Open Mathematics

Research Article

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Injective hulls of many-sorted ordered algebras

https://doi.org/10.1515/math-2019-0101
Received July 10, 2019; accepted September 17, 2019

Abstract: This paper is devoted to the study of injectivity for ordered universal algebras. We first characterize injectives in the category $\mathsf{OAL}_{\Sigma}^{\leqslant}$ of ordered Σ-algebras with lax morphisms as sup-Σ-algebras. Second, we show that every ordered Σ-algebra has an $\sigma^{\leqslant}$-injective hull, and give its concrete form.

Keywords: ordered Σ-algebra, lax morphism, sup-lattice, injective hull

MSC: 06Fxx, 08B30, 08C05

1 Introduction

There are quite a lot of papers investigating injective hulls for algebras. Here we only mention some of them which our current manuscript is related to. Injective hulls for posets were studied by Banaschewski and Bruns ([1], 1967) where they got that the injective hull of a poset is its MacNeille completion. After that, Bruns and Lakser, and independently Horn and Kimura constructed injective hulls of semilattices ([2], 1970 and [3], 1971), and their results were soon applied into S-systems over a semilattice by Johnson, Jr., and McMorris ([4], 1972). By the conclusion of Schein ([5], 1974) that there are no non-trivial injectives in the category of semigroups, it took a long time to make further progress for the theory of injective hulls on both discrete and ordered (general) semigroups. In 2012, Lambek, Barr, Kennison and Raphael ([6]) studied a kind of category of pomonoids in which the usual category of pomonoids is a subcategory, and found injective hulls for pomonoids. Later on, Zhang and Laan generalized their results, first to the posemigroup case ([7], 2014), and later to S-posets ([8], 2015). In 2017, Xia, Zhao, and Han ([9]) obtained almost the same constructions as in [7], but they described it in a different way. A related approach to injectivity can be found in [10] and [11] for quantum B-algebras (which cover the majority of implicational algebras). Recently, injective hulls were constructed for quantum B-modules [12], as well as for S-semigroups and semicategories [13].

It is natural to study injectivity with respect to homomorphisms that are order-embeddings in the case of ordered algebras. However, in many cases it turns out that the only injectives with respect to this class are the trivial ones (see, e.g., [14] for the case of lattices or [6] for the case of posemigroups). If we admit more morphisms, one will get non-trivial injectives (see Section 3 for details).

The purpose of this paper is to study injectivity on universal ordered algebras which are many-sorted. We will deal with heterogeneous algebras, as in [15]. Let us first fix terminology and notation.

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Throughout the paper, $\Sigma = \langle S, O\rangle$ will be a fixed but arbitrary signature, where S is a set of sorts, O is a family of operation symbols. A *Σ-algebra A* is an S-indexed family of sets $A_s,\ s \in S$, equipped with operations $o_A : A_{s_1} \times \cdots \times A_{s_n} \to A_s$ for each operation symbol o of rank $s_1 \cdots s_n \to s,\ n \in \mathbb{N}$.

A homomorphism $h : A \to B$ of Σ-algebras is an S-indexed family of mappings $h_s : A_s \to B_s,\ s \in S$, such that for any $o \in O$ with rank $s_1 \cdots s_n \to s,\ n \in \mathbb{N}$, and $x_{s_i} \in A_{s_i},\ i = 1, \ldots, n$, we have

$$h_s(o_A(x_{s_1}, \ldots, x_{s_n})) = o_B(h_{s_1}(x_{s_1}), \ldots, h_{s_n}(x_{s_n})).$$

A Σ-algebra A is said to be *ordered* if for each sort $s \in S,\ A_s$ is a poset, and $o_A : A_{s_1} \times \cdots \times A_{s_n} \to A_s$ preserves ordering for each $o \in O$ with rank $s_1 \cdots s_n \to s,\ n \in \mathbb{N}$.

Let $A,\ B$ be ordered Σ-algebras, $h : A \to B$ an S-indexed family of mappings $h_s : A_s \to B_s,\ s \in S$. We say that h is a *lax morphism* if each h_s is monotone and

$$o_B(h_{s_1}(x_{s_1}), \ldots, h_{s_n}(x_{s_n})) \leqslant h_s(o_A(x_{s_1}, \ldots, x_{s_n})),$$

for $o \in O$ with rank $s_1 \cdots s_n \to s,\ x_{s_i} \in A_{s_i},\ i = 1, \ldots, n,\ n \in \mathbb{N}$. All ordered Σ-algebras together with their lax morphisms form a category which we denote by $\mathsf{OAL}_\Sigma^{\leqslant}$.

For an ordered Σ-algebra A, *Σ-terms of sort s*, $s \in S$ are inductively defined as following:

(1) Every $x_s \in A_s$ is a Σ-term of sort s;
(2) If $x_{s_i} \in A_{s_i}$ are Σ-terms of sorts $s_i,\ i = 1, \ldots, n,\ n \in \mathbb{N}$, and $o \in O$ with rank $s_1 \cdots s_n \to s$ then $o(x_{s_1}, \ldots, x_{s_n})$ is a Σ-term of sort s;
(3) Σ-terms of sort s are those and only those which we get from (1) and (2) in a finite number of steps.

Recall that for a class of morphisms $\mathcal{M}$ in a category $\mathcal{C}$, an object Q from $\mathcal{C}$ is called *$\mathcal{M}$-injective* in $\mathcal{C}$ provided that for any morphism $h : A \to B$ in $\mathcal{M}$ and any morphism $f : A \to Q$ in $\mathcal{C}$ there exists a morphism $g : B \to Q$ in $\mathcal{C}$ such that $gh = f$. A category $\mathcal{C}$ is said to have *enough $\mathcal{M}$-injectives* if for every object C of $\mathcal{C}$, there exists an $\mathcal{M}$-morphism from C to an $\mathcal{M}$-injective object.

A morphism $\eta : A \to B$ in $\mathcal{M}$ is called *$\mathcal{M}$-essential* if every morphism $\psi : B \to C$ in $\mathcal{C}$, for which the composite $\psi\eta$ is in $\mathcal{M}$, is itself in $\mathcal{M}$. An object $H \in \mathcal{C}$ is called an *$\mathcal{M}$-injective hull* of an object $A \in \mathcal{C}$ if H is $\mathcal{M}$-injective and there exists an $\mathcal{M}$-essential morphism $A \to H$ (see [16, Def. 9.22]).

An order-embedding is a mapping $h : A \to B$ between posets $(A, \leqslant_A)$ and $(B, \leqslant_B)$ such that $a \leqslant_A b$ iff $h(a) \leqslant_B h(b)$, for all $a, b \in A$. For the case of ordered Σ-algebras, a lax morphism $h : A \to B$ of ordered Σ-algebras is said to be an *order-embedding* if for every $s \in S$, h_s is an order-embedding between posets A_s and B_s.

In this paper we will study injectivity in classes of ordered Σ-algebras with respect to two classes of order-embeddings. The first of them is the class σ of order-embeddings that are homomorphisms. The other class is a weak version in terms of lax morphisms.

