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Random uniform exponential attractor for stochastic non-autonomous reaction-diffusion equation with multiplicative noise in $\mathbb{R}^3$

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Abstract: We first introduce the concept of the random uniform exponential attractor for a jointly continuous non-autonomous random dynamical system (NRDS) and give a theorem on the existence of the random uniform exponential attractor for a jointly continuous NRDS. Then we study the existence of the random uniform exponential attractor for reaction-diffusion equation with quasi-periodic external force and multiplicative noise in $\mathbb{R}^3$.

Keywords: random uniform exponential attractor, non-autonomous random dynamical system, non-autonomous stochastic reaction-diffusion equation

MSC: 37L55, 35B41, 35B40

1 Introduction

The concept of the exponential attractor was introduced by A. Eden et al., which is a compact positively invariant set with finite fractal dimension and attracts trajectories exponentially fast, see [1]. It can describe the asymptotic behavior of trajectories of autonomous dynamical system or the solutions to dissipative autonomous evolution equations. In contrast to a global attractor, the exponential attractor has finite fractal dimension, if it exists, the asymptotic behavior of infinite dimensional dynamical systems can be characterized by the dynamics on the finite dimensional compact set (i.e., exponential attractor). Besides, exponential attractors are stable under perturbation because of the exponential rate of convergence of trajectories to it. We should note that an exponential attractor is not necessarily unique since it is not invariant, and includes a global attractor in general.

By the notion of pullback attraction, the concept of the exponential attractor can be extended to the case of non-autonomous dynamical system, called pullback exponential attractor, see [2–6] and the references therein. An alternative extension to the case of non-autonomous dynamical system of the concept of the exponential attractor was based on the work [7] of Chepyzhov and Vishik (see also [8, Chapter 4]), in which they introduced an approach to study a family of non-autonomous evolution equations of the form

$$\frac{du}{dt} = \mathcal{G}_{\sigma(t)}(u), \quad \sigma \in \Sigma, \tag{1}$$

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where $u \in E$ (Banach space) and Σ is an appropriate compact symbol space. More precisely, they constructed a semigroup (skew-product semiflow) associated with (1) on extended phase space $\Sigma \times E$ and used the theory of semigroup to study the longtime behavior of solutions of non-autonomous evolution equations. Newly extended attractors for non-autonomous dynamical system are called uniform exponential attractors, see [9–13] and the references therein. By its definition, a uniform exponential attractor is time independent exponentially attracting compact set and has finite fractal dimension. Because we regard extended phase space $\Sigma \times E$ as a whole, symbol space Σ require to be compact and finite dimensional when we use semigroup on $\Sigma \times E$ to study the existence of a uniform exponential attractor of non-autonomous evolution equations, this is why we choose k-dimensional torus $\mathbb{T}^k,\ k \in \mathbb{Z}_+$ (corresponding to the hull of quasi-periodic functions, see [8]) as the symbol space.

Our aim in this article is to extend the uniform exponential attractor to the random uniform exponential attractor for a jointly continuous non-autonomous random dynamical system, and give a theorem on the existence of a random uniform exponential attractor for a jointly continuous NRDS. By definition, a random uniform exponential attractor is a random compact set with finite fractal dimension, which (pullback) attracts uniformly every element of attraction universe $\mathcal{D}$ with exponential rate. We emphasize that the attraction universe considered in the definition is autonomous attraction universe (it contains only autonomous random set, see [14, 15]). We also emphasize that the random uniform exponential attractor has no (positive) invariance property along the sample path. According to [16], we have a criterion for the existence of a random exponential attractor for a continuous random dynamical system on a separable Hilbert space. Other results on existence criteria of a random exponential attractor for a random dynamical system can be found in [17, 18]. With the help of the concept of skew-product cocycle (i.e., random dynamical system) introduced in [15], we consider the existence of the random exponential attractor for a continuous skew-product cocycle (generated by a jointly continuous NRDS ϕ and a base flow θ) on the extended phase space $\mathbb{T}^k \times E$, and project this random exponential attractor onto the phase space E. Then we obtain the random uniform exponential attractor for the jointly continuous NRDS ϕ. Consequently, we formulate Theorem 2.8 on the existence of a random uniform exponential attractor for a jointly continuous NRDS.

As an application of Theorem 2.8, we will consider the following reaction-diffusion equation with quasi-periodic external force and multiplicative noise on $\mathbb{R}^3$,

$$\begin{cases} du + (\lambda u - \Delta u)dt = (f(x,u) + g(x, \widetilde{\sigma}(t)))dt + bu \circ dW(t), \quad t > 0, \\ u(x,0) = u_0(x), \quad x \in \mathbb{R}^3, \end{cases} \tag{2}$$

where $u = u(x,t)$ is real-valued functions defined on $\mathbb{R}^3 \times [0, +\infty)$, the coefficients λ, b are positive constants. $W(t)$ is a two-sided real-valued Brownian motion on a probability space which will be specified later. The symbol " $\circ$ " means that the stochastic integration in system is in the Stratonovich sense. $\widetilde{\sigma}(t) = (\mathbf{x}t + \sigma)\mathrm{mod}(\mathbb{T}^k)$, where $\sigma \in \mathbb{T}^k$, $\mathbf{x} = (x_1, \ldots, x_k) \in \mathbb{R}^k$ is a fixed vector satisfying that $x_1, \ldots, x_k$ are rationally independent. The functions g, f are assumed to satisfy some conditions.

There have been many works concerning random attractors for stochastic reaction-diffusion equation, see [19, Introduction] for detailed summary. It is worth noting that Zhou [19] considered the existence of random exponential attractor for non-autonomous stochastic reaction-diffusion equation with multiplicative noise in $\mathbb{R}^3$. In the setting of f, g in [19], we here further assume that the external force g is quasi-periodic and prove the existence of a random uniform exponential attractor.

This paper is organized as follows. In the next section, we show some preliminaries and give a theorem on the existence of a random uniform exponential attractor for a jointly continuous NRDS. In section 3, we study the existence of a random uniform exponential attractor for the non-autonomous stochastic reaction-diffusion equation (2) defined on $\mathbb{R}^3$.

2 Existence of random uniform exponential attractors

In this section, we present some notations and provide a criterion concerning the existence of a random uniform exponential attractor for a jointly continuous NRDS.

Let X be a separable Hilbert space with norm $\|\cdot\|_X$ and Borel σ-algebra $\mathcal{B}(X)$, d_X denotes the metric induced from norm $\|\cdot\|_X$, the Hausdorff semi-distance between two nonempty subsets F_1, F_2 in X is defined by $\mathrm{dist}_X(F_1, F_2) = \sup_{u\in F_1} \inf_{v\in F_2} \|u - v\|_X$.

Let $\mathbb{T}^k$ be the k-dimensional torus:

$$\mathbb{T}^k = \{\sigma = (\sigma_1, \dots, \sigma_k) : \sigma_j \in [-\pi, \pi], \ \forall j = 1, \dots, k\}$$

with the identification

$$(\sigma_1, \dots, \sigma_{j-1}, -\pi, \sigma_{j+1}, \dots, \sigma_l) \sim (\sigma_1, \dots, \sigma_{j-1}, \pi, \sigma_{j+1}, \dots, \sigma_l), \quad \forall j = 1, \dots, l,$$

and the topology, metric induced from the topology, metric on $\mathbb{R}^k$. Thus, the norm in $\mathbb{T}^k$ is given by

$$\|\sigma\|_{\mathbb{T}^k} = \left(\sum_{j=1}^{k} \sigma_j^2\right)^{1/2}, \quad \forall \sigma = (\sigma_1, \dots, \sigma_k) \in \mathbb{T}^k.$$

Let $\mathbf{x} = (x_1, \dots, x_k) \in \mathbb{R}^k$ be a fixed vector such that $x_1, \dots, x_k$ are rationally independent, i.e., if there exist integers $l_1, \dots, l_k$ such that $\sum_{j=1}^{n} l_j x_j = 0$, then $l_j = 0$ for $j = 1, \dots, k$. For $t \in \mathbb{R}$, define

$$\theta_t \sigma = (\mathbf{x}t + \sigma) \,\mathrm{mod}(\mathbb{T}^k), \quad \sigma \in \mathbb{T}^k, \tag{3}$$

then $\{\theta_t\}_{t\in\mathbb{R}}$ is a translation group on $\mathbb{T}^k$ with

$$\theta_t \mathbb{T}^k = \mathbb{T}^k, \quad \forall t \in \mathbb{R} \tag{4}$$

and

$$(t, \sigma) \to \theta_t \sigma \ \text{ is continuous.} \tag{5}$$

When the time symbol is quasiperiodic, we consider $\mathbb{T}^k$ as the symbol space (see [8]). $\mathcal{B}(\mathbb{T}^k)$ denotes the Borel σ-algebra of $\mathbb{T}^k$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and let $(\Omega, \mathcal{F}, \mathbb{P}, \{\vartheta\}_{t\in\mathbb{R}})$ be an ergodic metric dynamical system, where $\{\vartheta\}_{t\in\mathbb{R}}$ satisfies: (i) ϑ_0 is the identity on Ω; (ii) $\vartheta_s \circ \vartheta_t = \vartheta_{s+t}, \forall t, s \in \mathbb{R}$; (iii) $(t, \omega) \to \vartheta_t \omega$ is $(\mathcal{B}(\mathbb{R}) \times \mathcal{F}, \mathcal{F})$-measurable; (iv) $\mathbb{P}$-preserving: $\mathbb{P}(\vartheta_t F) = \mathbb{P}(F), \forall t \le 0, \ F \in \mathcal{F}$; (v) if for any $F \in \mathcal{F}$, provided $\mathbb{P}(\vartheta_t^{-1} F \triangle F) = 0$, it holds $\mathbb{P}(F) = 0$ or $1, \forall t \in \mathbb{R}$; (vi) $\vartheta_t \Omega = \Omega, \forall t \in \mathbb{R}$ (see [20]).

Two groups $\{\theta_t\}_{t\in\mathbb{R}}$ and $\{\vartheta\}_{t\in\mathbb{R}}$ are called base flows.

Definition 2.1. *A (autonomous) random dynamical system (RDS) on X with base flow $\{\vartheta\}_{t\in\mathbb{R}}$ is defined as a mapping $\psi(t, \omega, x) : \mathbb{R}^+ \times \Omega \times X \to X$ satisfying*

(i) ψ is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$-measurable;
(ii) $\psi(0, \omega, \cdot)$ is the identity on X for each $\omega \in \Omega$;
(iii) it holds the cocycle property $\psi(t + s, \omega, \cdot) = \psi(t, \vartheta_s \omega, \cdot) \circ \psi(s, \omega, \cdot), \forall t, s \ge 0, \omega \in \Omega$.

A RDS is said to be continuous if for each $t \in \mathbb{R}^+$, $\omega \in \Omega$, the mapping $\psi(t, \omega, \cdot)$ is continuous.

Definition 2.2. *A non-autonomous random dynamical system (NRDS) on X with base flows $\{\vartheta\}_{t\in\mathbb{R}}$ on Ω and $\{\theta_t\}_{t\in\mathbb{R}}$ on $\mathbb{T}^k$ is defined as a mapping $\phi(t, \omega, \sigma, x) : \mathbb{R}^+ \times \Omega \times \mathbb{T}^k \times X \to X$ satisfying*

(i) ϕ is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(\mathbb{T}^k) \times \mathcal{B}(X), \mathcal{B}(X))$-measurable;
(ii) $\phi(0, \omega, \sigma, \cdot)$ is the identity on X for each $\sigma \in \mathbb{T}^k$ and $\omega \in \Omega$;
(iii) it holds the cocycle property $\phi(t + s, \omega, \sigma, \cdot) = \phi(t, \vartheta_s \omega, \theta_s \sigma, \cdot) \circ \phi(s, \omega, \sigma, \cdot), \forall t, s \ge 0, \omega \in \Omega, \sigma \in \mathbb{T}^k$.

A NRDS is said to be continuous if for each $t \in \mathbb{R}^+$, $\omega \in \Omega$ and $\sigma \in \mathbb{T}^k$, the mapping $\phi(t, \omega, \sigma, \cdot)$ is continuous. It is called jointly continuous in $\mathbb{T}^k$ and X if the mapping $\phi(t, \omega, \cdot, \cdot)$ is continuous for each $t \in \mathbb{R}^+$ and $\omega \in \Omega$. We obtain the general definition of NRDS by replacing torus $\mathbb{T}^k$ with general symbol space Σ in Definition 2.2 (see [15, Definition 2.1]).

Definition 2.3. *A (autonomous) random set $D(\cdot)$ in X is a multi-valued map $D : \Omega \to 2^X \setminus \emptyset$ such that for each $x \in X$ the map $\omega \to d_X(x, D(\omega))$ is measurable. It is said that the (autonomous) random set is bounded (resp. closed or compact) if $D(\omega)$ is bounded (resp. closed or compact) for a.e. $\omega \in \Omega$.*

We often write $D(\cdot)$ as D or $\{D(\omega)\}_{\omega\in\Omega}$. Given two random sets D_1, D_2, we write $D_1 \subseteq D_2$ if $D_1(\omega) \subseteq D_2(\omega)$ for a.e. $\omega \in \Omega$.

Definition 2.4. *A random set $D(\cdot)$ in X is called tempered with respect to $\{\vartheta\}_{t\in\mathbb{R}}$, if for a.e. $\omega \in \Omega$,*

$$e^{-\beta t}\|D(\vartheta_{-t}\omega)\|_X \overset{t\to\infty}{\longrightarrow} 0, \quad \forall \beta > 0,$$

where $\|D(\omega)\|_X = \sup_{x\in D(\omega)} \|x\|$.

Hereafter, we denote by $\mathcal{D}(X)$ the collection of all tempered bounded random subset of X. For simplicity, we identify " *a.e.* $\omega \in \Omega$" and "$\omega \in \Omega$" unless otherwise stated.

Define the extended space $\mathbb{X} \doteq \mathbb{T}^k \times X$ with norm:

$$\|\mathcal{X}\|_{\mathbb{X}} = \left(\|\sigma\|^2_{\mathbb{T}^k} + \|x\|^2_X\right)^{1/2}, \quad \forall \mathcal{X} = \{\sigma\} \times \{x\} \in \mathbb{X}. \tag{6}$$

and Borel σ-algebra $\mathcal{B}(\mathbb{X})$.

Obviously, any subset $B \subseteq \mathbb{X}$ has the form $\mathbb{B} = \cup_{\sigma\in\mathbb{T}^k}\{\sigma\} \times B(\sigma)$, where $B(\sigma)$ (possibly empty) is called the σ-section of $\mathbb{B}$. Let $P_\sigma\mathbb{B} \doteq B(\sigma)$, $\forall\, \mathbb{B} \subseteq \mathbb{X}$, and let

$$P_X\mathbb{B} = \bigcup_{\sigma\in\mathbb{T}^k} P_\sigma\mathbb{B} = \{x \in X : \text{ there is some } \sigma \in \mathbb{T}^k \text{ such that } \{\sigma\} \times \{x\} \in \mathbb{B}\}.$$

Then P_X is the projection from $\mathbb{X}$ to X. Denote by $P_{\mathbb{T}^k}$ the projection from $\mathbb{X}$ to $\mathbb{T}^k$.

Definition 2.5 (see [15]). *A set-valued mapping $\mathbb{B}(\cdot) : \Omega \to 2^{\mathbb{X}}\setminus\emptyset$ is called a random set in $\mathbb{X}$ if for each $\mathcal{X} \in \mathbb{X}$ the mapping $\omega \to d_{\mathbb{X}}(\mathcal{X}, \mathbb{B}(\omega))$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}^+))$-measurable. If, moreover, $\mathbb{B}$ satisfies*

$$P_\sigma(\mathbb{B}(\omega)) \neq \emptyset, \quad \forall\, \sigma \in \mathbb{T}^k,\ \omega \in \Omega, \tag{7}$$

and

$$P_X(\mathbb{B}) \in \mathcal{D}(X),$$

where $\mathcal{D}(X)$ is the collection of all tempered bounded random subset of X, then it is said to be proper random set.

