}$. If u is a critical point of $J(u)$ on $\tilde{U}$, then u has a minimal period pT.*

Proof. Assume, for the sake of contradiction, that u exists a minimal period $\frac{pT}{\phi}$. It comes from (F_5) that $\phi \geq l_p$.

Similarly, for any $u \in \tilde{U}$,

$$u_k = \sum_{j=1}^{\left[\frac{pT-\phi}{2\phi}\right]} b_j \sin\frac{\omega\phi j}{p}k,$$

and then

$$J(u) = \frac{1}{2}x^\star Mx - \sum_{k=1}^{pT} F(k,u_k) \geq 2\sin^2\frac{\omega\phi}{2p}\|x\|^2 - \sum_{k=1}^{pT} F(k,u_k),$$

where $x = (\Delta x_1, \Delta x_2, \cdots, \Delta x_{pT})^\star$. It is easy to see that

$$\|x\|^2 = \sum_{k=1}^{pT}\left(\Delta^{n-2}u_{k+1} - \Delta^{n-2}u_k\right)^2 \geq 4\sin^2\frac{\omega\phi}{2p}\sum_{k=1}^{pT}\left(\Delta^{n-2}u_k\right)^2 \geq \cdots \geq \left(4\sin^2\frac{\omega\phi}{2p}\right)^{n-1}\|u\|^2.$$

We have

$$\begin{aligned}
J(u) &\geq \frac{1}{2}\left(4\sin^2\frac{\omega\phi}{2p}\right)^n \|u\|^2 - \left(\sum_{k=1}^{pT} f^2(k,0)\right)^{\frac{1}{2}} \|u\| - \frac{\epsilon}{2}\|u\|^2 \\
&\geq -\frac{1}{2\left[\left(4\sin^2\frac{\omega\phi}{2p}\right)^n - \epsilon\right]}\sum_{k=1}^{pT} f^2(k,0) \\
&\geq -\frac{p}{2\left[\left(4\sin^2\frac{\omega\phi}{2p}\right)^n - \epsilon\right]}\sum_{k=1}^{pT} f^2(k,0) \\
&= \Psi_\phi,
\end{aligned}$$

which is a contradiction to the assumption $J(u) < \Psi_{l_p}$. The result is obtained. □

4 Proofs of the existence results

In this section, we shall give the proofs of the existence results by making use of the variational method.

Proof of Theorem 2.1. We shall prove that (1.1) has at least one periodic solution with minimal period pT. Lemmas 3.1-3.3 imply that (1.1) has at least one pT-periodic solution. Hence, by Lemma 3.4, it suffices to prove that

$$J(u) < \Psi_{l_p}, \forall u \in \tilde{U}.$$

According to the condition (F_3), we have

$$F(k,u)=f(k,0)u+\frac{1}{2}\times\frac{\partial^2 F(k,\zeta u)}{\partial u^2}u^2\geq f(k,0)u+\frac{\varepsilon}{2}u^2,\forall|u|\leq\eta.$$

Then

$$J(u)=\frac{1}{2}\sum_{k=1}^{pT}\left(\Delta^2u_k,\Delta^2u_k\right)-\sum_{n=1}^{pT}F(k,u_k)\leq\frac{1}{2}\sum_{k=1}^{pT}\left(\Delta^2u_k,\Delta^2u_k\right)-\frac{\varepsilon}{2}\sum_{k=1}^{pT}u_k^2-\sum_{k=1}^{pT}f(k,0)u_k.$$

Make a choice that

$$u_k=\eta\sin\frac{\omega k}{p}.$$

Combining with $f(-k,0)=f(k,0)$ and $f(k+T,0)=f(k,0)$, we have

$$f(k,0)=\sum_{j=1}^{[\frac{T-1}{2}]}a_j\sin\frac{2j\pi}{T}k=\sum_{j=1}^{[\frac{T-1}{2}]}a_j\sin\frac{2j\pi}{pT}pk,$$

where a_j is a constant. Therefore

$$\sum_{k=1}^{pT}f(k,0)u_k=\sum_{j=1}^{[\frac{T-1}{2}]}\eta a_j\sum_{k=1}^{pT}\sin\frac{2j\pi}{pT}pk\cdot\sin\frac{2\pi}{pT}k=0.$$

Similarly, we get

$$J(u)\leq 2\left[\left(4\sin^2\frac{\omega}{2p}\right)^n-\varepsilon\right]\|u\|^2.$$

Obviously,

$$\|u\|=\eta\left(\frac{p\pi}{\omega}\right)^{\frac{1}{2}}.$$

Thereby,

$$J(u)=\frac{2\left[\left(4\sin^2\frac{\omega}{2p}\right)^n-\varepsilon\right]\eta^2p\pi}{\omega}<\Psi_{l_p}.$$

The desired result follows. □

Proof of Corollary 2.1. For the reason that p is a positive prime integer, it is easy to see that $l_p=p$. Therefore

$$\sum_{k=1}^{pT}f^2(k,0)<\frac{4\pi\eta^2\left[\left(4\sin^2\frac{\omega}{2}\right)^n-\epsilon\right]\left[\varepsilon-\left(4\sin^2\frac{\omega}{2p}\right)^n\right]}{\omega}.$$

Due to Theorem 2.1, the conclusion of Corollary 2.1 is obviously true. The proof of Corollary 2.1 is finished.

Remark 3.1. Similarly to the proofs of Theorem 2.1 and Corollary 2.1, we can also prove Theorem 2.2 and Corollary 2.2. For simplicity, we omit their proofs.

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