We denote by T_Σ^S the set of all Σ-terms t^s of sort $s,\ s \in S$. Let $\sigma^{\leqslant}$ be the class of mappings $h : A \to B$ between ordered Σ-algebras that satisfy the following conditions:

(1) h is a lax morphism;
(2) for every $t^s \in T_\Sigma^S$, $a_{s_i} \in A_{s_i},\ i = 1, \ldots, n,\ n \in \mathbb{N}$,

$$t_B^s(h_{s_1}(a_{s_1}), \ldots, h_{s_n}(a_{s_n})) \leqslant h_s(a) \Rightarrow t_A^s(a_{s_1}, \ldots, a_{s_n}) \leqslant a.$$

Morphisms in $\sigma^{\leqslant}$ will be called *lax order-embeddings*. Using the term $t = x$, we see that every lax order-embedding is an order-embedding between ordered Σ-algebras.

Lemma 1. *For a class of ordered Σ-algebras, $\sigma \subseteq \sigma^{\leqslant}$.*

Proof. It is clear that a morphism h from σ is a lax morphism. Assume that $t_B^s(h_{s_1}(a_{s_1}), \ldots, h_{s_m}(a_{s_m})) \leqslant h_s(a)$. Then $h_s(t_A^s(a_{s_1}, \ldots, a_{s_m})) \leqslant h_s(a)$ since h is a homomorphism, and hence $t_A^s(a_{s_1}, \ldots, a_{s_m}) \leqslant a$ by the fact that h is an order-embedding. □

By a *sup-Σ-algebra*, we mean an ordered Σ-algebra whose carriers are also sup-lattices and whose operations are sup-lattice homomorphisms in each variable separately. A homomorphism of sup-Σ-algebras is a homomorphism of ordered Σ-algebras whose components are sup-lattice homomorphisms.

Let A be a sup-Σ-algebra. A *nucleus* j on A is an S-indexed family of closure operators which is a lax morphism.

Let A be a sup-Σ-algebra, j a nucleus on A with an S-indexed family of closure operators $j_s,\ s \in S$. Then A_j, which is an S-indexed family of $A_{s_{j_s}}$, is a sup-Σ-algebra under the operation induced from A :

$$o_{A_j}(a_{s_1}, \ldots, a_{s_n}) = j_s(o_A(a_{s_1}, \ldots, a_{s_n})),$$

where $o \in O$ with rank $s_1 \cdots s_n \to s$, $n \in \mathbb{N}$, $a_{s_i} \in A_{s_{i j_{s_i}}}$, $i \in \{1, \ldots, n\}$, and by the fact that $A_{s_{j_s}}$ is a complete lattice under joins

$$\bigvee M = j_s\left(\bigvee M\right),$$

for every $M \subseteq A_{s_{j_s}}$, $s \in S$.

2 σ-injective and $\sigma^{\leqslant}$-injective presentations

This section contributes to the observation of σ-injectives and $\sigma^{\leqslant}$-injectives in the category $\mathsf{OAL}_{\Sigma}^{\leqslant}$.

Theorem 2. *Every sup-Σ-algebra is $\sigma^{\leqslant}$-injective and therefore σ-injective in the category $\mathsf{OAL}_{\Sigma}^{\leqslant}$.*

Proof. Let Q be a sup-Σ-algebra. It is enough to show that Q is $\sigma^{\leqslant}$-injective in $\mathsf{OAL}_{\Sigma}^{\leqslant}$. Suppose that $h : A \to B$ is a lax order-embedding and $f : A \to Q$ is a morphism in $\mathsf{OAL}_{\Sigma}^{\leqslant}$. We have to find a lax morphism $g : B \to Q$ such that the following diagram commutes.

$$\begin{tikzcd} A \arrow[rr, "h"] \arrow[dr, "f"'] & & B \arrow[dl, dashed, "g"] \\ & Q & \end{tikzcd}$$

Let $g : B \to Q$ be an S-indexed family of mappings $g_s : B_s \to Q_s,\ s \in S$, which is defined by

$$\begin{aligned} g_s(b_s) = \bigvee\{t_Q^s(f_{s_1}(a_{s_1}), \ldots, f_{s_m}(a_{s_m})) \mid t_B^s(h_{s_1}(a_{s_1}), \ldots, h_{s_m}(a_{s_m})) \leqslant b_s,\\ t^s \in T_{\Sigma}^S, s \in S,\ a_{s_i} \in A_{s_i},\ i = 1, \ldots, m,\ m \in \mathbb{N}\}, \end{aligned}$$

for any $b_s \in B_s$. Let us write the above join shortly as

$$\bigvee_{h^{b_s}} t_Q^s(f_{s_1}(a_{s_1}), \ldots, f_{s_m}(a_{s_m})).$$

It is clear that g_s is monotone. Let us show that g is a lax morphism, that is, the inequality

$$o_Q(g_{s_1}(b_{s_1}), \ldots, g_{s_n}(b_{s_n})) \leqslant g_s(o_B(b_{s_1}, \ldots, b_{s_n})),$$

holds for any $o \in O$ with rank $s_1 \cdots s_n \to s$, $n \in \mathbb{N}$, $b_{s_i} \in B_{s_i}$, $i = 1, \ldots, n$. Since Q is a sup-Σ-algebra whose operations are sup-lattice homomorphisms in each variable separately, it turns out that

$$\begin{aligned} o_Q(g_{s_1}(b_{s_1}), \ldots, g_{s_n}(b_{s_n})) &= o_Q\left(\bigvee_{h^{b^{s_1}}} t_Q^{s_1}\left(f_{s_{11}}(a_{s_{11}}), \ldots, f_{s_{1m_1}}(a_{s_{1m_1}})\right), \ldots, \bigvee_{h^{b^{s_n}}} t_Q^{s_n}\left(f_{s_{n1}}(a_{s_{n1}}), \ldots, f_{s_{nm_n}}(a_{s_{nm_n}})\right)\right) \\ &= \bigvee_{\substack{h^{b_{s_1}} \\ \cdots \\ h^{b_{s_n}}}} o_Q\left(t_Q^{s_1}\left(f_{s_{11}}(a_{s_{11}}), \ldots, f_{s_{1m_1}}(a_{s_{1m_1}})\right), \ldots, t_Q^{s_n}\left(f_{s_{n1}}(a_{s_{n1}}), \ldots, f_{s_{nm_n}}(a_{s_{nm_n}})\right)\right) \end{aligned}$$

$$\leqslant \bigvee_{h^{o_B(b_{s_1},\dots,b_{s_n})}} t_Q^{s_m}\left(f_{s_1}(a_{s_1}),\dots,f_{s_m}(a_{s_m})\right)$$
$$= g_s(o_B(b_{s_1},\dots,b_{s_n})).$$

Note that the inequality holds because the inequalities

$$t_Q^{s_1}\left(h_{s_{11}}(a_{s_{11}}),\dots,h_{s_{1m_1}}(a_{s_{1m_1}})\right) \leqslant b_{s_1}$$
$$\vdots$$
$$t_Q^{s_n}\left(h_{s_{n1}}(a_{s_{n1}}),\dots,h_{s_{nm_n}}(a_{s_{nm_n}})\right) \leqslant b_{s_n}$$

imply that

$$o_B\left(t_Q^{s_1}\left(h_{s_{11}}(a_{s_{11}}),\dots,h_{s_{1m_1}}(a_{s_{1m_1}})\right),\dots,t_Q^{s_n}\left(h_{s_{n1}}(a_{s_{n1}}),\dots,h_{s_{nm_n}}(a_{s_{nm_n}})\right)\right) \leqslant o_B(b_{s_1},\dots b_{s_n}),$$

and

$$o_B(t^{s_1}(x_{11},\dots,x_{1m_1}),\dots,t^{s_n}(x_{n1},\dots,x_{nm_n}))$$

is a Σ-term of sort s.