By [15], condition (7) is equivalent to

$$P_{\mathbb{T}^k}(\mathbb{B}(\omega)) \equiv \mathbb{T}^k, \quad \forall\, \omega \in \Omega,$$

which implies the stochastic perturbation happens only to the X-component.

By Definition 2.4, a random set $\mathbb{B}$ in $\mathbb{X}$ is tempered if it satisfies $e^{-\beta t}\|\mathbb{B}(\vartheta_{-t}\omega)\|_{\mathbb{X}} \overset{t\to\infty}{\longrightarrow} 0$, $\forall \beta > 0$. Let

$$\mathcal{D}_{0,\mathbb{X}} = \left\{\mathbb{B} : \mathbb{B} \text{ is a bounded tempered random set in } \mathbb{X}\right\},$$

$$\mathcal{D}_{1,\mathbb{X}} = \left\{\mathbb{B} : \mathbb{B} = \mathbb{T}^k \times B = \{\mathbb{T}^k \times B(\omega)\}_{\omega\in\Omega} \text{ and } B \in \mathcal{D}(X)\right\}, \tag{8}$$

and

$$\mathcal{D}_{\mathbb{X}} = \left\{\mathbb{B} : \mathbb{B} \text{ is a proper random set in } \mathbb{X}\right\}$$

then $\mathcal{D}_{1,\mathbb{X}} \subset \mathcal{D}_{\mathbb{X}} \subset \mathcal{D}_{0,\mathbb{X}}$ and for any element $\mathbb{B} \in \mathcal{D}_{\mathbb{X}}$, there exist an element $\mathbb{B}_1 \in \mathcal{D}_{1,\mathbb{X}}$ such that $\mathbb{B} \subseteq \mathbb{B}_1$.

For K-dimensional subspace X_K of X ($K \in \mathbb{N}$), we define the bounded projections $\mathbb{P}_{k+K} : \mathbb{X} \to \mathbb{X}_K = \mathbb{T}^k \times X_K$ and $\mathbb{Q}_{k+K} : \mathbb{X} \to \mathbb{X}_K^{\perp} = \mathbb{T}^k \times X_K^{\perp}$ as

$$\mathbb{P}_{k+K}\mathcal{X} = \{\sigma\} \times \{P_K x\}, \quad \mathbb{Q}_{k+K}\mathcal{X} = \{0\} \times \{Q_K x\}, \quad \forall \mathcal{X} = \{\sigma\} \times \{x\} \in \mathbb{X} \tag{9}$$

where $P_K : X \to X_K$ is K-dimensional orthogonal projection from X into X_K, $Q_K = I_X - P_K$ and $\mathbb{Q}_{k+K} = I_{\mathbb{X}} - \mathbb{P}_{k+K}$, where $I_{\mathbb{X}}$ is the identity operator on $\mathbb{X}$.

To study the existence of a random uniform exponential attractor for a NRDS on a separable Hilbert space, we need to introduce a skew-product cocycle on extended space $\mathbb{X}$ (see [15, Section 4.1]). Given an NRDS ϕ, define a mapping $\pi : \mathbb{R}^+ \times \Omega \times \mathbb{X} \to \mathbb{X}$ by

$$\pi(t, \omega, \{\sigma\} \times \{x\}) = \{\theta_t\sigma\} \times \{\phi(t, \omega, \sigma, x)\}. \tag{10}$$

Then the mapping π is a RDS, namely, satisfying

(i) π is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(\mathbb{X}), \mathcal{B}(\mathbb{X}))$-measurable;
(ii) $\pi(0, \omega, \mathcal{X}) = \mathcal{X}, \quad \forall \omega \in \Omega, \mathcal{X} \in \mathbb{X}$;
(iii) the cocycle property $\pi(t + s, \omega, \mathcal{X}) = \pi(t, \vartheta_s\omega, \pi(s, \omega, \mathcal{X})), \forall t, s \geq 0, \omega \in \Omega, \mathcal{X} \in \mathbb{X}$.

The RDS π is called the skew-product cocycle generated by ϕ and θ. Note that π is continuous, that is, the mapping $\mathcal{X} \to \pi(\cdot, \cdot, \mathcal{X})$ is continuous in $\mathbb{X}$, if and only if ϕ is jointly continuous in $\mathbb{T}^k$ and X.

Now we define the random uniform exponential attractor for continuous NRDS $\{\phi(t, \omega, \sigma)\}_{t\geq 0, \omega\in\Omega, \sigma\in\mathbb{T}^k}$ on a separable Hilbert space X.

Definition 2.6. *A random set $\{\mathcal{M}(\omega)\}_{\omega\in\Omega}$ in X is called a $\mathcal{D}(X)$-random uniform exponential attractor for the continuous NRDS $\{\phi(t, \omega, \sigma)\}_{t\geq 0, \omega\in\Omega, \sigma\in\mathbb{T}^k}$ on X if there is a set of full measure $\tilde{\Omega} \in \mathcal{F}$ such that for every $\omega \in \tilde{\Omega}$, it holds that*

(i) Compactness: $\mathcal{M}(\omega)$ is compact set.
(ii) Finite-dimensionality: there exists a random variable $\xi_\omega (< \infty)$ such that $\dim_f \mathcal{M}(\omega) \leq \xi_\omega < \infty$, where $\dim_f \mathcal{M}(\omega)$ is the fractal dimension of $\mathcal{M}(\omega)$.
(iii) Exponential attraction: there exists a constant $\alpha > 0$ such that for any $B \in \mathcal{D}(X)$, there exist random variables $\bar{t}_B(\omega) \geq 0$, $\bar{Q}(\omega, \|B\|_X) > 0$ satisfying

$$\sup_{\sigma\in\mathbb{T}^k} dist_X(\phi(t, \vartheta_{-t}\omega, \theta_{-t}\sigma)B(\vartheta_{-t}\omega), \mathcal{M}(\omega)) \leq \bar{Q}(\omega, \|B\|_X)e^{-\alpha t}, \quad t \geq \bar{t}_B(\omega). \tag{11}$$

Remark 2.7. *By definition the random uniform exponential attractor has no (positive) invariance property along the sample path.*

We make the following assumptions on the continuous skew-product $\{\pi(t, \omega)\}_{t\geq 0, \omega\in\Omega}$ on extended space $\mathbb{X}$ defined in (10):

(A1) there exists a tempered closed random set $\{\chi(\omega)\}_{\omega\in\Omega}$ of $\mathbb{X}$ such that for any $\omega \in \Omega$,
(a11) the diameter $\|\chi(\omega)\|_{\mathbb{X}}$ of $\chi(\omega)$ is bounded by a tempered random variable R_ω, i.e.,

$$\sup_{\mathcal{X}, \mathcal{Y}\in\chi(\omega)} ||\mathcal{X} - \mathcal{Y}||_{\mathbb{X}} \leq R_\omega < \infty,$$

where $R_{\vartheta_t\omega}$ is continuous in t for all $t \in \mathbb{R}$;
(a12) $\chi(\omega)$ is positively invariant with respect to $\{\vartheta_t\}_{t\in\mathbb{R}}$ in the sense that $\pi(t, \vartheta_{-t}\omega)\chi(\vartheta_{-t}\omega) \subseteq \chi(\omega)$, for all $t \geq 0$;
(a13) $\chi(\omega)$ is pullback absorbing in the sense that for any family of set $\mathbb{B} \in \mathcal{D}_{\mathbb{X}}$, there exist $T_{\mathbb{B}} = T_{\mathbb{B}}(\omega) \geq 0$ such that $\pi(t, \vartheta_{-t}\omega)\mathbb{B}(\vartheta_{-t}\omega) \subseteq \chi(\omega)$ for $t \geq T_{\mathbb{B}}$;

(A2) there exist positive numbers $\bar{\lambda}$, $\bar{\delta}$, random variables $\bar{C}_0(\omega)$, $\bar{C}_1(\omega) \geq 0$ and $(k+K)$-dimensional projector $\mathbb{P}_{k+K}\colon \mathbb{X} \to \mathbb{P}_{k+K}\mathbb{X}$ ($\dim(\mathbb{P}_{k+K}\mathbb{X}) = (k+K) \in \mathbb{N}$) such that for every $\omega \in \Omega$ and any $\mathcal{X}, \mathcal{Y} \in \chi(\omega)$,

$$\|\pi(t,\omega)\mathcal{X} - \pi(t,\omega)\mathcal{Y}\|_{\mathbb{X}} \leq e^{\int_0^{\frac{8\ln 2}{\bar{\lambda}}} \bar{C}_0(\theta_s\omega)ds}\|\mathcal{X}-\mathcal{Y}\|_{\mathbb{X}}, \quad \forall t \in [0, \frac{8\ln 2}{\bar{\lambda}}] \tag{12}$$

and

$$\left\|(I_{\mathbb{X}} - \mathbb{P}_{k+K})\left(\pi(\frac{8\ln 2}{\bar{\lambda}},\omega)\mathcal{X} - \pi(\frac{8\ln 2}{\bar{\lambda}},\omega)\mathcal{Y}\right)\right\|_{\mathbb{X}} \leq (e^{-8\ln 2 + \int_0^{\frac{8\ln 2}{\bar{\lambda}}} \bar{C}_1(\theta_s\omega)ds} + \frac{\bar{\delta}}{2}e^{\int_0^{\bar{t}_0} \bar{C}_0(\theta_s\omega)ds})\|\mathcal{X}-\mathcal{Y}\|_{\mathbb{X}}, \tag{13}$$

where $\bar{\lambda}$, K are independent of ω;

(A3) $\bar{C}_0(\omega)$, $\bar{C}_1(\omega)$, $\bar{\lambda}$, $\bar{\delta}$ satisfy:

$$\begin{cases} 0 \leq \mathbf{E}[\bar{C}_0^2(\omega)] < \infty, \quad 0 \leq \mathbf{E}[\bar{C}_1(\omega)] \leq \frac{\bar{\lambda}}{16}, \\ 0 < \bar{\delta} \leq \min\left\{\frac{1}{16}, e^{-\frac{2}{\ln\frac{3}{2}}\left(\frac{128(\ln 2)^2}{\bar{\lambda}^2}\mathbf{E}[\bar{C}_0^2(\omega)] + \frac{64(\ln 2)^2}{\bar{\lambda}}\mathbf{E}[\bar{C}_0(\omega)]\right)}\right\}, \end{cases} \tag{14}$$

where "$\mathbf{E}$"denotes the expectation.

Theorem 2.8. *Assume that conditions $(A1)$-$(A3)$ hold. Then the continuous skew-product cocycle π acting on $\mathbb{X}$ generated by jointly continuous NRDS $\{\phi(t,\omega,\sigma)\}_{t\geq 0,\omega\in\Omega,\sigma\in\mathbb{T}^k}$ with base flow $\{\vartheta_t\}_{t\in\mathbb{R}}$ possesses a $\mathcal{D}_{\mathbb{X}}$-random exponential attractor $\mathbb{E} \subseteq \chi$. Moreover, $P_X\mathbb{E}$ is the $\mathcal{D}(X)$-random uniform exponential attractor of jointly continuous NRDS $\{\phi(t,\omega,\sigma)\}_{t\geq 0,\omega\in\Omega,\sigma\in\mathbb{T}^k}$.*

Proof. The existence of $\mathcal{D}_{\mathbb{X}}$-random exponential attractor $\mathbb{E}$ for $\{\pi(t,\omega)\}_{t\geq 0,\omega\in\Omega}$ follows from Theorem 2.1 in [16]. We claim that $P_{\mathbb{T}^k}(\mathbb{E}(\omega)) = \mathbb{T}^k$, a.e. $\omega \in \Omega$. If otherwise, $\mathbb{P}\{\omega : P_{\mathbb{T}^k}(\mathbb{E}(\omega)) \neq \mathbb{T}^k\} > 0$. Let $\mathfrak{I} = \{\omega : P_{\mathbb{T}^k}(\mathbb{E}(\omega)) \neq \mathbb{T}^k\}$. Then $P_{\mathbb{T}^k}\mathbb{E}(\omega) = A(\omega)$, $\omega \in \mathfrak{I}$, where $A(\omega) \subsetneqq \mathbb{T}^k$. There exist an element $\sigma_0(\omega) \in \mathbb{T}^k \backslash A(\omega)$ such that $\mathrm{dist}_{\mathbb{T}^k}(\mathbb{T}^k, A(\omega)) \geq \mathrm{dist}_{\mathbb{T}^k}(\sigma_0(\omega), A(\omega)) > 0$, $\omega \in \mathfrak{I}$, this is contrary to the exponential attraction of exponential attractor $\mathbb{E}$. In other words, there exist a subset $\mathfrak{I}$ of Ω such that $P(\mathfrak{I}) > 0$ and the exponential attraction of exponential attractor $\mathbb{E}$ fails to hold for $\omega \in \mathfrak{I}$, which contradicts the definition of exponential attractor; The compactness and measurable of $P_X\mathbb{E}$ follows from $\mathbb{E}$ directly. Obviously, $\dim_f P_X\mathbb{E}(\omega) \leq \dim_f \mathbb{E}(\omega) \leq \frac{2(k+K)\ln(\frac{2\sqrt{k+K}}{\delta}+1)}{\ln\frac{4}{3}} < \infty$, $\forall \omega \in \Omega$, since $\frac{\ln N_\epsilon(\bigcup_{\sigma\in\mathbb{T}^k} P_\sigma\mathbb{E}(\omega))}{-\ln\epsilon} \leq \frac{\ln N_\epsilon(\bigcup_{\sigma\in\mathbb{T}^k}\{\sigma\}\times P_\sigma\mathbb{E}(\omega))}{-\ln\epsilon}$, $\forall \omega \in \Omega$; We next show the uniform exponential attraction of $P_X\mathbb{E}$ with respect to $\mathcal{D}(X)$. Note that for each $x \in X$, $\sigma \in \mathbb{T}^k$, $\omega \in \Omega$, and $t \geq 0$, we have

$$\begin{aligned} \mathrm{dist}_X(x, P_X\mathbb{E}(\omega)) &= \inf_{\sigma'\in\mathbb{T}^k} \mathrm{dist}_X(x, P_{\sigma'}\mathbb{E}(\omega)) \\ &\leq \inf_{\sigma'\in\mathbb{T}^k}\left(\mathrm{dist}_X(x, P_{\sigma'}\mathbb{E}(\omega)) + \|\sigma_1 - \sigma_2\|_{\mathbb{T}^k}\right) \\ &= \mathrm{dist}_{\mathbb{X}}\left(\{\sigma\}\times\{x\}, \cup_{\sigma'\in\mathbb{T}^k}\{\sigma'\}\times\{P_{\sigma'}\mathbb{E}(\omega)\}\right). \end{aligned}$$

For $\forall D \in \mathcal{D}(X)$, since $\mathbb{D}$ with $\mathbb{D}(\omega) = \mathbb{T}^k \times D(\omega)$ belongs to $\mathcal{D}_{\mathbb{X}}$, $\mathbb{D}$ is attracted exponentially by $\mathbb{E}$. Thus we have for each $t \geq 0$, $\omega \in \Omega$, $\sigma \in \mathbb{T}^k$,

$$\begin{aligned} &\sup_{\sigma\in\mathbb{T}^k} \mathrm{dist}_X\left(\phi(t,\vartheta_{-t}\omega,\theta_{-t}\sigma)D(\vartheta_{-t}\omega),\ P_X\mathbb{E}(\omega)\right) \\ &\quad\leq \sup_{\sigma\in\mathbb{T}^k} \mathrm{dist}_{\mathbb{X}}\left(\{\sigma\}\times\phi(t,\vartheta_{-t}\omega,\theta_{-t}\sigma)D(\vartheta_{-t}\omega),\ \cup_{\sigma\in\mathbb{T}^k}\{\sigma'\}\times P_{\sigma'}\mathbb{E}(\omega)\right) \\ &\quad= \sup_{\sigma\in\mathbb{T}^k} \mathrm{dist}_{\mathbb{X}}\left(\pi(t,\vartheta_{-t}\omega)\{\theta_{-t}\sigma\}\times D(\vartheta_{-t}\omega),\ \mathbb{E}(\omega)\right) \\ &\quad= \mathrm{dist}_{\mathbb{X}}\left(\pi(t,\vartheta_{-t}\omega)\mathbb{D}(\vartheta_{-t}\omega),\ \mathbb{E}(\omega)\right) \leq \check{b}(\omega,\mathbb{D})e^{-\ln\frac{\ln\frac{4}{3}}{4\bar{t}_0}t}, \quad t \geq \tilde{T}(\omega,\mathbb{D}), \end{aligned} \tag{15}$$

which indicates the uniform exponential attraction of $P_X\mathbb{E}$. The proof is complete. □

2.1 Application to stochastic reaction-diffusion equation

In this section, we apply Theorem 2.8 to the reaction-diffusion equation with quasi-periodic external force and multiplicative noise on $\mathbb{R}^3$. Namely, we consider the equation (2). The functions g, f are assumed to satisfy the following conditions:

(H1) $\omega_i \to g(\cdot, \omega_1, \dots, \omega_k)$ is 2π-periodic, $i = 1, \dots, k$, $g \in C(\mathbb{R}^3 \times \mathbb{T}^k, \mathbb{R})$, $g(x, \cdot) \in C(\mathbb{T}^k, L^2(\mathbb{R}^3))$ with $\|g\|^2 = \sup_{\sigma\in\mathbb{T}^k} \|g(\cdot, \sigma)\|^2 < \infty$. $g(x, 0_{\mathbb{T}^k}) = 0$. There exists $0 \le h(x) \in L^2(\mathbb{R}^3)$ such that

$$|g(x, \tilde{\sigma}_1(t)) - g(x, \tilde{\sigma}_2(t))| \le h(x)\|\sigma_1 - \sigma_2\|_{\mathbb{T}^k}. \tag{16}$$

(H2) There exist real positive constants $c_1, c_2, c_3 > 0$ and integral functions $\beta_1 \in L^1(\mathbb{R}^3, \mathbb{R}_+)$, $\beta_2, \beta_3 \in L^2(\mathbb{R}^3, \mathbb{R}_+)$, $\beta_4 \in L^3(\mathbb{R}^3, \mathbb{R}_+)$ such that

$$\begin{cases} uf(x,u) \le \beta_1(x), \quad |f(x,u)| \le c_1|u|^3 + \beta_2(x), \\ \left|\frac{\partial f}{\partial x}(x,u)\right| \le \beta_3(x), \quad \frac{\partial f}{\partial u}(x,u) \le c_2, \\ \left|\frac{\partial f}{\partial u}(x,u)\right| \le c_3u^2 + \beta_4(x), \end{cases} \quad \forall x \in \mathbb{R}^3,\ u \in \mathbb{R}. \tag{17}$$

Hereafter, let $(\cdot,\cdot)$, $\|\cdot\|$ and $(\cdot,\cdot)_1$, $\|\cdot\|_1$ denote the inner products and norms of $L^2(\mathbb{R}^3)$ and $H^1(\mathbb{R}^3)$, respectively, where

$$(v_1, v_2) = \int_{\mathbb{R}^3} v_1(x)v_2(x)dx, \quad (v_1, v_2)_1 = (\nabla v_1, \nabla v_2) + (v_1, v_2).$$

2.2 Setting of the problem

In the sequel, we will use the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where

$$\Omega = \{\omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0\},$$

$\mathcal{F}$ is the Borel σ-algebra on Ω generated by the open compact topology, and $\mathbb{P}$ represents the Wiener measure on $\mathcal{F}$. The Brownian motion has a realization $W(t) = W(t, \omega) = \omega(t)$ for $\omega \in \Omega$, $t \in \mathbb{R}$. Define

$$\vartheta_t\omega(\cdot) = \omega(t + \cdot) - \omega(t), \forall\omega \in \Omega,\ t \in \mathbb{R},$$

then $\mathbb{P}$ is ergodic and invariant under ϑ (see [20, 21]). It is known that $z(\vartheta_t\omega) = -\int_{-\infty}^0 e^s(\vartheta_t\omega)(s)ds$ $(t \in \mathbb{R})$ is a stationary solution of one-dimensional equation $dz + zdt = dW(t)$. From [22], we know that for $\omega \in \Omega$, $t \mapsto z(\vartheta_t\omega)$ is continuous in t and

$$\lim_{t\to\pm\infty} \frac{|z(\vartheta_t\omega)|}{|t|} = \lim_{t\to\pm\infty} \frac{1}{t}\int_0^t z(\vartheta_s\omega)ds = 0. \tag{18}$$

Considering variable transformation $v(t) = e^{-bz(\vartheta_t\omega)}u(t)$. Then (2) is equivalent to the following system with random coefficients

$$\begin{cases} \frac{dv(t)}{dt} = \Delta v(t) - \lambda v(t) + bz(\vartheta_t\omega)v(t) + e^{-bz(\vartheta_t\omega)}f(x, e^{bz(\vartheta_t\omega)}v(t)) + e^{-bz(\vartheta_t\omega)}g(x, \widetilde{\sigma}(t)), \\ v(x, 0) = v_0(x) = e^{-bz(\omega)}u_0(x), \quad x \in \mathbb{R}^3. \end{cases} \tag{19}$$

We known from [19] for each $\sigma \in \mathbb{T}^k$, $\omega \in \Omega$, $v_0 \in L^2(\mathbb{R}^3)$, the unique solution $v(t, \omega, \sigma, v_0)$ of (19) exists globally for $t \in [0, \infty)$ and $v(\cdot, \omega, \sigma, v_0) \in C([0, +\infty); L^2(\mathbb{R}^3)) \cap L^2_{loc}([0, +\infty); H^1(\mathbb{R}^3))$. Moreover, $v(\cdot, \omega, \sigma, v_0)$ is $(\mathcal{F}, \mathcal{B}(L^2(\mathbb{R}^3)))$-measurable in ω and continuous in σ and v_0. Thus the mapping of solutions generates a NRDS $\phi : \mathbb{R}^+ \times \Omega \times \mathbb{T}^k \times L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$, i.e., $\phi(t, \omega, \sigma, v_0) = v(t, \omega, \sigma, v_0)$, which is continuous both in initial value and symbols.

From now on, let $\mathcal{D} = \mathcal{D}(L^2(\mathbb{R}^3))$ be the collection of all tempered bounded random sets of $L^2(\mathbb{R}^3)$, i.e.,

$$\mathcal{D} = \left\{D : D \text{ is the bounded random sets in } L^2(\mathbb{R}^3) \text{ satisfying } e^{-\alpha t}\|D(\vartheta_{-t}\omega)\|^2 \xrightarrow{t\to\infty} 0,\ \forall\alpha > 0,\ \omega \in \Omega\right\}.$$

We will prove the existence of $\mathcal{D}$-random uniform exponential attractor of ϕ in this paper.

2.3 Boundedness of solution

For every $\omega \in \Omega$, $\sigma \in \mathbb{T}^k$ and $t \geq 0$, let $v(r) = v(r, \vartheta_{-t}\omega, \sigma, v_0(\vartheta_{-t}\omega))$ $(r \geq 0)$ be a solution of (19) with symbol σ and initial value $v_0(\vartheta_{-t}\omega) \in L^2(\mathbb{R}^3)$.

Lemma 2.9. *For every $D \in \mathcal{D}$ and $\omega \in \Omega$, there exist $T_0 = T_0(\omega, D) \geq 0$ and a tempered random variable $M_0^2(\omega) > 0$ such that for $v_0(\vartheta_{-t}\omega) \in D(\vartheta_{-t}\omega)$,*

$$\|v(t, \vartheta_{-t}\omega, \sigma, v_0(\vartheta_{-t}\omega))\|^2 + 2\int_0^t e^{\int_l^t (2bz(\vartheta_{s-t}\omega)-\lambda)ds}\|\nabla v(l)\|^2 dl \leq M_0^2(\omega), \quad t \geq T_0$$

holds uniformly for $\sigma \in \mathbb{T}^k$.

Proof. Taking the inner product of (19) with $v(r)$, we have for $r \geq 0$,

$$\begin{aligned}\frac{d}{dt}\|v(r)\|_E^2 &+ \left(2\lambda - 2bz(\vartheta_{r-t}\omega)\right)\|v(r)\|^2 + 2\|\nabla v(r)\|^2 \\ &= e^{-bz(\vartheta_{r-t}\omega)} f(x, e^{bz(\vartheta_{r-t}\omega)}v(r), v(r)) - e^{-bz(\vartheta_{r-t}\omega)}(g(x, \widetilde{\sigma}(r)), v(r)),\end{aligned}$$

where $\widetilde{\sigma}(r) = (\mathbf{x}r + \sigma)\,\mathrm{mod}(\mathbb{T}^k) \in \mathbb{T}^k$. By

$$\beta e^{-bz(\vartheta_{r-t}\omega)}\int_{\mathbb{R}^3} f(x, e^{bz(\vartheta_{r-t}\omega)}v)v\,dx \leq \|\beta_1\|_{L^1(\mathbb{R}^3)}e^{-2bz(\vartheta_{r-t}\omega)}, \qquad \text{(by (17))}$$

$$2e^{-bz(\vartheta_{r-t}\omega)}g(x, \widetilde{\sigma}(r), v(r)) \leq \frac{2}{\lambda}e^{-2bz(\vartheta_{r-t}\omega)}\|g\|^2 + \frac{\lambda}{2}\|v(r)\|^2,$$

it follows that for $r \geq 0$,

$$\frac{d}{dt}\|v(r)\|^2 + 2\|\nabla v(r)\|^2 \leq \left(2bz(\vartheta_{r-t}\omega) - \lambda\right)\|v(r)\|^2 + c_4 e^{-2bz(\vartheta_{r-t}\omega)}, \tag{20}$$

where $c_4 = \frac{2}{\lambda}\|g\|^2 + 2\|\beta_1\|_{L^1}$. Applying Gronwall inequality to (20) on $[0, r]$ $(r \geq 0)$, we obtain

$$\begin{aligned}\|v(r, \vartheta_{-t}\omega, \sigma, v_0(\vartheta_{-t}\omega))\|^2 &+ 2\int_0^r e^{\int_l^r (2bz(\vartheta_{s-t}\omega)-\lambda)ds}\|\nabla v(l)\|^2 dl \\ &\leq e^{\int_0^r (2bz(\vartheta_{s-t}\omega)-\lambda)ds}\|v_0(\vartheta_{-t}\omega)\|^2 + c_4\int_0^r e^{\int_l^r (2bz(\vartheta_{s-t}\omega)-\lambda)ds - 2bz(\vartheta_{l-t}\omega)}dl.\end{aligned} \tag{21}$$

Let $r = t$ in (21), we get

$$\begin{aligned}\|v(t, \vartheta_{-t}\omega, \sigma, v_0(\vartheta_{-t}\omega))\|^2 &+ 2\int_0^t e^{\int_l^t (2bz(\vartheta_{s-t}\omega)-\lambda)ds}\|\nabla v(l)\|^2 dl \\ &\leq e^{\int_{-t}^0 (2bz(\vartheta_s\omega)-\lambda)ds}\|v_0(\vartheta_{-t}\omega)\|^2 + c_4\int_{-t}^0 e^{\int_l^0 (2bz(\vartheta_s\omega)-\lambda)ds - 2bz(\vartheta_l\omega)}dl\end{aligned} \tag{22}$$

For $\forall D \in \mathcal{D}$, let

$$T_0(D, \omega) = \min\{t : e^{\int_{-t}^0 (2bz(\vartheta_s\omega)-\lambda)ds}\sup_{v \in D(\vartheta_{-t}\omega)}\|v\|^2 \leq 1\}.$$

Take $M_0^2(\omega) = 1 + c_4K_0(\omega)$, where $K_0(\omega) = \int_{-\infty}^0 e^{\int_l^0 (2bz(\vartheta_s\omega)-\lambda)ds - 2bz(\vartheta_l\omega)}dl$, then $M_0^2(\omega)$ is tempered. The proof is completed. □

Lemma 2.10. *For each $\omega \in \Omega$, let $B_0(\omega) = \{u \in L^2(\mathbb{R}^3) : \|u\| \le M_0(\omega)\}$. Then for each $\omega \in \Omega$, there exist $T_1 = T_1(B_0, \omega) \ge 1$ and a tempered random variable $M_1^2(\omega) > 0$ such that for $v_0(\vartheta_{-t}\omega) \in B_0(\vartheta_{-t}\omega)$,*

$$\|\nabla v(t, \vartheta_{-t}\omega, \sigma, v_0(\vartheta_{-t}\omega))\|^2 \le M_1^2(\omega), \qquad t \ge T_1(B_0, \omega)$$

and

$$\|v(t, \vartheta_{-t}\omega, \sigma, v_0(\vartheta_{-t}\omega))\|^2 + \|\nabla v(t, \vartheta_{-t}\omega, \sigma, v_0(\vartheta_{-t}\omega))\|^2 \le M_0^2(\omega) + M_1^2(\omega), \quad t \ge T_0(B_0, \omega) + T_1(B_0, \omega)$$

hold uniformly for $\sigma \in \mathbb{T}^k$.

Proof. By $v_0(\vartheta_{-t}\omega) \in B_0(\vartheta_{-t}\omega)$ and (21), we obtain for $t \ge 1$,

$$\begin{aligned}\|v(t-1, \vartheta_{-t}\omega, \sigma, v_0(\vartheta_{-t}\omega))\|^2 &+ 2\int_0^{t-1} e^{\int_l^{t-1}(2bz(\vartheta_{s-t}\omega)-\lambda)ds}\|\nabla v(l)\|^2 dl \\ &\le e^{\int_{-t}^{-1}(2bz(\vartheta_s\omega)-\lambda)ds}\|v_0(\vartheta_{-t}\omega)\|^2 + c_4\int_{-t}^{-1} e^{\int_l^{-1}(2bz(\vartheta_s\omega)-\lambda)ds-2bz(\vartheta_l\omega)}dl. \end{aligned} \tag{23}$$

Using Gronwall inequality to (20) on $[t-1, t]$, we have

$$\|v(t)\|^2 + 2\int_{t-1}^{t} e^{\int_l^t(2bz(\vartheta_{s-t}\omega)-\lambda)ds}\|\nabla v(l)\|^2 dl \le e^{\int_{-1}^{0}(2bz(\vartheta_s\omega)-\lambda)ds}\|v(t-1)\|^2 + c_4\int_{-1}^{0} e^{\int_l^0(2bz(\vartheta_s\omega)-\lambda)ds-2bz(\vartheta_l\omega)}dl,$$

which and (23) implies

$$\begin{aligned}\|v(t)\|^2 &+ 2\int_{t-1}^{t} e^{\int_l^t(2bz(\vartheta_{s-t}\omega)-\lambda)ds}\|\nabla v(l)\|^2 dl \\ &\le e^{\int_{-t}^{0}(2bz(\vartheta_s\omega)-\lambda)ds}\|v_0(\vartheta_{-t}\omega)\|^2 + c_4\int_{-\infty}^{0} e^{\int_l^0(2bz(\vartheta_s\omega)-\lambda)ds-2bz(\vartheta_l\omega)}dl. \end{aligned} \tag{24}$$

Notice that

$$2\int_{t-1}^{t} e^{\int_l^t(2bz(\vartheta_{s-t}\omega)-\lambda)ds}\|\nabla v(l)\|^2 dl \ge 2e^{-\int_{-1}^{0} 2b|z(\vartheta_s\omega)|ds-\lambda}\int_{t-1}^{t}\|\nabla v(l)\|^2 dl. \tag{25}$$

We derive from (24) and (25) that

$$\begin{aligned}2\int_{t-1}^{t}\|\nabla v(l)\|^2 dl &\le e^{\int_{-1}^{0} 2b|z(\vartheta_s\omega)|ds+\lambda+\int_{-t}^{0}(2bz(\vartheta_s\omega)ds-\lambda)ds}\|v_0(\vartheta_{-t}\omega)\|^2 \\ &+ c_4 e^{\int_{-1}^{0} 2b|z(\vartheta_s\omega)|ds+\lambda}\int_{-\infty}^{0} e^{\int_l^0(2bz(\vartheta_s\omega)ds-\lambda)ds-2bz(\vartheta_l\omega)}dl.\end{aligned}$$

Take

$$T_1(B_0, \omega) = \min\{t : e^{\int_{-1}^{0} 2b|z(\vartheta_s\omega)|ds+\lambda+\int_{-t}^{0}(2bz(\vartheta_s\omega)ds-\lambda)ds}M_0^2(\vartheta_{-t}\omega) \le 1\} \ge 1.$$

Let $\tilde{M}_1^2(\omega) = 1 + c_4e^{\int_{-1}^{0} 2b|z(\vartheta_s\omega)|ds+\lambda}K_0(\omega)$, then $2\int_{t-1}^{t}\|\nabla v(l)\|^2 dl \le \tilde{M}_1^2(\omega)$, for $t \ge T_1(B_0, \omega)$.