To complete the proof, let us show that $g_s h_s = f_s$ for any $s \in S$. Take $a_s \in A_s$, by the fact that

$$g_s h_s(a_s) = \bigvee_{h^{h_s(a_s)}} t_Q^s(f_{s_1}(a_{s_1}),\dots,f_{s_m}(a_{s_m}))$$

and

$$t_B^s(h_{s_1}(a_{s_1}),\dots,h_{s_m}(a_{s_m})) \leqslant h_s(a_s),$$

we obtain that $t_A^s(a_{s_1},\dots,a_{s_m}) \leqslant a_s$ since $h \in \sigma^{\leqslant}$. This indicates that

$$t_Q^s(f_{s_1}(a_{s_1}),\dots,f_{s_n}(a_{s_m})) \leqslant f_s(t_A^s(a_{s_1},\dots,a_{s_m})) \leqslant f_s(a_s).$$

Therefore $g_s h_s(a_s) \leqslant f_s(a_s)$. On the other hand, using the term $t^s = x_s$, we see that $f_s(a_s)$ is in the join that defines $g_s h_s(a_s)$, consequently $g_s h_s = f_s$ as required. □

Let A be an ordered Σ-algebra. We write $\mathscr{P}(A)$ as an S-indexed family of posets $(\mathscr{P}(A)_s, \subseteq)$, where $\mathscr{P}(A)_s$ is the set of all down-sets of A_s for any sort $s \in S$. The operations on $\mathscr{P}(A)$ are derived from A in the following sense:

$$o_{\mathscr{P}(A)}(D_{s_1},\dots,D_{s_n}) = o_A(D_{s_1},\dots,D_{s_n})\downarrow \tag{2.1}$$

for any $o \in O$ with rank $s_1 \cdots s_n \to s$, $D_{s_i} \in \mathscr{P}(A)_{s_i},\ i = 1,\dots,n,\ n \in \mathbb{N}$, where

$$o_A(D_{s_1},\dots,D_{s_n}) := \{o_A(d_{s_1},\dots,d_{s_n}) \mid d_{s_1} \in D_{s_1},\dots,d_{s_n} \in D_{s_n}\}.$$

It is routine to check that $\mathscr{P}(A)$ equipped with the operations defined in (2.1) is a sup-Σ-algebra, and thus injective by Theorem 2.

The following lemma is easy to verify.

Lemma 3. *Let A be an ordered Σ-algebra. Then for any $s \in S,\ n \in \mathbb{N},\ o \in O$ with rank $s_1 \cdots s_n \to s$, and $a_{s_i} \in A_{s_i},\ i = 1,\dots,n$, we have*

$$o_A(a_{s_1}\downarrow,\dots,a_{s_n}\downarrow)\downarrow = o_A(a_{s_1},\dots,a_{s_n})\downarrow,$$

and therefore

$$t_A^s(a_{s_1}\downarrow,\dots,a_{s_n}\downarrow)\downarrow = t_A^s(a_{s_1},\dots,a_{s_n})\downarrow$$

for any $t^s \in T_\Sigma^S$.

Proposition 4. *Let A be an ordered Σ-algebra. Then $\mathscr{P}(A)$ is $\sigma^{\leqslant}$-injective and σ-injective in the category $\mathsf{OAL}_{\Sigma}^{\leqslant}$, which A can be embedded in.*

Proof. Given an ordered Σ-algebra A, the injectivity of $\mathscr{P}(A)$ follows by the fact that $\mathscr{P}(A)$ is a sup-Σ-algebra. Let us consider $\eta : A \to \mathscr{P}(A)$, which is an S-indexed family of mappings $\eta_s : A_s \to \mathscr{P}(A)_s,\ s \in S$, where $\eta_s(a_s) = a_s{\downarrow}$, for any $a_s \in A_s$. It is clear that for each sort s, η_s is an order-embedding of the poset $(A_s, \leqslant_{A_s})$ into the poset $(\mathscr{P}(A)_s, \subseteq)$. Moreover, for any $s \in S,\ o \in O$ with rank $s_1 \cdots s_n \to s,\ n \in \mathbb{N}$ and $a_{s_i} \in A_{s_i},\ i = 1, \dots, n$, Lemma 3 gives that

$$\begin{aligned} o_{\mathscr{P}(A)}(\eta_{s_1}(a_{s_1}), \dots, \eta_{s_n}(a_{s_n})) &= o_A(a_{s_1}{\downarrow}, \dots, a_{s_n}{\downarrow}){\downarrow} \\ &= o_A(a_{s_1}, \dots, a_{s_n}){\downarrow} \\ &= \eta_s(o_A(a_{s_1}, \dots, a_{s_n})). \end{aligned}$$

So $\eta : A \to \mathscr{P}(A)$ is a homomorphism of ordered Σ-algebras, thereby η belongs to both σ and $\sigma^{\leqslant}$. □

Lemma 5. *In the category $\mathsf{OAL}_{\Sigma}^{\leqslant}$, every retract of a sup-Σ-algebra is a sup-Σ-algebra.*

Proof. Let Q be a sup-Σ-algebra and let A be a retract of Q. Then there exist lax morphisms $\iota : A \to Q$ and $f : Q \to A$, where ι and f are relative S-indexed family of mappings $\iota_s : A_s \to Q_s$ and $f_s : Q_s \to A_s$ respectively, fulfilling $f_s g_s = id_{A_s}$ for all $s \in S$, where id_{A_s} is the identity mapping on A_s. Clearly, A_s is a sup-lattice for all $s \in S$.