Taking the inner product of (19) with $-\Delta v(r)$, we have

$$\frac{d}{dt}\|\nabla v(r)\|^2 + 2\|\Delta v(r)\|^2 + (2\lambda - 2bz(\vartheta_{r-t}\omega))\|\nabla v(r)\|^2$$

$$= -2e^{-bz(\vartheta_{r-t}\omega)}\int_{\mathbb{R}^3} f(x, e^{bz(\vartheta_{r-t}\omega)}v(r))\Delta v(r)dx - 2e^{-bz(\vartheta_{r-t}\omega)}(g(x,\widetilde{\sigma}(r)), \Delta v(r)). \tag{26}$$

By

$$-e^{-bz(\vartheta_{r-t}\omega)}(g(x,\widetilde{\sigma}(r)), \Delta v(r)) \le \frac{1}{2}e^{-2bz(\vartheta_{r-t}\omega)}\|g\|^2 + \frac{1}{2}\|\Delta v(r)\|^2$$

and

$$\begin{aligned}-2e^{-bz(\vartheta_{r-t}\omega)}\int_{\mathbb{R}^3} f(x, e^{bz(\vartheta_{r-t}\omega)}v(r))\Delta v(r)dx &= 2e^{-bz(\vartheta_{r-t}\omega)}\int_{\mathbb{R}^3}(\frac{\partial f}{\partial x}(x, e^{bz(\vartheta_{r-t}\omega)}v)\nabla v + e^{bz(\vartheta_{r-t}\omega)}\frac{\partial f}{\partial u}|\nabla v|^2)dx\\ &\le e^{-2bz(\vartheta_{r-t}\omega)}\|\beta_3\|^2 + (1+2c_2)\|\nabla v\|^2, \qquad \text{(by (17))}\end{aligned}$$

it follows that

$$\frac{d}{dt}\|\nabla v\|^2 + \|\Delta v\|^2 = (2bz(\vartheta_{r-t}\omega) - \lambda)\|\nabla v\|^2 + (1+2c_2)\|\nabla v\|^2 + c_5e^{-2bz(\vartheta_{r-t}\omega)}, \tag{27}$$

where $c_5 = \|\beta_3\|^2 + \|g\|^2$. Take $t \ge T_1(B_0, \omega) \ge 1$ and $s \in [t-1, t]$. Integrating (27) over $[s, t]$, we obtain

$$\begin{aligned}&\|\nabla v(t, \vartheta_{-t}\omega, \sigma, v_0(\vartheta_{-t}\omega))\|^2 - \|\nabla v(s, \vartheta_{-t}\omega, \sigma, v_0(\vartheta_{-t}\omega))\|^2\\ &\le c_5\int_s^t e^{-2bz(\vartheta_{l-t}\omega)}dl + \int_s^t (1 + 2b|z(\vartheta_{l-t}\omega)| + 2c_2)\|\nabla v(l)\|^2 dl. \end{aligned}\tag{28}$$

Integrating (28) with respect to s over $[t-1, t]$, we arrive that for $t \ge T_1(B_0, \omega)$,

$$\|\nabla v(t, \vartheta_{-t}\omega, \sigma, v_0(\vartheta_{-t}\omega))\|^2 \le c_5\int_{-1}^0 e^{-2bz(\vartheta_l\omega)}dl + (1 + b\max_{-1\le l\le 0}|z(\vartheta_l\omega)| + c_2)\tilde{M}_1^2(\omega) \doteq M_1^2(\omega).$$

The proof is complete. □

Lemma 2.11. *For each $\omega \in \Omega$, let $B_1 = \{u \in H^1(\mathbb{R}^3) : \|u\|_1^2 \le M_0^2(\omega) + M_1^2(\omega)\}$. Then for each $\omega \in \Omega$, there exist $T_2(B_1, \omega) \ge 0$ and a tempered random variable $M_2^2(\omega) > 0$ such that for $v_0(\vartheta_{-t}\omega) \in B_1(\vartheta_{-t}\omega)$,*

$$\|\nabla v(t, \vartheta_{-t}\omega, \sigma, v_0(\vartheta_{-t}\omega))\|^2 + \|v(t, \vartheta_{-t}\omega, \sigma, v_0(\vartheta_{-t}\omega))\|^2 \le M_2^2(\omega), \quad t \ge T_2(B_0, \omega)$$

holds uniformly for $\sigma \in \mathbb{T}^k$.

Proof. According to (20) and (27), we derive for $r \ge 0$,

$$\frac{d}{dt}(\|\nabla v\|^2 + \|v\|^2) \le (2bz(\vartheta_{r-t}\omega) - \lambda)(\|\nabla v\|^2 + \|v\|^2) + (1+2c_2)\|\nabla v\|^2 + (c_4 + c_5)e^{-2bz(\vartheta_{r-t}\omega)}. \tag{29}$$

Applying Gronwall inequality to (29) over $[0, t]$, we get

$$\begin{aligned}&\|\nabla v(t, \vartheta_{-t}\omega, \sigma, v_0(\vartheta_{-t}\omega))\|^2 + \|v(t, \vartheta_{-t}\omega, \sigma, v_0(\vartheta_{-t}\omega))\|^2\\ &\le e^{\int_0^t(2bz(\vartheta_{l-t}\omega)-\lambda)dl}(\|\nabla v_0(\vartheta_{-t}\omega)\|^2 + \|v_0(\vartheta_{-t}\omega)\|^2)\\ &\quad + \int_0^t e^{\int_l^t(2bz(\vartheta_{s-t}\omega)-\lambda)ds}\left((1+2c_2)\|\nabla v(l)\|^2 + (c_4+c_5)e^{-2bz(\vartheta_{l-t}\omega)}\right)dl. \end{aligned}\tag{30}$$

It follows from (22) and (30) that

$$\|\nabla v(t)\|^2 + \|v(t)\|^2 \le \left(1 + \frac{1+2c_2}{2}\right)e^{\int_{-t}^0(2bz(\vartheta_l\omega)-\lambda)ds}(M_0^2(\vartheta_{-t}\omega) + M_1^2(\vartheta_{-t}\omega)) + c_6K_0(\omega),$$

where $c_6 = \frac{3}{2}c_4 + c_2c_4 + c_5$. For every $\omega \in \Omega$, let $M_2^2(\omega) = 1 + c_6K_0(\omega)$ and

$$T_2(B_1, \omega) = \min\{t : (1 + \frac{1+2c_2}{2})e^{\int_{-t}^0(2bz(\vartheta_l\omega)-\lambda)ds}(M_0^2(\vartheta_{-t}\omega) + M_1^2(\vartheta_{-t}\omega)) \le 1\} \ge 0.$$

The proof is completed. □

Lemma 2.12. *For every $\omega \in \Omega$, let $B_2(\omega) = \{u \in H^1(\mathbb{R}^3) : \|u\|_1 \le M_2(\omega)\} \subset H^1(\mathbb{R}^3)$. Then for each $\omega \in \Omega$, the following hold uniformly for $\sigma \in \mathbb{T}^k$,*

$$\phi(t, \vartheta_{-t}\omega, \sigma)B_0(\vartheta_{-t}\omega) \subseteq B_0(\omega) \subset L^2(\mathbb{R}^3), \quad t \ge T_0(B_0, \omega) \tag{31}$$

$$\phi(t, \vartheta_{-t}\omega, \sigma)B_0(\vartheta_{-t}\omega) \subseteq B_2(\omega) \subset L^2(\mathbb{R}^3), \quad t \ge T_0(B_0, \omega) + T_1(B_0, \omega) + T_2(B_1, \omega) \tag{32}$$

Proof. It is a direct consequence of Lemma 2.9-2.11. □

2.4 Estimation on tail of solutions

For every $\omega \in \Omega$, let $T_\star(\omega) = T_0(B_0, \omega) + T_1(B_0, \omega) + T_2(B_1, \omega)$ and

$$B_3(\omega) = \bigcup_{t \ge T_\star(\omega)} \phi(t, \vartheta_{-t}\omega, \mathbb{T}^k)B_0(\vartheta_{-t}\omega),$$

where $\phi(t, \vartheta_{-t}\omega, \mathbb{T}^k)B_0(\vartheta_{-t}\omega) = \cup_{\sigma \in \mathbb{T}^k}\phi(t, \vartheta_{-t}\omega, \sigma)B_0(\vartheta_{-t}\omega)$, then $B_3(\omega) \subset B_0(\omega) \cap B_2(\omega)$. Set

$$\tilde{\mathbb{B}}(\vartheta_{-s}\omega) = \overline{\cup_{t \ge \max\{T_\star(\vartheta_{-s}\omega), T_\star(\omega)\}}\pi(t, \vartheta_{-t-s}\omega)\mathbb{T}^k \times B_3(\vartheta_{-t-s}\omega)}, \qquad s \ge 0, \quad \omega \in \Omega, \tag{33}$$

where π is the skew-product cocycle generated by ϕ and θ. Evidently, $\tilde{\mathbb{B}}$ belongs to $\mathcal{D}_{\mathbb{X}}$, since $P_{L^2(\mathbb{R}^3)}\tilde{\mathbb{B}} \subseteq B_3$, where $P_{L^2(\mathbb{R}^3)}$ denotes the projection from $\mathbb{T}^k \times L^2(\mathbb{R}^3)$ to $L^2(\mathbb{R}^3)$.

We assert that $\tilde{\mathbb{B}}$ is positive invariant, i.e., $\pi(t, \vartheta_{-t}\omega)\tilde{\mathbb{B}}(\vartheta_{-t}\omega) \subseteq \tilde{\mathbb{B}}(\omega)$, for $\forall\omega \in \Omega$. Indeed, for any $\omega \in \Omega$, take $\{\sigma\} \times \{x\} \in \cup_{s \ge \max\{T_\star(\vartheta_{-t}\omega), T_\star(\omega)\}}\pi(s, \vartheta_{-s-t}\omega)\mathbb{T}^k \times B_3(\vartheta_{-s-t}\omega)$ arbitrarily, then there exist $\hat{t} \ge \max\{T_\star(\vartheta_{-t}\omega), T_\star(\omega)\}$, $\hat{\sigma} \in \mathbb{T}^k$ and $\hat{x} \in B_3(\vartheta_{-\hat{t}-t}\omega)$ such that

$$\{\sigma\} \times \{x\} = \pi(\hat{t}, \vartheta_{-\hat{t}-t}\omega)\{\hat{\sigma}\} \times \{\hat{x}\} = \{\theta_{\hat{t}}\hat{\sigma}\} \times \{\phi(\hat{t}, \vartheta_{-\hat{t}-t}\omega, \hat{\sigma}, \hat{x})\},$$

therefore, for $\forall t \ge 0$,

$$\begin{aligned}\pi(t, \vartheta_{-t}\omega)\{\sigma\} \times \{x\} &= \pi(t, \vartheta_{-t}\omega)\pi(\hat{t}, \vartheta_{-\hat{t}-t}\omega)\{\hat{\sigma}\} \times \{\hat{x}\} \\ &= \pi(\hat{t} + t, \vartheta_{-\hat{t}-t}\omega)\{\hat{\sigma}\} \times \{\hat{x}\} \\ &\in \pi(\hat{t} + t, \vartheta_{-\hat{t}-t}\omega)\{\hat{\sigma}\} \times B_3(\vartheta_{-\hat{t}-t}\omega).\end{aligned}$$

Note that $\hat{t} + t \ge T_\star(\omega) + t$, by (33), we have $\pi(t, \vartheta_{-t}\omega)\{\sigma\} \times \{x\} \in \tilde{\mathbb{B}}(\omega)$. Considering the continuity of π and the closeness of $\tilde{\mathbb{B}}$, we derive $\pi(t, \vartheta_{-t}\omega)\tilde{\mathbb{B}}(\vartheta_{-t}\omega) \subseteq \tilde{\mathbb{B}}(\omega)$. Moreover, for $\forall r \ge 0$, $t \ge 0$, $\omega \in \Omega$, we have

$$\pi(r, \vartheta_{-t}\omega)\tilde{\mathbb{B}}(\vartheta_{-t}\omega) \subseteq \tilde{\mathbb{B}}(\vartheta_{r-t}\omega), \tag{34}$$

which along with $P_{L^2(\mathbb{R}^3)}\tilde{\mathbb{B}} \subseteq B_3$ imply for $\forall r \ge 0$, $t \ge 0$, $\omega \in \Omega$,

$$\phi(r, \vartheta_{-t}\omega, \sigma, x) \in B_3(\vartheta_{r-t}\omega), \quad \forall\{\sigma\} \times \{x\} \in \tilde{\mathbb{B}}(\vartheta_{-t}\omega). \tag{35}$$

Choosing a smooth increasing function $\xi \in C^1(\mathbb{R}^+, \mathbb{R})$ such that

$$\begin{cases} \xi(s) = 0, & 0 \le s \le 1; \\ 0 \le \xi(s) \le 1, & 1 \le s \le 2; \\ \xi(s) = 1, & 2 \le s < +\infty; \\ |\xi'(s)| \le \tilde{C}, & \forall s \in \mathbb{R}^+ \text{ and some constant } \tilde{C} > 0. \end{cases}$$

Lemma 2.13. *For every $\omega \in \Omega$, $R \ge 1$, $t \ge 0$, let $v(r) = v(r, \vartheta_{-t}\omega, \sigma, v_0(\vartheta_{-t}\omega))$ be the solution of* (19) *with $\{\sigma\} \times \{v_0(\vartheta_{-t}\omega)\} \in \tilde{\mathbb{B}}(\vartheta_{-t}\omega)$. Then*

(i) there exist a tempered random variable $K_1(\omega)$ and a function $\gamma_{(\cdot)}$ such that

$$\int_{\mathbb{R}^3} \xi(\frac{|x|^2}{R^2})|v(t, \vartheta_{-t}\omega, \sigma, v_0(\vartheta_{-t}\omega))|^2 dx \le e^{\int_{-t}^0 (2bz(\vartheta_s\omega)-\lambda)ds} M_2^2(\vartheta_{-t}\omega) + \frac{K_1(\omega)}{R} + \gamma_R K_0(\omega), \tag{36}$$