For any $o \in O$ with rank $s_1 \cdots s_n \to s,\ n \in \mathbb{N}$ and $M \subseteq A_{s_i},\ i \in \{1, \dots, n\}$, let us show that

$$o_A\left(a_{s_1}, \dots, \bigvee M, \dots, a_{s_n}\right) = \bigvee_{m \in M} o_A(a_{s_1}, \dots, m, \dots, a_{s_n}),$$

for $a_{s_k} \in A_{s_k},\ k = \{1, \dots, n\}\backslash\{i\}$. For this purpose, assume that u is an upper bound of the set of all elements $o_A(a_{s_1}, \dots, m, \dots, a_{s_n}),\ m \in M$. Then $\iota_s(u) \geqslant \bigvee\limits_{m \in M} \iota_s(o_A(a_{s_1}, \dots, m, \dots, a_{s_n}))$. Hence we have

$$\begin{aligned} o_A\left(a_{s_1}, \dots, \bigvee M, \dots, a_{s_n}\right) &= o_A\left(a_{s_1}, \dots, \bigvee_{m \in M} f_{s_i}(\iota_{s_i}(m)), \dots, a_{s_n}\right) \\ &\leqslant o_A\left(a_{s_1}, \dots, f_{s_i}\left(\bigvee_{m \in M} \iota_{s_i}(m)\right), \dots, a_{s_n}\right) \\ &= o_A\left(f_{s_1}(\iota_{s_1}(a_{s_1})), \dots, f_{s_i}\left(\bigvee_{m \in M} \iota_{s_i}(m)\right), \dots, f_{s_n}(\iota_{s_n}(a_{s_n}))\right) \\ &\leqslant f_s\left(o_Q\left(\iota_{s_1}(a_{s_1}), \dots, \bigvee_{m \in M} \iota_{s_i}(m), \dots, \iota_{s_n}(a_{s_n})\right)\right) \\ &= f_s\left(\bigvee_{m \in M} o_Q(\iota_{s_1}(a_{s_1}), \dots, \iota_{s_i}(m), \dots, \iota_{s_n}(a_{s_n}))\right) \\ &\leqslant f_s\left(\bigvee_{m \in M} \iota_s(o_A(a_{s_1}, \dots, m, \dots, a_{s_n}))\right) \\ &\leqslant f_s(\iota_s(u)) = u. \end{aligned}$$

□

By the above preparations, we eventually conclude that there are enough σ-injectives and $\sigma^{\leqslant}$-injectives in the category $\mathsf{OAL}_{\Sigma}^{\leqslant}$, which are exactly sup-$\Sigma$-algebras.

Theorem 6. *For an ordered Σ-algebra A, the following statements are equivalent:*

(1) *A is $\sigma^{\leqslant}$-injective in $\mathsf{OAL}_{\Sigma}^{\leqslant}$,*

(2) *A is σ-injective in* $\mathsf{OAL}_{\Sigma}^{\leqslant}$,
(3) *A is a sup-Σ-algebra.*

Proof. (1) $\Rightarrow$ (2) follows from Lemma 1, and (3) $\Rightarrow$ (1) follows from Theorem 2.

(2) $\Rightarrow$ (3) holds by the reason that $\eta : A \to \mathscr{P}(A)$ given in Proposition 4 has a left inverse, and thus A is a retract of $\mathscr{P}(A)$. Therefore A is a sup-Σ-algebra by Lemma 5. □

3 Some applications

Example 7. A *posemigroup* $(S, \otimes, \leqslant)$ is a semigroup $(S, \otimes)$ equipped with a partial ordering $\leqslant$ which is compatible with the semigroup multiplication, that is, $s \otimes s' \leqslant t \otimes t'$ whenever $s \leqslant t$, $s' \leqslant t'$ in S.

A mapping $f : S \to T$ between posemigroups $(S, \otimes, \leqslant)$ and $(T, \otimes, \leqslant)$is said to be *submultiplicative* if $f(a) \otimes f(a') \leqslant f(a \otimes a')$ for all $a, a' \in S$.

A *quantale* (cf. [17]) is a posemigroup $(S, \otimes, \leqslant)$ such that

(1) the poset $(S, \leqslant)$ is a complete lattice,
(2) $s \otimes (\bigvee M) = \bigvee\{s \otimes m \mid m \in M\}$ and $(\bigvee M) \otimes s = \bigvee\{m \otimes s \mid m \in M\}$ for each subset M of S and each $s \in S$.

Then quantales are injectives in the category $\mathsf{PoSgr}_{\leqslant}$ where objects are posemigroups and morphisms are submultiplicative order-preserving mappings ([7]).

Example 8. Let $(S, \otimes, \leqslant)$ be a posemigroup. A poset $(A, \leqslant)$ together with a mapping $S \times A \to A$ (under which a pair (s, a) maps to an element of A denoted by $s \star a$) is called an *S-poset*, denoted by ${}_SA$, if for any $a, b \in A,\ s, t \in S$,

(1) $(s \otimes t) \star a = s \star (t \star a)$,
(2) $a \leqslant b,\ s \leqslant t \Rightarrow s \star a \leqslant t \star b$.

If S is a pomonoid, then ${}_SA$ fulfils

$$1 \star a = a,\ \forall a \in A.$$

An S-poset A_S is called an *S-quantale* if

(1) the poset A is a complete lattice,
(2) $s \star (\bigvee M) = \bigvee\{s \star m \mid m \in M\}$ for each subset M of A and each $s \in S$.

A mapping $f : {}_SA \to {}_SB$ of S-posets is said to be *S-submultiplicative* if $s \star f(a) \leqslant f(s \star a)$ for any $a \in A,\ s \in S$. Let S be a pomonoid, $\mathsf{Pos}_S^{\leqslant}$ be the category where objects are right S-posets and morphisms are S-submultiplicative order-preserving mappings, $\mathcal{E}_{\leqslant}$ be the class of morphisms $e : {}_SA \to {}_SB$ in the category $\mathsf{Pos}_S^{\leqslant}$ which satisfy the following condition: $s \star e(a) \leqslant e(a')$ implies $s \star a \leqslant a'$ for all $a, a' \in A$ and $s \in S$. Evidently, each morphism in $\mathcal{E}_{\leqslant}$ is an order-embedding. Then $\mathcal{E}_{\leqslant}$-injectives in the category $\mathsf{Pos}_S^{\leqslant}$ are exactly S-quantales ([8]).

Example 9. Let $(S, \otimes, \leqslant)$ be a posemigroup. A posemigroup $(A, \cdot, \leqslant)$, which is also an S-poset, where the action is denoted by $\star$, is called an *S-semigroup*, if for any $a, b \in A,\ s, t \in S$,

$$s \star (a \cdot b) = (s \star a) \cdot b = a \cdot (s \star b).$$

An *S-semigroup quantale* is an S-semigroup $({}_SA, \cdot, \star)$ such that $(A, \cdot, \bigvee)$ is a quantale and $({}_SA, \star)$ is an S-quantale.

An order-preserving mapping $f\colon {}_SA \to {}_SB$ of S-semigroups is called a *subhomomorphism* if it is both submultiplicative in posemigroups, i.e.,

$$f(a_1) \cdot f(a_2) \leqslant f(a_1 \cdot a_2)$$

for all $a_1, a_2 \in A$, and S-submultiplicative in S-posets, i.e.,

$$s \star f(a) \leqslant f(s \star a)$$

for all $a \in A$, $s \in S$. The category of S-semigroups with subhomomorphisms as morphisms is denoted by $\mathsf{Ssgr}_{\leqslant}$.

Let $\mathcal{E}_{\leqslant}$ be the class of morphisms $e : {}_SA \to {}_SB$ in the category $\mathsf{Ssgr}_{\leqslant}$ which satisfy the following conditions:

$$s \star (e(a_1) \cdot \ldots \cdot e(a_n)) \leqslant e(a) \Rightarrow s \star (a_1 \cdot \ldots \cdot a_n) \leqslant a,$$

and

$$e(a_1) \cdot \ldots \cdot e(a_n) \leqslant e(a) \Rightarrow a_1 \cdot \ldots \cdot a_n \leqslant a,$$

for all $a_1, a_2, \ldots, a_n, a \in A$, $s \in S$. Then $\mathcal{E}_{\leqslant}$-injectives in the category $\mathsf{Ssgr}_{\leqslant}$ are indeed S-semigroup quantales ([13]).

Example 10. Let us denote by Pos the category of posets and order-preserving mappings and Sup the category of sup-lattices and sup-preserving mappings. An *ordered semicategory* (*category*) is a locally small semicategory (category) such that *hom*-sets are partially ordered and composition on both sides is order-preserving.

A *lax semifunctor* $F\colon \mathcal{C} \to \mathcal{D}$ of ordered categories is given by functions

$$F\colon\ ob\,\mathcal{C} \to ob\,\mathcal{D} \text{ and } F_{X,Y}\colon \mathcal{C}(X, Y) \to \mathcal{D}(FX, FY)$$

for all $X \in ob\,\mathcal{C}$ (with $F_{X,Y}$ usually written only as F), such that
(1) $F_{X,Y}$ is monotone;
(2) $Fg \circ Ff \leqslant F(g \circ f)$,
for all $X, Y, Z \in ob\,\mathcal{C}, f\colon X \to Y, g\colon Y \to Z$ in $\mathcal{C}$. A lax semifunctor $F\colon \mathcal{C} \to \mathcal{D}$ is a *semifunctor* if
(2 =) $Fg \circ Ff = F(g \circ f)$,
for all $X, Y, Z \in ob\,\mathcal{C}, f\colon X \to Y, g\colon Y \to Z$ in $\mathcal{C}$.

Let $\mathcal{S}$ be a small ordered semicategory. An $\mathcal{S}$-*module* is a semifunctor $A\colon \mathcal{S} \to \mathsf{Pos}$ of ordered semicategories.

An $\mathcal{S}$-*morphism* is a lax natural map between $\mathcal{S}$-modules A and B, that is, a family $\alpha = \{\alpha_X \in \mathsf{Pos}(AX, BX) \mid X \in \mathcal{S}\}$ of order-preserving mappings such that for every $f\colon X \to Y$ in $\mathcal{S}$, we have $B(f) \circ \alpha_X \leqslant \alpha_Y \circ A(f)$.

An $\mathcal{S}$-$\mathcal{Q}$-*module* is an $\mathcal{S}$-module A such that for every two objects $X, Y \in \mathcal{S}$ and for every $f\colon X \to Y$ we have that AX and AY are sup-lattices and $A(f)$ is a sup-preserving mapping, i.e., A yields a semifunctor of ordered semicategories into Sup.

The category $\mathcal{S} - Mod$ of $\mathcal{S}$-modules has $\mathcal{S}$-modules as objects and lax natural maps as morphisms.

An *order embedding* ε between $\mathcal{S}$-modules A and B is an $\mathcal{S}$-morphism $\varepsilon\colon A \to B$ such that $\varepsilon_X\colon AX \to BX$ is an order embedding in posets for all $X \in \mathcal{S}$. Let $\mathcal{E}_{\leqslant_{\mathcal{S}}}$ be the class of order embeddings $\varepsilon : A \longrightarrow B$ in the category $\mathcal{S} - Mod$ which satisfy the following conditions:

$$B(f)(\varepsilon_X(a)) \leqslant \varepsilon_Y(b) \Longrightarrow A(f)(a) \leqslant b,$$

for all $a \in A(X)$, $b \in A(Y)$ and all $f\colon X \to Y$ in $\mathcal{S}$. Then $\mathcal{S}$-$\mathcal{Q}$-modules are $\mathcal{E}_{\leqslant_{\mathcal{S}}}$-injectives in the category $\mathcal{S} - Mod$ of $\mathcal{S}$-modules ([13]).

4 Constructions

In this section, we find a way to construct $\sigma^{\leqslant}$-injective hulls of ordered Σ-algebras in the category $\mathsf{OAL}_{\Sigma}^{\leqslant}$.

We first introduce the concept of unary polynomial functions. Let A be an ordered Σ-algebra. A unary polynomial function $p^{\tilde{s}}$ over A_s for any sort $s \in S$ is of the form $t_A^{\tilde{s}}(a_{s_1}, \ldots, a_{s_{i-1}}, \_, a_{s_{i+1}}, \ldots, a_{s_n})$, where $t^{\tilde{s}}$

is a Σ-term of sort $\widetilde{s}$, $a_{s_k} \in A_{s_k}$, $k \in \{1, \ldots, n\} \setminus \{i\}$, $n \in \mathbb{N}$. In this situation, $p^{\widetilde{s}}$ is indeed a mapping from A_s to $A_{\widetilde{s}}$. We denote by $P^1_{A_s}$ the set of all unary polynomial functions over A_s.

In particular, for $o \in O$ with rank $s_1 \cdots s_n \to s$, $n \in \mathbb{N}$, the function

$$o_A(a_{s_1}, \ldots, a_{s_{i-1}}, \_, a_{s_{i+1}}, \ldots, a_{s_n}) : A_{s_i} \to A_s,$$

which we write as o_{s_i}, is called an *elementary translation* with dependences a_{s_j} from A_{s_j} for $j \in \{1, \ldots, n\} \setminus \{i\}$.

Let $\mathrm{j} : \mathscr{P}(A) \to \mathscr{P}(A)$ be an S-indexed family of mappings $\mathrm{j}_s : \mathscr{P}(A)_s \to \mathscr{P}(A)_s$, $s \in S$, where j_s is defined by

$$\mathrm{j}_s(D_s) = \{x \in A_s \mid p^{\widetilde{s}}(D_s) \subseteq a{\downarrow} \Rightarrow p^{\widetilde{s}}(x) \leqslant a \text{ for all } p^{\widetilde{s}} \in P^1_{A_s},\ a \in A_{\widetilde{s}},\ \widetilde{s} \in S\} \tag{4.1}$$

for any $D_s \in \mathscr{P}(A)_s$, and the $p^{\widetilde{s}}(D_s)$ are given by $\{p^{\widetilde{s}}(d) \mid d \in D_s\}$. It is straightforward to verify that $\mathrm{j}_s(D_s)$ is a down-set in $\mathscr{P}(A)_s$. Furthermore, we claim that j is a nucleus on $\mathscr{P}(A)$.