(ii) there exist $c_7 > 0$, a tempered random variable $K_2(\omega)$ and a function $Y_{(\cdot)}$ such that

$$\int_{\mathbb{R}^3} \xi(\frac{|x|^2}{R^2})|\nabla v(t, \vartheta_{-t}\omega, \sigma, v_0(\vartheta_{-t}\omega))|^2 dx \le c_7 e^{\int_{-t}^{0}(2bz(\vartheta_s\omega)-\lambda)ds} M_2^2(\vartheta_{-t}\omega) + \frac{K_2(\omega)}{R} + Y_R K_0(\omega), \tag{37}$$

(iii) For $\forall \varepsilon > 0$, there exist $T(\varepsilon, \omega) > 0$ such that

$$\begin{aligned}\int_{\mathbb{R}^3} \xi(\frac{|x|^2}{R^2}) \left(|v(t, \vartheta_{-t}\omega, \sigma, v_0(\vartheta_{-t}\omega))|^2 + |\nabla v(t, \vartheta_{-t}\omega, \sigma, v_0(\vartheta_{-t}\omega))|^2\right) dx \\ \le \varepsilon + \frac{K_1(\omega) + K_2(\omega)}{R} + Y_R K_0(\omega), \qquad t \ge T(\varepsilon, \omega).\end{aligned} \tag{38}$$

Proof. By (35), we know for $\forall r \ge 0$, $t \ge 0$, $\omega \in \Omega$,

$$v(r) \in B_3(\vartheta_{r-t}\omega) \subseteq B_0(\vartheta_{r-t}\omega) \cap B_2(\vartheta_{r-t}\omega) \subset H_1(\mathbb{R}^3), \tag{39}$$

$$\|v(r)\|^2 + \|\nabla v(r)\|^2 \le M_2^2(\vartheta_{r-t}\omega), \tag{40}$$

$$v_0(\vartheta_{-t}\omega) \in B_3(\vartheta_{-t}\omega), \quad \|v_0(\vartheta_{-t}\omega)\|^2 + \|\nabla v_0(\vartheta_{-t}\omega)\|^2 \le M_2^2(\vartheta_{-t}\omega). \tag{41}$$

(i) Taking the inner product of (19) with $\xi(\frac{|x|^2}{R^2})v$ in $L^2(\mathbb{R}^3)$, we have

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^3} \xi(\frac{|x|^2}{R^2})|v|^2 dx + (2\lambda - 2bz(\vartheta_{r-t}\omega)) \int_{\mathbb{R}^3} \xi(\frac{|x|^2}{R^2})|v|^2 dx \\ = 2\int_{\mathbb{R}^3} \xi(\frac{|x|^2}{R^2})(\Delta v) v dx + 2e^{-bz(\vartheta_{r-t}\omega)} \int_{\mathbb{R}^3} \xi(\frac{|x|^2}{R^2}) f(x, e^{bz(\vartheta_{r-t}\omega)} v) v dx \\ +2e^{-bz(\vartheta_{r-t}\omega)} \int_{\mathbb{R}^3} \xi(\frac{|x|^2}{R^2}) g(x, \widetilde{\sigma}(r)) v dx.\end{aligned} \tag{42}$$

Similar to (3.26)-(3.28) in [19], we have

$$2\int_{\mathbb{R}^3} \xi(\frac{|x|^2}{R^2})(\Delta v) v dx \le -2\int_{\mathbb{R}^3} \xi(\frac{|x|^2}{R^2})|\nabla v|^2 dx + \frac{2\sqrt{2}\tilde{C}}{R^2} M_2^2(\vartheta_{r-t}\omega), \tag{43}$$

$$2e^{-bz(\vartheta_{r-t}\omega)} \int_{\mathbb{R}^3} \xi(\frac{|x|^2}{R^2}) f(x, e^{bz(\vartheta_{r-t}\omega)} v) v dx \le 2e^{-2bz(\vartheta_{r-t}\omega)} \int_{\mathbb{R}^3} \xi(\frac{|x|^2}{R^2}) \beta_1(x) dx, \tag{44}$$

$$2e^{-bz(\vartheta_{r-t}\omega)}\int_{\mathbb{R}^3} \xi(\frac{|x|^2}{R^2}) g(x, \widetilde{\sigma}(r)) v dx \le \frac{\lambda}{4}\int_{\mathbb{R}^3} \xi(\frac{|x|^2}{R^2})|v|^2 dx + \frac{e^{-2bz(\vartheta_{r-t}\omega)}}{\lambda}\int_{\mathbb{R}^3} \xi(\frac{|x|^2}{R^2}) g^2(x, \widetilde{\sigma}(r)) dx. \tag{45}$$

It follows from (42)-(45) that

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^3} \xi(\frac{|x|^2}{R^2})|v|^2 dx + 2\int_{\mathbb{R}^3} \xi(\frac{|x|^2}{R^2})|\nabla v|^2 dx \le (2bz(\vartheta_{r-t}\omega) - \lambda) \int_{\mathbb{R}^3} \xi(\frac{|x|^2}{R^2})|v|^2 dx \\ +\frac{2\sqrt{2}\tilde{C}}{R} M_2^2(\vartheta_{r-t}\omega) + 2e^{-2bz(\vartheta_{r-t}\omega)} \int_{\mathbb{R}^3} \xi(\frac{|x|^2}{R^2}) \left(\beta_1(x) + \frac{1}{\lambda} g^2(x, \widetilde{\sigma}(r))\right) dx.\end{aligned} \tag{46}$$

Using Gronwall inequality to (46) over $[0, t]$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^3} \xi(\frac{|x|^2}{R^2})|v(t, \vartheta_{-t}\omega, \sigma, v_0(\vartheta_{-t}\omega))|^2 dx + 2\int_0^t e^{\int_l^t (2bz(\vartheta_{s-t}\omega)-\lambda)ds} \int_{\mathbb{R}^3} \xi(\frac{|x|^2}{R^2})|\nabla v(l)|^2 dx dl \\ \le e^{\int_{-t}^{0}(2bz(\vartheta_s\omega)-\lambda)ds} M_2^2(\vartheta_{-t}\omega) + \frac{K_1(\omega)}{R} + \gamma_R K_0(\omega), \quad \text{(refer to (3.30) in [19])}\end{aligned} \tag{47}$$

where

$$K_1(\omega) = 2\sqrt{2}\tilde{C}\int_{-\infty}^{0} e^{\int_l^0 (2bz(\vartheta_s\omega)-\lambda)ds} M_2^2(\vartheta_l\omega)dl,$$

$$\gamma_R = 2\int_{|x|\geq R} \beta_1(x)dx + \frac{2}{\lambda}\sup_{\sigma\in\mathbb{T}^k}\int_{|x|\geq R} g^2(x,\sigma)dx.$$

(ii) Taking the inner product of (19) with $-\xi(\frac{|x|^2}{R^2})\Delta v$ in $L^2(\mathbb{R}^3)$, we derive

$$\begin{aligned}
-\int_{\mathbb{R}^3}\frac{dv}{dt}\xi(\frac{|x|^2}{R^2})\Delta v dx + \int_{\mathbb{R}^3}\xi(\frac{|x|^2}{R^2})|\Delta v|^2dx = (\lambda - bz(\vartheta_{r-t}\omega))\int_{\mathbb{R}^3}\xi(\frac{|x|^2}{R^2})(\Delta v)vdx\\
-e^{-bz(\vartheta_{r-t}\omega)}\int_{\mathbb{R}^3}\xi(\frac{|x|^2}{R^2})f(x,e^{bz(\vartheta_{r-t}\omega)}v)\Delta v dx - e^{-bz(\vartheta_{r-t}\omega)}\int_{\mathbb{R}^3}\xi(\frac{|x|^2}{R^2})g(x,\widetilde{\sigma}(r))\Delta v dx. \qquad (48)
\end{aligned}$$

By embedding $H^1(\mathbb{R}^3)\hookrightarrow L^6(\mathbb{R}^3)$ [23], there exists $C_0>0$ such that

$$\|v\|_{L^6(\mathbb{R}^3)} \leq C_0\|v\|_1 = C_0(\|\nabla v\|^2 + \|v\|^2)^{1/2}, \quad \forall v\in H^1(\mathbb{R}^3).$$

Thus, $\|v(r)\|_{L^6(\mathbb{R}^3)} \leq C_0(\|v(r)\|^2 + \|\nabla v(r)\|^2)^{1/2} \leq C_0M_2(\vartheta_{r-t}\omega)$, which together with (17) implies

$$\|e^{-bz(\vartheta_{r-t}\omega)}f(x,e^{bz(\vartheta_{r-t}\omega)}v)\|^2 \leq 2c_1^2C_0^6e^{4bz(\vartheta_{r-t}\omega)}M_2^6(\vartheta_{r-t}\omega) + 2\|\beta_2\|^2e^{-2bz(\vartheta_{r-t}\omega)}.$$

By (19), we have

$$\begin{aligned}
\left\|\frac{dv}{dt}\right\|^2 &\leq 4\left(\|\Delta v\|^2 + (bz(\vartheta_{r-t}\omega)-\lambda)^2\|v(r)\|^2 + \|e^{-bz(\vartheta_{r-t}\omega)}f(x,e^{bz(\vartheta_{r-t}\omega)}v)\|^2 + e^{-2bz(\vartheta_{r-t}\omega)}\|g\|^2\right)\\
&\leq 4\|\Delta v\|^2 + c_8(1+z^2(\vartheta_{r-t}\omega))M_2^2(\vartheta_{r-t}\omega) + 8c_1^2C_0^6e^{4bz(\vartheta_{r-t}\omega)}M_2^6(\vartheta_{r-t}\omega) + 4e^{-2bz(\vartheta_{r-t}\omega)}(2\|\beta_2\|^2 + \|g\|^2).
\end{aligned}$$

Consequently,

$$\begin{aligned}
-\int_{\mathbb{R}^3}\frac{dv}{dt}\xi(\frac{|x|^2}{R^2})\Delta v dx \geq \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}\xi(\frac{|x|^2}{R^2})|\nabla v|^2dx - \frac{c_9}{R}\|\Delta v\|^2 - \frac{c_{10}}{R}e^{-2bz(\vartheta_{r-t}\omega)}\\
-\frac{c_{10}}{R}\left(M_2^2(\vartheta_{r-t}\omega) + z^2(\vartheta_{r-t}\omega)M_2^2(\vartheta_{r-t}\omega) + e^{4bz(\vartheta_{r-t}\omega)}M_2^6(\vartheta_{r-t}\omega)\right). \quad \text{(refer to (3.32) in [19])} \qquad (49)
\end{aligned}$$

$$(\lambda - bz(\vartheta_{r-t}\omega))\int_{\mathbb{R}^3}\xi(\frac{|x|^2}{R^2})(\Delta v)vdx \leq (bz(\vartheta_{r-t}\omega)-\lambda)\int_{\mathbb{R}^3}\xi(\frac{|x|^2}{R^2})|\nabla v|^2dx + \frac{\sqrt{2}\tilde{C}}{R}(\lambda + b|z(\vartheta_{r-t}\omega)|)M_2^2(\vartheta_{r-t}\omega). \quad (50)$$

$$\begin{aligned}
-e^{-bz(\vartheta_{r-t}\omega)}&\int_{\mathbb{R}^3}\xi(\frac{|x|^2}{R^2})f(x,e^{bz(\vartheta_{r-t}\omega)}v)\Delta v dx\\
&\leq \frac{1}{2}e^{-2bz(\vartheta_{r-t}\omega)}\int_{\mathbb{R}^3}\xi(\frac{|x|^2}{R^2})\beta_3^2dx + c_{11}\int_{\mathbb{R}^3}\xi(\frac{|x|^2}{R^2})|\nabla v|^2dx\\
&+\frac{c_{12}}{R}\left(M_2^2(\vartheta_{r-t}\omega) + e^{4bz(\vartheta_{r-t}\omega)}M_2^6(\vartheta_{r-t}\omega) + e^{-2bz(\vartheta_{r-t}\omega)}\right). \quad \text{(refer to (3.34) in [19])} \qquad (51)
\end{aligned}$$

$$-e^{-bz(\vartheta_{r-t}\omega)}\int_{\mathbb{R}^3}\xi(\frac{|x|^2}{R^2})g(x,\widetilde{\sigma}(r))\Delta v dx \leq \frac{1}{2}e^{-2bz(\vartheta_{r-t}\omega)}\int_{\mathbb{R}^3}\xi(\frac{|x|^2}{R^2})g^2(x,\widetilde{\sigma}(r))dx + \int_{\mathbb{R}^3}\xi(\frac{|x|^2}{R^2})|\Delta v|^2dx. \quad (52)$$

It follows from (48)-(52) that

$$\frac{d}{dt}\int_{\mathbb{R}^3}\xi(\frac{|x|^2}{R^2})|\nabla v|^2dx \leq (2bz(\vartheta_{r-t}\omega) - \lambda)\int_{\mathbb{R}^3}\xi(\frac{|x|^2}{R^2})|\nabla v|^2dx + c_{13}\int_{\mathbb{R}^3}\xi(\frac{|x|^2}{R^2})|\nabla v|^2dx$$

$$+\frac{c_{14}}{R}\left(M_2^2(\vartheta_{r-t}\omega)+z^2(\vartheta_{r-t}\omega)M_2^2(\vartheta_{r-t}\omega)+e^{4bz(\vartheta_{r-t}\omega)}M_2^6(\vartheta_{r-t}\omega)\right)+\frac{c_{15}}{R}\|\Delta v\|^2+\left(\frac{c_{16}}{R}+\tilde{\gamma}_R\right)e^{-2bz(\vartheta_{r-t}\omega)}, \tag{53}$$

where $\tilde{\gamma}_R = \frac{1}{2}\sup_{\sigma\in\mathbb{T}^k}\int_{|x|\geq R} g^2(x,\sigma)dx + \int_{|x|\geq R}\beta_3^2(x)dx$. Applying Gronwall inequality to (27) over $[0,t]$, we have

$$\|\nabla v(t,\vartheta_{-t}\omega,\sigma,v_0(\vartheta_{-t}\omega))\|^2 + \int_0^t e^{\int_l^t(2bz(\vartheta_{s-t}\omega)-\lambda)ds}\|\Delta v(l)\|^2dl$$
$$\leq e^{\int_{-t}^0(2bz(\vartheta_s\omega)-\lambda)ds}M_2^2(\vartheta_{-t}\omega)+(1+2c_2)\int_{-t}^0 e^{\int_l^0(2bz(\vartheta_s\omega)-\lambda)ds}M_2^2(\vartheta_l\omega)dl + c_5K_0(\omega). \tag{54}$$

Using Gronwall inequality to (53) over $[0,t]$, by (47) and (54), we get

$$\int_{\mathbb{R}^3}\xi(\frac{|x|^2}{R^2})|\nabla v(t,\vartheta_{-t}\omega,\sigma,v_0(\vartheta_{-t}\omega))|^2dx$$
$$\leq e^{\int_0^t(2bz(\vartheta_{l-t}\omega)-\lambda)dl}\|\nabla v_0(\vartheta_{-t}\omega)\|^2$$
$$+c_{13}\int_0^t e^{\int_l^t(2bz(\vartheta_{s-t}\omega)-\lambda)ds}\int_{\mathbb{R}^3}\xi(\frac{|x|^2}{R^2})|\nabla v|^2dx$$
$$+\frac{c_{14}}{R}\int_0^t e^{\int_l^t(2bz(\vartheta_{s-t}\omega)-\lambda)ds}\left(M_2^2(\vartheta_{l-t}\omega)+z^2(\vartheta_{l-t}\omega)M_2^2(\vartheta_{l-t}\omega)+e^{4bz(\vartheta_{l-t}\omega)}M_2^6(\vartheta_{l-t}\omega)\right)dl$$
$$+\frac{c_{15}}{R}\int_0^t e^{\int_l^t(2bz(\vartheta_{s-t}\omega)-\lambda)ds}\|\Delta v(l)\|^2dl+\left(\frac{c_{16}}{R}+\tilde{\gamma}_R\right)\int_0^t e^{\int_l^t(2bz(\vartheta_{s-t}\omega)-\lambda)ds-2bz(\vartheta_{l-t}\omega)}dl$$
$$\leq c_7e^{\int_{-t}^0(2bz(\vartheta_l\omega)-\lambda)dl}M_2^2(\vartheta_{-t}\omega)+\frac{c_{17}}{R}(K_0(\omega)+K_1(\omega)+K_3(\omega))+c_{18}(\gamma_R+\tilde{\gamma}_R)K_0(\omega),$$

where $K_3(\omega)=\int_{-\infty}^0 e^{\int_l^0(2bz(\vartheta_s\omega)-\lambda)ds}\left(M_2^2(\vartheta_l\omega)+z^2(\vartheta_l\omega)M_2^2(\vartheta_l\omega)+e^{4bz(\vartheta_l\omega)}M_2^6(\vartheta_l\omega)\right)dl$. Let

$$Y_R=c_{18}(\gamma_R+\tilde{\gamma}_R),\quad K_2(\omega)=c_{17}(K_0(\omega)+K_1(\omega)+K_3(\omega)), \tag{55}$$

then (37) holds.