Proposition 11. *Let A be an ordered Σ-algebra, $\mathrm{j} : \mathscr{P}(A) \to \mathscr{P}(A)$ be an S-indexed family of mappings j_s defined as in (4.1). Then j is a nucleus on $\mathscr{P}(A)$ satisfying $\mathrm{j}_s(a_s{\downarrow}) = (a_s{\downarrow})$ for any $a_s \in A_s$, $s \in S$.*

Proof. Let us first show that for any $s \in S$, j_s is a closure operator on $\mathscr{P}(A)_s$. It is obvious that j_s is increasing and monotone, in particular $\mathrm{j}_s(D_s) \subseteq \mathrm{j}_s(\mathrm{j}_s(D_s))$ for every $D_s \in \mathscr{P}(A)_s$. Take $u \in \mathrm{j}_s(\mathrm{j}_s(D_s))$ and assume that $p^{\widetilde{s}}(D_s) \subseteq a{\downarrow}$ for some $a \in A_{\widetilde{s}}$ and $p^{\widetilde{s}} \in P^1_{A_s}$, $\widetilde{s} \in S$. Since for any $v \in \mathrm{j}_s(D_s)$, we have $p(v) \leqslant a$, it follows that $p^{\widetilde{s}}(\mathrm{j}_s(D_s)) \subseteq a{\downarrow}$. Thus $p^{\widetilde{s}}(u) \leqslant a$ and $u \in \mathrm{j}_s(D_s)$. Consequently, $\mathrm{j}_s(\mathrm{j}_s(D_s)) = \mathrm{j}_s(D_s)$ as needed.

We next show that j is a lax morphism on $\mathscr{P}(A)$, namely, we need to prove $o_{\mathscr{P}(A)}(\mathrm{j}_{s_1}(D_{s_1}), \ldots, \mathrm{j}_{s_n}(D_{s_n})) \subseteq \mathrm{j}_s(o_{\mathscr{P}(A)}(D_{s_1}, \ldots, D_{s_n}))$ for any $s \in S$, $o \in O$ with rank $s_1 \cdots s_n \to s$, $n \in \mathbb{N}$ and $D_{s_i} \in \mathscr{P}(A)_{s_i}$, $i = 1, \ldots, n$. It is sufficient to show that the inclusion

$$o_{\mathscr{P}(A)}\left(D_{s_1}, \ldots, D_{s_{i-1}}, \mathrm{j}_{s_i}(D_{s_i}), D_{s_{i+1}}, \ldots, D_{s_n}\right) \subseteq \mathrm{j}_s\left(o_{\mathscr{P}(A)}(D_{s_1}, \ldots, D_{s_n})\right) \tag{4.2}$$

is satisfied, that is, the inclusion

$$o_A\left(D_{s_1}, \ldots, D_{s_{i-1}}, \mathrm{j}_{s_i}(D_{s_i}), D_{s_{i+1}}, \ldots, D_{s_n}\right){\downarrow} \subseteq \mathrm{j}_s\left(o_A(D_{s_1}, \ldots, D_{s_n}){\downarrow}\right) \tag{4.3}$$

holds. For this aim, take $z \in A_s$ such that $z \leqslant o_A(d_{s_1}, \ldots, d_{s_{i-1}}, \overline{d}_{s_i}, d_{s_{i+1}}, \ldots, d_{s_n})$, where $d_{s_j} \in D_{s_j}$, $j \in \{1, \ldots, n\} \setminus \{i\}$, and $\overline{d}_{s_i} \in \mathrm{j}_{s_i}(D_{s_i})$. Then $z \leqslant o_{s_i}(\overline{d}_{s_i})$. Suppose that for a unary polynomial function $p^{\widetilde{s}}$ over A_s, $p^{\widetilde{s}}(o_A(D_{s_1}, \ldots, D_{s_n}){\downarrow}) \subseteq a{\downarrow}$ for some $a \in A_{\widetilde{s}}$, then we have

$$p^{\widetilde{s}}\left(o_{s_i}(D_{s_i})\right) \subseteq p^{\widetilde{s}}\left(o_A(D_{s_1}, \ldots, D_{s_n})\right) \subseteq p^{\widetilde{s}}\left(o_A(D_{s_1}, \ldots, D_{s_n}){\downarrow}\right) \subseteq a{\downarrow}.$$

Since $p^{\widetilde{s}} o_{s_i} \in P^1_{A_{s_i}}$, we obtain $p^{\widetilde{s}} o_{s_i}(\overline{d}_{s_i}) \leqslant a$ because $\overline{d}_{s_i} \in \mathrm{j}_{s_i}(D_{s_i})$. Therefore, $p^{\widetilde{s}}(z) \leqslant p^{\widetilde{s}}(o_{s_i}(\overline{d}_{s_i})) \leqslant a$. We have shown that $z \in \mathrm{j}_s(o_A(D_{s_1}, \ldots, D_{s_n}){\downarrow})$, and hence the inclusion (4.3) holds.

Applying this fact n times and using idempotency of j_s, it follows that

$$\begin{aligned} o_{\mathscr{P}(A)}\left(\mathrm{j}_{s_1}(D_{s_1}), \ldots, \mathrm{j}_{s_n}(D_{s_n})\right) &\subseteq \mathrm{j}_s\left(o_{\mathscr{P}(A)}(D_{s_1}, \mathrm{j}_{s_2}(D_{s_2}), \ldots, \mathrm{j}_{s_n}(D_{s_n}))\right) \\ &\subseteq \mathrm{j}_s\left(o_{\mathscr{P}(A)}(D_{s_1}, D_{s_2}, \mathrm{j}_{s_3}(D_{s_3}), \ldots, \mathrm{j}_{s_n}(D_{s_n}))\right) \\ &\ \ \vdots \\ &\subseteq \mathrm{j}_s\left(o_{\mathscr{P}(A)}(D_{s_1}, D_{s_2}, \ldots, D_{s_n})\right), \end{aligned}$$

which imply that j is a lax morphism and hence a nucleus on $\mathscr{P}(A)$. Finally, by the definition of j_s and the fact the id_{A_s} is a Σ-term of sort s, we immediately get that $\mathrm{j}_s(a_s{\downarrow}) = a_s{\downarrow}$, for all $a_s \in A_s$. □

Given an ordered Σ-algebra A, Proposition 11 presents a sup-Σ-algebra $\mathscr{P}(A)_{\mathrm{j}}$, which is a quotient of $\mathscr{P}(A)$ through the nucleus j, where $\mathscr{P}(A)_{\mathrm{j}}$ is an S-indexed family of $\mathscr{P}(A)_{\mathrm{j}_s}$, and

$$\mathscr{P}(A)_{\mathrm{j}_s} = \{D_s \in \mathscr{P}(A)_s \mid \mathrm{j}_s(D_s) = D_s\}$$

for any $s \in S$. We note that the operations on $\mathscr{P}(A)_{\mathrm{j}_s}$ are given by

$$o_{\mathscr{P}(A)_{\mathrm{j}}}(D_{s_1}, \ldots, D_{s_n}) = \mathrm{j}_s\left(o_{\mathscr{P}(A)}(D_{s_1}, \ldots, D_{s_n})\right) = \mathrm{j}_s\left(o_A(D_{s_1}, \ldots, D_{s_n}){\downarrow}\right),$$

for $o \in O$ with rank $s_1 \cdots s_n \to s$, $s \in S$, $n \in \mathbb{N}$, and $D_{s_i} \in \mathscr{P}(A)_{\mathrm{j}_{s_i}}$, $i = 1, \ldots, n$.

Lemma 12. [18, Lemma 2.2.6] *Let j be a nucleus on a sup-Σ-algebra A with an S-indexed family of closure operators j_s, $s \in S$. If for $n \in \mathbb{N}$, $o \in O$ with rank $s_1 \cdots s_n \to s$, $x_{s_i}, x'_{s_i} \in A_{s_i}$, we have $x_{s_i} \leqslant x'_{s_i} \leqslant j_{s_i}(x_{s_i})$, for any $i \in \{1, \ldots, n\}$, then*

$$j_s\left(o_A(x'_{s_1}, \ldots, x'_{s_n})\right) = j_s\left(o_A(x_{s_1}, \ldots, x_{s_n})\right).$$

By Lemma 12, we immediately get the following result.

Lemma 13. *If j is a nucleus on a sup-Σ-algebra A, then for any $o \in O$ with rank $s_1 \cdots s_n \to s$, $s \in S$, $n \in \mathbb{N}$, and $a_{s_i} \in A_{s_i}$, $i = 1, \ldots, n$, one has that*

$$j_s\left(o_A(j_{s_1}(a_{s_1}), \ldots, j_{s_n}(a_{s_n}))\right) = j_s(o_A(a_{s_1}, \ldots, a_{s_n})).$$

The following result can be easily obtained from Lemma 13.

Lemma 14. *Let A be an ordered Σ-algebra, j be the nucleus on $\mathscr{P}(A)$ with an S-indexed family of closure operators j_s, $s \in S$, defined in (4.1). Then for any $o \in O$ with rank $s_1 \cdots s_n \to s$, $n \in \mathbb{N}$, and $D_{s_i} \in \mathscr{P}(A)_{s_i}$, $i = 1, \ldots, n$, we have*

$$\mathrm{j}_s\left(o_{\mathscr{P}(A)}(D_{s_1}, \ldots, D_{s_n})\right) = \mathrm{j}_s\left(o_{\mathscr{P}(A)}(\mathrm{j}_{s_1}(D_{s_1}), \ldots, \mathrm{j}_{s_n}(D_{s_n}))\right).$$

Therefore, for every $t^s \in T^S_\Sigma$, $s \in S$, $D_{s_i} \in \mathscr{P}(A)_{s_i}$, $i = 1, \ldots, n$, $n \in \mathbb{N}$, we have

$$\mathrm{j}_s\left(t^s_{\mathscr{P}(A)}(D_{s_1}, \ldots, D_{s_n})\right) = \mathrm{j}_s\left(t^s_{\mathscr{P}(A)}\left(\mathrm{j}_{s_1}(D_{s_1}), \ldots, \mathrm{j}_{s_n}(D_{s_n})\right)\right).$$

As a result, we obtain a corollary.

Corollary 15. *Let A be an ordered Σ-algebra. Then*

$$t^s_{\mathscr{P}(A)_\mathrm{j}}(D_{s_1}, \ldots, D_{s_n}) = \mathrm{j}_s\left(t^s_{\mathscr{P}(A)}(D_{s_1}, \ldots, D_{s_n})\right) = \mathrm{j}_s\left(t^s_A(D_{s_1}, \ldots, D_{s_n})\downarrow\right)$$

for every $t^s \in T^S_\Sigma$, $s \in S$, $D_{s_i} \in \mathscr{P}(A)_{\mathrm{j}_{s_i}}$, $i = 1, \ldots, n$, $n \in \mathbb{N}$.

Now we assert that we are ready to construct a suitable injective hull for every ordered Σ-algebra in the category $\mathsf{OAL}^{\leqslant}_\Sigma$.

Theorem 16. *Let A be an ordered Σ-algebra. Then $\mathscr{P}(A)_\mathrm{j}$ is the $\sigma^{\leqslant}$-injective hull of A in the category $\mathsf{OAL}^{\leqslant}_\Sigma$.*

Proof. It is clear that $\mathscr{P}(A)_\mathrm{j}$ is $\sigma^{\leqslant}$-injective. Consider the mapping $\eta : A \to \mathscr{P}(A)_\mathrm{j}$, which is given by an S-indexed family of mappings $\eta_s : A_s \to \mathscr{P}(A)_{\mathrm{j}_s}$, where $\eta_s(a_s) = a_s\downarrow$, for any $a_s \in A_s$, $s \in S$. Then obviously η is an order-embedding. Moreover, η is indeed a homomorphism and thus belongs to σ and $\sigma^{\leqslant}$, respectively. In fact, for any $s \in S$, $o \in O$ with rank $s_1 \cdots s_n \to s$, $n \in \mathbb{N}$, and $a_{s_i} \in A_{s_i}$, $i = 1, \ldots, n$, by Lemma 3, we have

$$\begin{aligned} o_{\mathscr{P}(A)_\mathrm{j}}\left(\eta_{s_1}(a_{s_1}), \ldots, \eta_{s_n}(a_{s_n})\right) &= o_{\mathscr{P}(A)_\mathrm{j}}(a_{s_1}\downarrow, \ldots, a_{s_n}\downarrow) \\ &= \mathrm{j}_s(o_A(a_{s_1}\downarrow, \ldots, a_{s_n}\downarrow)\downarrow) \\ &= \mathrm{j}_s(o_A(a_{s_1}, \ldots, a_{s_n})\downarrow) \\ &= o_A(a_{s_1}, \ldots, a_{s_n})\downarrow \\ &= \eta_s(o_A(a_{s_1}, \ldots, a_{s_n})). \end{aligned}$$

It remains to show that η is $\sigma^{\leqslant}$-essential in the category $\mathsf{OAL}^{\leqslant}_\Sigma$. Let $\psi : \mathscr{P}(A)_\mathrm{j} \to B$ be a morphism of ordered Σ-algebras in $\mathsf{OAL}^{\leqslant}_\Sigma$ such that $\psi\eta$ is a lax order-embedding. We have to show that ψ is a lax order-embedding, as well, i.e., condition (2) in the definition of $\sigma^{\leqslant}$ is satisfied.

Assume that $t^s_B(\psi_{s_1}(D_{s_1}), \ldots, \psi_{s_m}(D_{s_m})) \subseteq \psi_s(D_s)$ in B, for $t^s \in T^S_\Sigma$, $s \in S$, where $D_{s_i} \in \mathscr{P}(A)_{\mathrm{j}_{s_i}}$, $i = 1, \ldots, m$, $m \in \mathbb{N}$ and $D_s \in \mathscr{P}(A)_{\mathrm{j}_s}$. Our aim is to establish the inclusion $t^s_{\mathscr{P}(A)_\mathrm{j}}(D_{s_1}, \ldots, D_{s_m}) \subseteq D_s = \mathrm{j}_s(D_s)$. Take $u \in t^s_{\mathscr{P}(A)_\mathrm{j}}(D_{s_1}, \ldots, D_{s_m}) = \mathrm{j}_s(t^s_A(D_{s_1}, \ldots, D_{s_m})\downarrow)$. Suppose that $p^{\tilde{s}}(D_s) \subseteq a\downarrow$ for some $p^{\tilde{s}} \in P^1_{A_s}$, $a \in$