(iii) Take $T(\varepsilon,\omega)=\min\{(1+c_7)e^{\int_{-t}^0(2bz(\vartheta_s\omega)-\lambda)ds}M_2^2(\vartheta_{-t}\omega)\leq\varepsilon\}<\infty$, which along with (36), (37) implies (38) holds. □

2.5 Existence of random uniform exponential attractor

For every $\omega\in\Omega$, $s\geq 0$ and $\varepsilon\geq 0$, set

$$\mathbb{B}(\vartheta_{-s}\omega)=\overline{\cup_{t\geq\max\{T_*(\vartheta_{-s}\omega),T_*(\omega),T_*(\vartheta_{-T(\varepsilon,\omega)}\omega)\}+T(\varepsilon,\omega)}\pi(t,\vartheta_{-t-s}\omega)\mathbb{T}^k\times B_3(\vartheta_{-t-s}\omega)}, \tag{56}$$

where $T(\varepsilon,\omega)$ is defined in (38). Then $\mathbb{B}\subseteq\tilde{\mathbb{B}}$ and $P_{L^2(\mathbb{R}^3)}\mathbb{B}\subseteq B_3$. Consequently, $\mathbb{B}\in\mathcal{D}_{\mathbb{X}}$. It is easy to prove that $\mathbb{B}$ possesses the following properties,

(a-11) for every $\omega\in\Omega$, $\mathbb{B}(\omega)\subseteq\mathbb{T}^k\times B_3(\omega)\subseteq\mathbb{T}^k\times(B_0(\omega)\cap B_2(\omega))$. Hence, the diameter of $\mathbb{B}(\omega)$ in $\mathbb{T}^k\times L^2(\mathbb{R}^3)$ is bounded by $(k(2\pi)^2+4M_0^2(\omega))^{1/2}$, where $M_0^2(\vartheta_t\omega)$ is continuous in $t\in\mathbb{R}$.

(a-12) $\mathbb{B}(\omega)$ is positive invariant, i.e., $\pi(t,\vartheta_{-t}\omega)\mathbb{B}(\vartheta_{-t}\omega)\subseteq\mathbb{B}(\omega)$, $\forall\,\omega\in\Omega$, $t\geq 0$. Moreover,

$$\pi(r,\vartheta_{-t}\omega)\mathbb{B}(\vartheta_{-t}\omega)\subseteq\mathbb{B}(\vartheta_{r-t}\omega),\quad\forall\,\omega\in\Omega,\ r\geq 0,\ t\geq 0. \tag{57}$$

(a-13) $\mathbb{B}$ is pullback absorbing in $\mathcal{D}_{\mathbb{X}}$. Indeed, note that for $\forall\, \mathbb{D} \in \mathcal{D}_{\mathbb{X}}$, there exists $\mathbb{D}_1 \in \mathcal{D}_{1,\mathbb{X}}$ such that $\mathbb{D} \subset \mathbb{D}_1$. On the other hand, for $\forall\, \mathbb{B}_1 \in \mathcal{D}_{1,\mathbb{X}}$, $\omega \in \Omega$, there exist $\tilde{t}(\mathbb{B}_1, \omega) > 0$ such that

$$\pi(t, \vartheta_{-t}\omega)\mathbb{B}_1(\vartheta_{-t}\omega) \subseteq \mathbb{B}(\omega), \quad t \ge \tilde{t}(\mathbb{B}_1, \omega),$$

thus $\mathbb{B}$ is pullback absorbing in $\mathcal{D}_{\mathbb{X}}$.

(a-14) for any $\{\sigma\} \times \{v\} \in \mathbb{B}(\omega)$, the following holds,

$$\int_{\mathbb{R}^3} \xi\Big(\frac{|x|^2}{R^2}\Big)(|v|^2 + |\nabla v|^2)dx \le \varepsilon + \frac{K_1(\omega) + K_2(\omega)}{R} + Y_R K_0(\omega). \tag{58}$$

In fact, for any $\{\sigma\} \times \{v\} \in \cup_{t \ge \max\{T_\star(\omega), T_\star(\vartheta_{-T(\varepsilon,\omega)}\omega)\} + T(\varepsilon,\omega)} \pi(t, \vartheta_{-t}\omega)\mathbb{T}^k \times B_3(\vartheta_{-t}\omega)$, there exists $\hat{t} \ge \max\{T_\star(\omega), T_\star(\vartheta_{-T(\varepsilon,\omega)}\omega)\} + T(\varepsilon, \omega)$, $\hat{\sigma} \in \mathbb{T}^k$, $\hat{v} \in B_3(\vartheta_{-\hat{t}}\omega)$ such that

$$\begin{aligned}
\{\sigma\} \times \{v\} &= \pi(\hat{t}, \vartheta_{-\hat{t}}\omega)\{\hat{\sigma}\} \times \{\hat{v}\} \\
&= \pi(T(\varepsilon, \omega), \vartheta_{-T(\varepsilon,\omega)}\omega)\, \pi(\hat{t} - T(\varepsilon, \omega), \vartheta_{-\hat{t}}\omega)\{\hat{\sigma}\} \times \{\hat{v}\} \\
&\in \pi(T(\varepsilon, \omega), \vartheta_{-T(\varepsilon,\omega)}\omega)\, \pi(\hat{t} - T(\varepsilon, \omega), \vartheta_{-\hat{t}}\omega)\{\hat{\sigma}\} \times B_3(\vartheta_{-\hat{t}}\omega) \\
&\subseteq \pi(T(\varepsilon, \omega), \vartheta_{-T(\varepsilon,\omega)}\omega)\tilde{\mathbb{B}}(\vartheta_{-T(\varepsilon,\omega)}\omega) \quad \Big(\text{since } \hat{t} - T(\varepsilon, \omega) \ge \max\{T_\star(\omega), T_\star(\vartheta_{-T(\varepsilon,\omega)}\omega)\}\Big).
\end{aligned}$$

Hence, there exist $\{\tilde{\sigma}\} \times \{\tilde{v}\} \in \tilde{\mathbb{B}}(\vartheta_{-T(\varepsilon,\omega)}\omega)$ such that $\{\sigma\} \times \{v\} = \{\theta_{T(\varepsilon,\omega)}\tilde{\sigma}\} \times \{\phi(T(\varepsilon, \omega), \vartheta_{-T(\varepsilon,\omega)}\omega, \tilde{\sigma}, \tilde{v})\}$. By the jointly continuity of ϕ and the closeness of $\mathbb{B}$, we conclude from (38) that (58) holds.

In order to prove the existence of a random uniform exponential attractor for ϕ, we need to prove the existence of a random exponential attractor for π. Consequently, we next present $\mathbb{B}$ satisfy $(A2)$, $(A3)$ in Theorem 2.8.

2.5.1 Lipschitz continuity of π

For any $r \ge 0$, $t \ge 0$, $\omega \in \Omega$, $\{\sigma_i\} \times \{v_0^i(\vartheta_{-t}\omega)\} \in \mathbb{B}(\vartheta_{-t}\omega)$, $i = 1, 2$, let

$$y(r) = v_1(r) - v_2(r), \quad v_i(r) = v(r, \vartheta_{-t}\omega, \sigma_i, v_0^i(\vartheta_{-t}\omega)), \quad i = 1,\ 2. \tag{59}$$

By (57), we have

$$v_i(r) \in B_3(\vartheta_{r-t}\omega), \quad \|v_i(r)\|_1 \le M_2(\vartheta_{r-t}\omega), \quad i = 1,\ 2. \tag{60}$$

It follows from (59) that

$$\begin{cases}
\frac{dy(r)}{dt} = \Delta y(r) + (bz(\vartheta_{r-t}\omega) - \lambda)y(r) + e^{-bz(\vartheta_{r-t}\omega)}\Big(f(x, e^{bz(\vartheta_{r-t}\omega)}v_1(r)) - f(x, e^{bz(\vartheta_{r-t}\omega)}v_2(r))\Big) \\
\qquad + e^{-bz(\vartheta_{r-t}\omega)}\left(g(x, \widetilde{\sigma_1}(r)) - g(x, \widetilde{\sigma_2}(r))\right), \\
y(0) = v_1(0) - v_2(0) = v_0^1(\vartheta_{-t}\omega) - v_0^2(\vartheta_{-t}\omega), \quad r \ge 0.
\end{cases} \tag{61}$$

The following theorem shows the Lipschitz continuity of π on $\mathbb{B}$.

Lemma 2.14. *For any $r \ge 0$, $t \ge 0$, $\omega \in \Omega$, $\{\sigma_i\} \times \{v_0^i(\vartheta_{-t}\omega)\} \in \mathbb{B}(\vartheta_{-t}\omega)$, $i = 1, 2$, the following holds*

$$\begin{aligned}
&\|\pi(r, \vartheta_{-t}\omega)\{\sigma_1\} \times \{v_0^1(\vartheta_{-t}\omega)\} - \pi(r, \vartheta_{-t}\omega)\{\sigma_2\} \times \{v_0^2(\vartheta_{-t}\omega)\}\|_{\mathbb{X}}^2 \\
&\quad \le e^{\int_0^r (2b|z(\vartheta_{s-t}\omega)| + \varepsilon e^{-2bz(\vartheta_{s-t}\omega)} + \frac{c_{19}}{\varepsilon})ds}\left(\|v_0^1(\vartheta_{-t}\omega) - v_0^2(\vartheta_{-t}\omega)\|^2 + \|\sigma_1 - \sigma_2\|_{\mathbb{T}^k}^2\right).
\end{aligned} \tag{62}$$

Particularly,

$$\begin{aligned}
&\|\pi(t, \vartheta_{-t}\omega)\{\sigma_1\} \times \{v_0^1(\vartheta_{-t}\omega)\} - \pi(t, \vartheta_{-t}\omega)\{\sigma_2\} \times \{v_0^2(\vartheta_{-t}\omega)\}\|_{\mathbb{X}}^2 \\
&\quad \le e^{\int_{-t}^0 (2b|z(\vartheta_s\omega)| + \varepsilon e^{-2bz(\vartheta_s\omega)} + \frac{c_{19}}{\varepsilon})ds}\left(\|v_0^1(\vartheta_{-t}\omega) - v_0^2(\vartheta_{-t}\omega)\|^2 + \|\sigma_1 - \sigma_2\|_{\mathbb{T}^k}^2\right).
\end{aligned} \tag{63}$$

Proof. Taking the inner product of (61) with $y(r)$, and by

$$\left(e^{-bz(\vartheta_{r-t}\omega)}(f(x,e^{bz(\vartheta_{r-t}\omega)}v_1(r))-f(x,e^{bz(\vartheta_{r-t}\omega)}v_2(r))),\ y(r)\right)\le c_2\|y(r)\|^2,\qquad \text{(by(17))}$$

$$\begin{aligned}\left(e^{-bz(\vartheta_{r-t}\omega)}(g(x,\widetilde{\sigma_1}(r))\ -g(x,\widetilde{\sigma_2}(r))),\ y(r)\right)&\\ &\le \|g(x,\widetilde{\sigma_1}(r))-g(x,\widetilde{\sigma_2}(r))\|\cdot e^{-bz(\vartheta_{r-t}\omega)}\|y(r)\|\\ &\le \frac{\|h\|^2}{2\varepsilon}\|\sigma_1-\sigma_2\|^2_{\mathbb{T}^k}+\frac{\varepsilon}{2}e^{-2bz(\vartheta_{r-t}\omega)}\|y(r)\|^2,\qquad \text{(hereafter, we set } 0<\varepsilon\le 1)\end{aligned}$$

then we have

$$\frac{d}{dt}(\|y\|^2+\|\sigma_1-\sigma_2\|^2_{\mathbb{T}^k})+2\|\nabla y\|^2\le(2b|z(\vartheta_{r-t}\omega)|+\varepsilon e^{-2bz(\vartheta_{r-t}\omega)}+\frac{c_{19}}{\varepsilon})(\|y\|^2+\|\sigma_1-\sigma_2\|^2_{\mathbb{T}^k})\tag{64}$$

where $c_{19}=2c_2+\|h\|^2$. Using Gronwall inequality to (64) over $[0,r]$, we arrive

$$\begin{aligned}\|y(r)\|^2+\|\sigma_1-\sigma_2\|^2_{\mathbb{T}^k}+2\int_0^r e^{\int_l^r(2b|z(\vartheta_{s-t}\omega)|+\varepsilon e^{-2bz(\vartheta_{s-t}\omega)}+\frac{c_{19}}{\varepsilon})dl}\|\nabla y(l)\|^2dl&\\ \le e^{\int_0^r(2b|z(\vartheta_{s-t}\omega)|+\varepsilon e^{-2bz(\vartheta_{s-t}\omega)}+\frac{c_{19}}{\varepsilon})ds}\left(\|y(0)\|^2+\|\sigma_1-\sigma_2\|^2_{\mathbb{T}^k}\right),&\end{aligned}\tag{65}$$

hence, (62)-(63) hold. We have thus proved the lemma. □

2.5.2 Decomposition of solution

Denote $\mathbb{U}_R=\{x\in\mathbb{R}^3:|x|<R\}$ the ball in $\mathbb{R}^3$, $0<R<\infty$. Consider the eigenvalue problem

$$\begin{cases}-\Delta\tilde{u}(x)=\mu\tilde{u}(x) & \text{in } \mathbb{U}_{2R}\\ \tilde{u}(x)=0 & \text{on } \partial\,\mathbb{U}_{2R}.\end{cases}$$

It is known that there are a family of eigenfunctions $\{\tilde{e}_{m,R}\}_{m\in\mathbb{N}}$, which form an orthonormal base of $L^2(\mathbb{U}_{2R})$ and $H_0^1(\mathbb{U}_{2R})$, and a family of eigenvalues $\{\mu_{m,R}\}_{m\in\mathbb{N}}$ such that

$$0<\mu_{1,R}\le\mu_{2,R}\le\cdots\le\mu_{m,R}\le\cdots,\quad \mu_{m,R}\to+\infty\quad as\quad m\to+\infty.\tag{66}$$

Moreover, for $m\in\mathbb{N}$, $-\Delta\tilde{e}_{m,R}=\mu_{m,R}\tilde{e}_{m,R}$, $\tilde{e}_{m,R}\in H^2(\mathbb{U}_{2R})\cap H_0^1(\mathbb{U}_{2R})$. Let

$$L_m^2(\mathbb{U}_{2R})=\text{span}\{\tilde{e}_{1,R},\tilde{e}_{2,R},\cdots,\tilde{e}_{m,R}\},\quad L_m^2(\mathbb{U}_{2R})^{\perp}=\text{span}\{\tilde{e}_{m+1,R},\tilde{e}_{m+2,R},\cdots\}.$$

and

$$\tilde{P}_{m,R}:L^2(\mathbb{U}_{2R})\to L_m^2(\mathbb{U}_{2R}),\quad \tilde{Q}_{m,R}:L^2(\mathbb{U}_{2R})\to L_m^2(\mathbb{U}_{2R})^{\perp},$$

then $\tilde{P}_{m,R}$ are m-dimensional orthonormal projector, and for $v\in L_m^2(\mathbb{U}_{2R})^{\perp}$

$$\mu_{m+1,R}\|\tilde{Q}_{m,R}v\|^2\le\|\nabla v\|^2.$$

Write

$$e_{m,R}(x)=\begin{cases}\tilde{e}_{m,R}(x), & |x|<2R\\ 0, & |x|\ge 2R\end{cases},\quad m\in\mathbb{N},$$

then $\{e_{m,R}\}_{m\in\mathbb{N}}$ is a family of orthonormal functions of $L^2(\mathbb{R}^3)$. For given $m\in\mathbb{N}$, let

$$L_{m,R}^2(\mathbb{R}^3)=\text{span}\{e_{1,R},e_{2,R},\cdots,e_{m,R}\},\quad L_{m,R}^2(\mathbb{R}^3)^{\perp}=\text{span}\{e_{m+1,R},e_{m+2,R},\cdots\},$$

$$P_{m,R}:L^2(\mathbb{R}^3)\to L_{m,R}^2(\mathbb{R}^3),\quad Q_{m,R}:L^2(\mathbb{R}^3)\to L_{m,R}^2(\mathbb{R}^3)^{\perp},$$

then $P_{m,R}$ is a m-dimensional projector from $L^2(\mathbb{R}^3)$ into $L^2_{m,R}(\mathbb{R}^3)$ and

$$\mu_{m+1,R}\|Q_{m,R}v\|^2 \le \|\nabla v\|^2 \le \|v\|_1^2, \quad \forall v \in L^2_{m,R}(\mathbb{R}^3)^{\perp}. \tag{67}$$

Let $y(r) = v_1(r, \vartheta_{-t}\omega, \sigma_1, v_0^1(\vartheta_{-t}\omega)) - v_2(r, \vartheta_{-t}\omega, \sigma_2, v_0^2(\vartheta_{-t}\omega))(r \ge 0)$ be the solution of (61) with initial value $y(0) = v_0^1(\vartheta_{-t}\omega) - v_0^2(\vartheta_{-t}\omega)$. Set

$$y_{1,m,R}(r) = P_{m,R}y(r) = \begin{cases} \tilde{P}_{m,R}y(r), & |x| < 2R \\ 0, & |x| \ge 2R \end{cases} \in L^2_{m,R}(R^3),$$

$$y_{2,m,R}(r) = Q_{m,R}y(r) = \begin{cases} \tilde{Q}_{m,R}y(r), & |x| < 2R \\ 0, & |x| \ge 2R \end{cases} \in L^2_{m,R}(R^3)^{\perp},$$

$$y_{3,m,R}(r) = (I - P_{m,R} - Q_{m,R})y(r) = \begin{cases} y(r), & |x| \ge 2R, \\ 0, & |x| < 2R. \end{cases}$$

Then

$$y(r) = y_{1,m,R}(r) + y_{2,m,R}(r) + y_{3,m,R}(r),$$

moreover,

$$(y_{1,m,R}(r), y_{2,m,R}(r)) = (y_{2,m,R}(r), y_{3,m,R}(r)) = (y_{1,m,R}(r), y_{3,m,R}(r)) = 0.$$

Next, we estimate $y_{2,m,R}(r), y_{3,m,R}(r)$.

Lemma 2.15. *For every $r \ge 0$, $t \ge 0$, $\omega \in \Omega$, $R \ge 0$ and $m \in \mathbb{N}$, there exists a random variable $C_1(\omega) \ge 0$ such that for any $\{\sigma_i\} \times \{v_0^i(\vartheta_{-t}\omega)\} \in \mathbb{B}(\vartheta_{-t}\omega)$, $i = 1, 2$, the following holds*

$$\begin{aligned}\|y_{2,m,R}(t)\| =& \|Q_{m,R}\phi(t, \vartheta_{-t}\omega, \sigma_1, v_0^1(\vartheta_{-t}\omega)) - Q_{m,R}\phi(t, \vartheta_{-t}\omega, \sigma_2, v_0^2(\vartheta_{-t}\omega))\| \\ \le& \left(e^{\int_{-t}^0 (bz(\vartheta_s\omega) - \lambda + \frac{\varepsilon}{2}e^{-2bz(\vartheta_s\omega)})ds} + H(\varepsilon, 2\lambda + \mu_{m+1,R})e^{\int_{-t}^0 C_1(\vartheta_s\omega)ds}\right) \\ &\times (\|v_0^1(\vartheta_{-t}\omega) - v_0^2(\vartheta_{-t}\omega)\|^2 + \|\sigma_1 - \sigma_2\|^2_{\mathbb{T}^k})^{\frac{1}{2}},\end{aligned} \tag{68}$$

where

$$H(\varepsilon, 2\lambda + \mu_{m+1,R}) = \sqrt{\frac{c_{21}}{\varepsilon(2\lambda + \mu_{m+1,R})} + \frac{c_{21}}{\sqrt{2\lambda + \mu_{m+1,R}}}}, \quad c_{21} > 0.$$

Proof. Taking the inner product of (61) with $y_{2,m,R}$ in $L^2(\mathbb{R}^3)$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|y_{2,m,R}\|^2 =& -\|\nabla y_{2,m,R}\|^2 + \left(bz(\vartheta_{r-t}\omega) - \lambda\right)\|y_{2,m,R}\|^2 \\ &+ \left(e^{-bz(\vartheta_{r-t}\omega)}(f(x, e^{bz(\vartheta_{r-t}\omega)}v_1(r)) - f(x, e^{bz(\vartheta_{r-t}\omega)}v_2(r))),\ y_{2,m,R}\right) \\ &+ \left(e^{-bz(\vartheta_{r-t}\omega)}(g(x, \widetilde{\sigma_1}(r)) - g(x, \widetilde{\sigma_2}(r))),\ y_{2,m,R}\right).\end{aligned} \tag{69}$$

Refer to (3.47) in [19], we have

$$\begin{aligned}&\left(e^{-bz(\vartheta_{r-t}\omega)}(f(x, e^{bz(\vartheta_{r-t}\omega)}v_1(r)) - f(x, e^{bz(\vartheta_{r-t}\omega)}v_2(r))),\ y_{2,m,R}\right) \\ &\le \frac{1}{2}c_{20}\left(e^{2bz(\vartheta_{r-t}\omega)}M_2^4(\vartheta_{r-t}\omega) + e^{-2bz(\vartheta_{r-t}\omega)}\right)\|y(r)\|^2 + \frac{1}{2}\|\nabla y_{2,m,R}\|^2,\end{aligned} \tag{70}$$

where $c_{20} > 0$ depends on c_3, C_0 and $\|\beta_4\|_{L^3(\mathbb{R}^3)}$. Note that

$$\left(e^{-bz(\vartheta_{r-t}\omega)}(g(x, \widetilde{\sigma_1}(r)) - g(x, \widetilde{\sigma_2}(r))),\ y_{2,m,R}\right) \le \frac{\varepsilon}{2}e^{-2bz(\vartheta_{r-t}\omega)}\|y_{2,m,R}\|^2 + \frac{\|h\|^2}{2\varepsilon}\|\sigma_1 - \sigma_2\|^2_{\mathbb{T}^k}, \tag{71}$$

$$\|\nabla y_{2,m,R}\|^2 \ge \mu_{m+1,R}\|y_{2,m,R}\|^2. \tag{72}$$

It follows from (69), (70), (71) and (72) that

$$\begin{aligned}\frac{d}{dt}\|y_{2,m,R}\|^2 \le &-\mu_{m+1,R}\|y_{2,m,R}\|^2 + 2\left(bz(\vartheta_{r-t}\omega)-\lambda\right)\|y_{2,m,R}\|^2 \\ &+ c_{20}\left(e^{2bz(\vartheta_{r-t}\omega)}M_2^4(\vartheta_{r-t}\omega)+e^{-2bz(\vartheta_{r-t}\omega)}\right)\|y(r)\|^2 + \varepsilon e^{-2bz(\vartheta_{r-t}\omega)}\|y_{2,m,R}\|^2 + \frac{\|h\|^2}{\varepsilon}\|\sigma_1-\sigma_2\|^2_{\mathbb{T}^k}.\end{aligned} \tag{73}$$

Putting (65) into (73), we get

$$\begin{aligned}\frac{d}{dt}\|y_{2,m,R}(r)\|^2 \le &\left(2bz(\vartheta_{r-t}\omega)-2\lambda-\mu_{m+1,R}+\varepsilon e^{-2bz(\vartheta_{r-t}\omega)}\right)\|y_{2,m,R}(r)\|^2 \\ &+ c_{21}\left(\frac{1}{\varepsilon} + (e^{2bz(\vartheta_{r-t}\omega)}M_2^4(\vartheta_{r-t}\omega)+e^{-2bz(\vartheta_{r-t}\omega)})e^{\int_0^r(2b|z(\vartheta_{s-t}\omega)|+\varepsilon e^{-2bz(\vartheta_{s-t}\omega)}+\frac{c_{19}}{\varepsilon})ds}\right) \\ &\times(\|y(0)\|^2+\|\sigma_1-\sigma_2\|^2_{\mathbb{T}^k})\end{aligned} \tag{74}$$

Using Gronwall inequality to (74) over $[0, t]$, we derive

$$\begin{aligned}\|y_{2,m,R}(t)\|^2 \le\ & e^{\int_{-t}^0(2bz(\vartheta_s\omega)-2\lambda-\mu_{m+1,R}+\varepsilon e^{-2bz(\vartheta_s\omega)})ds}\|y_{2,m,R}(0)\|^2 \\ &+ \frac{c_{21}}{\varepsilon}(\|y(0)\|^2+\|\sigma_1-\sigma_2\|^2_{\mathbb{T}^k})\int_{-t}^{0} e^{\int_l^0(2bz(\vartheta_s\omega)+\varepsilon e^{-2bz(\vartheta_s\omega)})ds}e^{(2\lambda+\mu_{m+1,R})l}dl \\ &+ c_{21}(\|y(0)\|^2+\|\sigma_1-\sigma_2\|^2_{\mathbb{T}^k})\int_{-t}^{0} e^{\int_{-t}^0(2bz(\vartheta_s\omega)+\varepsilon e^{-2bz(\vartheta_s\omega)}+\frac{c_{19}}{\varepsilon})ds} \\ &\times e^{(2\lambda+\mu_{m+1,R})l}(e^{2bz(\vartheta_l\omega)}M_2^4(\vartheta_l\omega)+e^{-2bz(\vartheta_l\omega)})dl \\ \le\ & e^{\int_{-t}^0(2bz(\vartheta_s\omega)-2\lambda-\mu_{m+1,R}+\varepsilon e^{-2bz(\vartheta_s\omega)})ds}(\|y(0)\|^2+\|\sigma_1-\sigma_2\|^2_{\mathbb{T}^k}) \\ &+ \frac{c_{21}}{\varepsilon}(\|y(0)\|^2+\|\sigma_1-\sigma_2\|^2_{\mathbb{T}^k})\frac{1}{2\lambda+\mu_{m+1,R}}e^{\int_{-t}^0(2b|z(\vartheta_s\omega)|+\varepsilon e^{-2bz(\vartheta_s\omega)})ds} \\ &+ c_{21}(\|y(0)\|^2+\|\sigma_1-\sigma_2\|^2_{\mathbb{T}^k})e^{\int_{-t}^0(2bz(\vartheta_s\omega)+\varepsilon e^{-2bz(\vartheta_s\omega)}+\frac{c_{19}}{\varepsilon})ds} \\ &\times\frac{1}{\sqrt{2\lambda+\mu_{m+1,R}}}e^{\int_{-t}^0(2e^{4bz(\vartheta_l\omega)}M_2^8(\vartheta_l\omega)+2e^{-4bz(\vartheta_l\omega)})dl} \qquad (\text{since } \sqrt{x}\le e^x) \\ \le\ & \left(e^{\int_{-t}^0(2bz(\vartheta_s\omega)-2\lambda+\varepsilon e^{-2bz(\vartheta_s\omega)})ds} + H^2(\varepsilon, 2\lambda+\mu_{m+1,R})e^{\int_{-t}^0 2C_1(\vartheta_s\omega)ds}\right) \\ &\times(\|y(0)\|^2+\|\sigma_1-\sigma_2\|^2_{\mathbb{T}^k}),\end{aligned} \tag{75}$$

where $C_1(\omega) = b|z(\omega)| + \frac{\varepsilon}{2}e^{-2bz(\omega)} + \frac{c_{19}}{2\varepsilon} + e^{4bz(\omega)}M_2^8(\omega) + e^{-4bz(\omega)}$ and

$$H(\varepsilon, 2\lambda+\mu_{m+1,R}) = \sqrt{\frac{c_{21}}{\varepsilon(2\lambda+\mu_{m+1,R})} + \frac{c_{21}}{\sqrt{2\lambda+\mu_{m+1,R}}}}.$$

The proof is completed. □

Lemma 2.16. *For $R > 2$ satisfying $h_R = \left(\int_{|x|\ge R} h^2(x)dx\right)^{1/2} \le \varepsilon$ $(0 < \varepsilon \le 1)$ and $t \ge 0$, $\omega \in \Omega$, $m \in \mathbb{N}$, there exists a random variable $C_2(\omega) \ge 0$ such that for any $\{\sigma_i\}\times\{v_0^i(\vartheta_{-t}\omega)\} \in \mathbb{B}(\vartheta_{-t}\omega)$, $i = 1, 2$, the following holds*

$$\begin{aligned}\|y_{3,m,R}(t)\| &= \left(\int_{|x|\ge R}|\phi(t,\vartheta_{-t}\omega,\sigma_1,v_0^1(\vartheta_{-t}\omega))-\phi(t,\vartheta_{-t}\omega,\sigma_2,v_0^2(\vartheta_{-t}\omega))|^2dx\right)^{\frac{1}{2}} \\ &\le\left(e^{\int_{-t}^0(bz(\vartheta_s\omega)-\lambda+\frac{h_R}{2}e^{-2bz(\vartheta_s\omega)})ds}+\sqrt{c_\lambda I_{\varepsilon,R}}e^{\int_{-t}^0 C_2(\vartheta_s\omega)ds}\right)(\|v_0^1(\vartheta_{-t}\omega)-v_0^2(\vartheta_{-t}\omega)\|^2+\|\sigma_1-\sigma_2\|^2_{\mathbb{T}^k})^{\frac{1}{2}}\end{aligned} \tag{76}$$

where

$$I_{\varepsilon,R} = \varepsilon + \frac{1}{R^2} + \frac{1}{R} + Y_{\frac{R}{2}} + \beta_{4,R} + h_R, \quad \beta_{4,R} = \left(\int\limits_{|x|\geq R} \beta_4^3(x)dx \right)^{\frac{1}{3}},$$

$$c_\lambda = \frac{1}{2\lambda} + \frac{1}{2\sqrt{\lambda}} + \frac{1}{\sqrt[4]{8\lambda}}.$$

Proof. By (3.51) in [19], we have

$$\|v\|^2_{L^6(\mathbb{R}^3\setminus\mathbb{U}_{2R})} \leq \frac{16C_0^2\tilde{C}^2}{R^2}\|v\|^2 + 2C_0^2 \int\limits_{\mathbb{R}^3} \xi(\frac{|x|^2}{R^2})(|\nabla v|^2 + |v|^2)dx, \quad v \in H^1(\mathbb{R}^3). \tag{77}$$

considering (57),(58), (60) and (77), we get

$$\begin{aligned} \|v_i(r)\|^2_{L^6(\mathbb{R}^3\setminus\mathbb{U}_R)} &\leq \frac{64C_0^2\tilde{C}^2}{R^2} M_2^2(\vartheta_{r-t}\omega) + 2C_0^2\left(\varepsilon + \frac{2K_1(\vartheta_{r-t}\omega) + 2K_2(\vartheta_{r-t}\omega)}{R} + Y_{\frac{R}{2}}K_0(\vartheta_{r-t}\omega)\right) \\ &\leq 2C_0^2\varepsilon + \left(\frac{1}{R^2} + \frac{1}{R} + Y_{\frac{R}{2}}\right) K_4(\vartheta_{r-t}\omega), \end{aligned} \tag{78}$$

where $K_4(\omega) = 64C_0^2\tilde{C}^2M_2^2(\omega) + 4C_0^2(K_0(\omega) + K_1(\omega) + K_2(\omega))$.