$A_{\widetilde{s}}$, $\widetilde{s} \in S$, we may further suppose that $p^{\widetilde{s}}$ has the form $t_A^{\widetilde{s}}(a_{\overline{s}_1}, \ldots, a_{\overline{s}_{i-1}}, \_, a_{\overline{s}_{i+1}}, \ldots, a_{\overline{s}_n})$, i.e., a function from A_s to $A_{\widetilde{s}_i}$, where $t_A^{\widetilde{s}}$ is a Σ-term of sort $\overline{s}$, $a_{\overline{s}_j} \in A_{\overline{s}_j}$, $j \in \{1, \ldots, n\} \setminus \{i\}$, $n \in \mathbb{N}$. If $p^{\widetilde{s}}(t_A^s(D_{s_1}, \ldots, D_{s_m})\downarrow) \subseteq a\downarrow$, then we conclude that $p^{\widetilde{s}}(u) \leqslant a$ because $u \in \mathrm{j}_s(t_A^s(D_{s_1}, \ldots D_{s_m})\downarrow)$, which means that $u \in \mathrm{j}_s(D_s) = D_s$ as needed. Therefore, to complete the proof, it remains to prove the implication

$$p^{\widetilde{s}}(D_s) \subseteq a\downarrow \Rightarrow p^{\widetilde{s}}(t_A^s(D_{s_1}, \ldots D_{s_m})\downarrow) \subseteq a\downarrow. \tag{4.4}$$

Since $p^{\widetilde{s}}(D_s) \subseteq a\downarrow$, we have $p^{\widetilde{s}}(D_s)\downarrow \subseteq a\downarrow$. By Lemma 3, it comes out that

$$\begin{aligned} p^{\widetilde{s}}(D_s)\downarrow &= t_A^{\widetilde{s}}(a_{\overline{s}_1}, \ldots, a_{\overline{s}_{i-1}},\ D_s,\ a_{\overline{s}_{i+1}}, \ldots a_{\overline{s}_n})\downarrow \\ &= t_A^{\widetilde{s}}(a_{\overline{s}_1}\downarrow, \ldots a_{\overline{s}_{i-1}}\downarrow,\ D_s,\ a_{\overline{s}_{i+1}}\downarrow, \ldots a_{\overline{s}_n}\downarrow)\downarrow \\ &\subseteq a\downarrow. \end{aligned}$$

We need to verify that

$$t_A^{\widetilde{s}}(a_{\overline{s}_1}, \ldots a_{\overline{s}_{i-1}},\ t_A^s(D_{s_1}, \ldots D_{s_m})\downarrow,\ a_{\overline{s}_{i+1}}, \ldots a_{\overline{s}_n}) \subseteq a\downarrow \tag{4.5}$$

Take $d_{s_1} \in D_{s_1}, \ldots d_{s_m} \in D_{s_m}$, then

$$\begin{aligned} & t_B^{\widetilde{s}}((\psi_{\overline{s}_1}\eta_{\overline{s}_1})(a_{\overline{s}_1}), \ldots, (\psi_{\overline{s}_{i-1}}\eta_{\overline{s}_{i-1}})(a_{\overline{s}_{i-1}}), t_B^s((\psi_{s_1}\eta_{s_1})(d_{s_1}), \ldots, \\ &\qquad (\psi_{s_m}\eta_{s_m})(d_{s_m})), (\psi_{\overline{s}_{i+1}}\eta_{\overline{s}_{i+1}})(a_{\overline{s}_{i+1}}), \ldots, (\psi_{\overline{s}_n}\eta_{\overline{s}_n})(a_{\overline{s}_n})) \\ &= t_B^{\widetilde{s}}(\psi_{\overline{s}_1}(a_{\overline{s}_1}\downarrow), \ldots, \psi_{\overline{s}_{i-1}}(a_{\overline{s}_{i-1}}\downarrow), t_B^s(\psi_{s_1}(d_{s_1}\downarrow), \ldots, \psi_{s_m}(d_{s_m}\downarrow)), \psi_{\overline{s}_{i+1}}(a_{\overline{s}_{i+1}}\downarrow), \ldots, \psi_{\overline{s}_n}(a_{\overline{s}_n}\downarrow)) \\ &\leqslant t_B^{\widetilde{s}}(\psi_{\overline{s}_1}(a_{\overline{s}_1}\downarrow), \ldots, \psi_{\overline{s}_{i-1}}(a_{\overline{s}_{i-1}}\downarrow), t_B^s(\psi_{s_1}(D_{s_1}), \ldots, \psi_{s_m}(D_{s_m})), \psi_{\overline{s}_{i+1}}(a_{\overline{s}_{i+1}}\downarrow), \ldots, \psi_{\overline{s}_n}(a_{\overline{s}_n}\downarrow)) \\ &\leqslant t_B^{\widetilde{s}}(\psi_{\overline{s}_1}(a_{\overline{s}_1}\downarrow), \ldots, \psi_{\overline{s}_{i-1}}(a_{\overline{s}_{i-1}}\downarrow),\ \psi_s(D_s),\ \psi_{\overline{s}_{i+1}}(a_{\overline{s}_{i+1}}\downarrow), \ldots \psi_{\overline{s}_n}(a_{\overline{s}_n}\downarrow)) \\ &\leqslant \psi_{\widetilde{s}}(t_{\mathscr{P}(A)_{\mathrm{j}}}^{\widetilde{s}}(a_{\overline{s}_1}\downarrow, \ldots, a_{\overline{s}_{i-1}}\downarrow, D_s, a_{\overline{s}_{i+1}}\downarrow, \ldots, a_{\overline{s}_n}\downarrow)) \\ &= \psi_{\widetilde{s}}(\mathrm{j}_{\widetilde{s}}(t_A^{\widetilde{s}}(a_{\overline{s}_1}\downarrow, \ldots, a_{\overline{s}_{i-1}}\downarrow,\ D_s,\ a_{\overline{s}_{i+1}}\downarrow, \ldots, a_{\overline{s}_n}\downarrow)\downarrow)) \\ &\leqslant \psi_{\widetilde{s}}(\mathrm{j}_{\widetilde{s}}(a\downarrow)) \\ &= \psi_{\widetilde{s}}(a\downarrow) \\ &= (\psi_{\widetilde{s}}\eta_{\widetilde{s}})(a). \end{aligned}$$

Since $\psi\eta$ is in $\sigma^{\leqslant}$, we achieve that

$$t_A^{\widetilde{s}}(a_{\overline{s}_1}, \ldots, a_{\overline{s}_{i-1}},\ t_A^s(d_{s_1}, \ldots d_{s_m}), a_{\overline{s}_{i+1}}, \ldots, a_{\overline{s}_n}) \leqslant a,$$

which indicates that (4.5) holds eventually.

□

Acknowledgement: Research of the first author was supported by the Natural Science Foundation of Guangdong Province, China (No. 2016A030313832), the Science and Technology Program of Guangzhou, China (No. 201607010190), the State Scholarship Fund, China (No. 201708440512).

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