Taking the inner product of (61) with $\xi(\frac{|x|^2}{R^2})y$ in $L^2(\mathbb{R}^3)$, and by

$$(\Delta y, \xi(\frac{|x|^2}{R^2})y) \leq \frac{2\sqrt{2}\tilde{C}}{R}\|y\| \cdot \|\nabla y\| \tag{79}$$

$$\begin{aligned} &\left(e^{-bz(\vartheta_{r-t}\omega)} \ (f(x, e^{bz(\vartheta_{r-t}\omega)}v_1(r)) - f(x, e^{bz(\vartheta_{r-t}\omega)}v_1(r))), \ \xi(\frac{|x|^2}{R^2})y\right) \\ &\quad \leq 16c_3C_0e^{bz(\vartheta_{r-t}\omega)}\left(2C_0^2\varepsilon + (\frac{1}{R^2} + \frac{1}{R} + Y_{\frac{R}{2}})K_4(\vartheta_{r-t}\omega)\right)(\|y\|^2 + \|y\| \cdot \|\nabla y\|) \\ &\quad\quad + 2C_0\beta_{4,R}e^{-bz(\vartheta_{r-t}\omega)}(\|y\|^2 + \|y\| \cdot \|\nabla y\|), \qquad \text{(refer to (3.54) in [19])} \end{aligned} \tag{80}$$

$$\begin{aligned} \|y(r)\|^2 + \|y(r)\| \cdot \|\nabla y(r)\| \leq \ & e^{\int_0^r (2b|z(\vartheta_{s-t}\omega)| + \varepsilon e^{-2bz(\vartheta_{s-t}\omega)} + \frac{c_{19}}{\varepsilon})ds}\left(\|y(0)\|^2 + \|\sigma_1 - \sigma_2\|^2_{\mathbb{T}^k}\right) \\ & + \|\nabla y(r)\| e^{\int_0^r (b|z(\vartheta_{s-t}\omega)| + \frac{\varepsilon}{2}e^{-2bz(\vartheta_{s-t}\omega)} + \frac{c_{19}}{2\varepsilon})ds}\left(\|y(0)\|^2 + \|\sigma_1 - \sigma_2\|^2_{\mathbb{T}^k}\right)^{\frac{1}{2}}. \quad \text{(by (65))} \end{aligned} \tag{81}$$

$$\begin{aligned} \left(e^{-bz(\vartheta_{r-t}\omega)}(g(x, \widetilde{\sigma_1}(r)) - g(x, \widetilde{\sigma_2}(r))), \ \xi(\frac{|x|^2}{R^2})y\right) &\leq \|\sqrt{\xi(\frac{|x|^2}{R^2})}h(x)\| \cdot \|\sigma_1 - \sigma_2\|_{\mathbb{T}^k} \cdot \|e^{-bz(\vartheta_{r-t}\omega)}\sqrt{\xi(\frac{|x|^2}{R^2})}y\| \\ &\leq \frac{1}{2}h_R\|\sigma_1 - \sigma_2\|^2_{\mathbb{T}^k} + \frac{1}{2}h_Re^{-2bz(\vartheta_{r-t}\omega)} \int\limits_{\mathbb{R}^3} \xi(\frac{|x|^2}{R^2})y^2dx, \end{aligned} \tag{82}$$

we arrive

$$\begin{aligned} \frac{d}{dt}\int\limits_{\mathbb{R}^3} \xi(\frac{|x|^2}{R^2})y^2dx \leq & \left(2bz(\vartheta_{r-t}\omega) - 2\lambda + h_Re^{-2bz(\vartheta_{r-t}\omega)}\right)\int\limits_{\mathbb{R}^3} \xi(\frac{|x|^2}{R^2})y^2dx \\ & + \left(I_{\varepsilon,R} + I_{\varepsilon,R}K_5(\vartheta_{r-t}\omega)e^{\int_0^r (2b|z(\vartheta_{s-t}\omega)| + \varepsilon e^{-2bz(\vartheta_{s-t}\omega)} + \frac{c_{19}}{\varepsilon})ds}\right)(\|y(0)\|^2 + \|\sigma_1 - \sigma_2\|^2_{\mathbb{T}^k}) \\ & + I_{\varepsilon,R}K_5(\vartheta_{r-t}\omega)e^{\int_0^r (b|z(\vartheta_{s-t}\omega)| + \frac{\varepsilon}{2}e^{-2bz(\vartheta_{s-t}\omega)} + \frac{c_{19}}{2\varepsilon})ds}(\|y(0)\|^2 + \|\sigma_1 - \sigma_2\|^2_{\mathbb{T}^k})^{\frac{1}{2}}\|\nabla y(r)\|, \end{aligned}$$

where

$$K_5(\omega) = c_{22}e^{b|z(\omega)|}(1 + M_2^2(\omega) + K_0(\omega) + K_1(\omega) + K_2(\omega)), \tag{83}$$

and $c_{22} > 0$ is a constant depending on c_0, C_3, $\tilde{C}$. Applying Gronwall inequality to the above equation over $[0, t]$, we have

$$
\begin{aligned}
\int\limits_{\mathbb{R}^3} \xi(\frac{|x|^2}{R^2}) y^2(t)dx \le & e^{\int_{-t}^{0}(2bz(\vartheta_l\omega)-2\lambda+h_R e^{-2bz(\vartheta_l\omega)})dl} \int\limits_{\mathbb{R}^3} \xi(\frac{|x|^2}{R^2}) y^2(0)dx \\
& + I_{\varepsilon,R} \int\limits_0^t e^{\int_l^t (2bz(\vartheta_{s-t}\omega)-2\lambda+h_R e^{-2bz(\vartheta_{s-t}\omega)})ds} (\|y(0)\|^2 + \|\sigma_1 - \sigma_2\|_{\mathbb{T}^k}^2) dl \\
& + I_{\varepsilon,R} \int\limits_0^t e^{\int_l^t (2bz(\vartheta_{s-t}\omega)-2\lambda+h_R e^{-2bz(\vartheta_{s-t}\omega)})ds} e^{\int_0^l (2bz(\vartheta_{s-t}\omega)+\varepsilon e^{-2bz(\vartheta_{s-t}\omega)+\frac{c_{19}}{\varepsilon}})ds} K_5(\vartheta_{l-t}\omega) \\
& \qquad \times (\|y(0)\|^2 + \|\sigma_1 - \sigma_2\|_{\mathbb{T}^k}^2) dl \\
& + I_{\varepsilon,R} \int\limits_0^t e^{\int_l^t (2bz(\vartheta_{s-t}\omega)-2\lambda+h_R e^{-2bz(\vartheta_{s-t}\omega)})ds} e^{\int_0^l (b|z(\vartheta_{s-t}\omega)|+\frac{\varepsilon}{2} e^{-2bz(\vartheta_{s-t}\omega)+\frac{c_{19}}{2\varepsilon}})ds} K_5(\vartheta_{l-t}\omega) \\
& \qquad \times (\|y(0)\|^2 + \|\sigma_1 - \sigma_2\|_{\mathbb{T}^k}^2)^{\frac{1}{2}} \cdot \|\nabla y(l)\| dl \\
\le & e^{\int_{-t}^{0}(2bz(\vartheta_l\omega)-2\lambda+h_R e^{-2bz(\vartheta_l\omega)})dl} (\|y(0)\|^2 + \|\sigma_1 - \sigma_2\|_{\mathbb{T}^k}^2) \\
& + \frac{1}{2\lambda} I_{\varepsilon,R} (\|y(0)\|^2 + \|\sigma_1 - \sigma_2\|_{\mathbb{T}^k}^2) e^{\int_{-t}^{0}(2b|z(\vartheta_s\omega)|+h_R e^{-2bz(\vartheta_s\omega)})ds} \\
& + \frac{1}{2\sqrt{\lambda}} I_{\varepsilon,R} (\|y(0)\|^2 + \|\sigma_1 - \sigma_2\|_{\mathbb{T}^k}^2) e^{\int_{-t}^{0}(2bz(\vartheta_s\omega)+(\varepsilon+h_R)e^{-2bz(\vartheta_s\omega)}+\frac{c_{19}}{\varepsilon})ds} e^{\int_{-t}^{0} K_5^2(\vartheta_s\omega)ds} \\
& + (\|y(0)\|^2 + \|\sigma_1 - \sigma_2\|_{\mathbb{T}^k}^2)^{\frac{1}{2}} I_{\varepsilon,R} \int\limits_0^t e^{\int_l^t (bz(\vartheta_{s-t}\omega)+\frac{h_R}{2} e^{-2bz(\vartheta_{s-t}\omega)})ds} e^{2\lambda(l-t)} \\
& \qquad \times e^{\int_l^t (b|z(\vartheta_{s-t}\omega)|+\frac{h_R}{2} e^{-2bz(\vartheta_{s-t}\omega)+\frac{c_{19}}{2\varepsilon}})ds} \\
& \qquad \times e^{\int_0^l (b|z(\vartheta_{s-t}\omega)|+\frac{\varepsilon}{2} e^{-2bz(\vartheta_{s-t}\omega)+\frac{c_{19}}{2\varepsilon}})ds} K_5(\vartheta_{l-t}\omega) \|\nabla y(l)\| dl.
\end{aligned}
\tag{84}
$$

Take $R > 2$ large enough such that $h_R \le \varepsilon$, since

$$
\begin{aligned}
& (\|y(0)\|^2 + \|\sigma_1 - \sigma_2\|_{\mathbb{T}^k}^2)^{\frac{1}{2}} I_{\varepsilon,R} \int\limits_0^t e^{\int_l^t (bz(\vartheta_{s-t}\omega)+\frac{h_R}{2} e^{-2bz(\vartheta_{s-t}\omega)})ds} e^{2\lambda(l-t)} e^{\int_l^t (b|z(\vartheta_{s-t}\omega)|+\frac{h_R}{2} e^{-2bz(\vartheta_{s-t}\omega)+\frac{c_{19}}{2\varepsilon}})ds} \\
& \qquad \times e^{\int_0^l (b|z(\vartheta_{s-t}\omega)|+\frac{\varepsilon}{2} e^{-2bz(\vartheta_{s-t}\omega)+\frac{c_{19}}{2\varepsilon}})ds} K_5(\vartheta_{l-t}\omega) \|\nabla y(l)\| dl \\
\le & (\|y(0)\|^2 + \|\sigma_1 - \sigma_2\|_{\mathbb{T}^k}^2)^{\frac{1}{2}} I_{\varepsilon,R} e^{\int_{-t}^{0} (b|z(\vartheta_s\omega)|+\frac{\varepsilon+h_R}{2} e^{-2bz(\vartheta_s\omega)+\frac{c_{19}}{2\varepsilon}})ds} \\
& \times \left(\int\limits_0^t e^{\int_l^t (2bz(\vartheta_{s-t}\omega)+h_R e^{-2bz(\vartheta_{s-t}\omega)}+\frac{c_{19}}{\varepsilon})ds} \|\nabla y(l)\|^2 dl \right)^{\frac{1}{2}} \left(\int\limits_0^t e^{4\lambda(l-t)} K_5(\vartheta_{l-t}\omega) dl \right)^{\frac{1}{2}} \quad (\text{by } h_R \le \varepsilon) \\
\le & \frac{1}{\sqrt[4]{8\lambda}} (\|y(0)\|^2 + \|\sigma_1 - \sigma_2\|_{\mathbb{T}^k}^2) I_{\varepsilon,R}\, e^{\int_{-t}^{0}(2b|z\vartheta_s(\omega)|+(\varepsilon+h_R)e^{-2bz(\vartheta_s\omega)}+\frac{c_{19}}{\varepsilon}+\frac{1}{2}K_5^4(\vartheta_s\omega))ds}, \quad (\text{by } (65))
\end{aligned}
\tag{85}
$$

thus

$$\int_{\mathbb{R}^3} \xi(\frac{|x|^2}{R^2})y^2(t)dx \le e^{\int_{-t}^{0}(2bz(\vartheta_s\omega)-2\lambda+h_R e^{-2bz(\vartheta_s\omega)})ds}(\|y(0)\|^2+\|\sigma_1-\sigma_2\|_{\mathbb{T}^k}^2)$$
$$+\frac{1}{2\lambda}I_{\varepsilon,R}(\|y(0)\|^2+\|\sigma_1-\sigma_2\|_{\mathbb{T}^k}^2)e^{\int_{-t}^{0}(2b|z(\vartheta_s\omega)|+h_R e^{-2bz(\vartheta_s\omega)})ds}$$
$$+\frac{1}{2\sqrt{\lambda}}I_{\varepsilon,R}(\|y(0)\|^2+\|\sigma_1-\sigma_2\|_{\mathbb{T}^k}^2)e^{\int_{-t}^{0}(2b|z(\vartheta_s\omega)|+(\varepsilon+h_R)e^{-2bz(\vartheta_s\omega)}+\frac{c_{19}}{\varepsilon}+K_5^2(\vartheta_s\omega))ds}$$
$$+\frac{1}{\sqrt[4]{8\lambda}}I_{\varepsilon,R}(\|y(0)\|^2+\|\sigma_1-\sigma_2\|_{\mathbb{T}^k}^2)\,e^{\int_{-t}^{0}(2b|z\vartheta_s(\omega)|+(\varepsilon+h_R)e^{-2bz(\vartheta_s\omega)}+\frac{c_{19}}{\varepsilon}+\frac{1}{2}K_5^4(\vartheta_s\omega))ds}$$
$$\le\left(e^{\int_{-t}^{0}(2bz(\vartheta_s\omega)-2\lambda+\varepsilon e^{-2bz(\vartheta_s\omega)})ds}+c_\lambda I_{\varepsilon,R}e^{\int_{-t}^{0}2C_2(\vartheta_s\omega)ds}\right)(\|y(0)\|^2+\|\sigma_1-\sigma_2\|_{\mathbb{T}^k}^2)$$

where

$$C_2(\omega)=b|z(\omega)|+\varepsilon e^{-2bz(\omega)}+\frac{c_{19}}{2\varepsilon}+\frac{1}{2}+\frac{1}{2}k_5^4(\omega). \tag{86}$$

The proof is completed. □

Lemma 2.17. *For $\omega\in\Omega$, $R>2$ satisfying $h_R\le\varepsilon$ $(0<\varepsilon\le1)$, $\{\sigma_i\}\times\{v_0^i(\vartheta_{-t}\omega)\}\in\mathbb{B}(\vartheta_{-t}\omega)$, $i=1,2$, there exist a random variable $C_3(\omega)\ge0$ and a $(k+m)$-dimensional projector*

$$\mathbb{P}_{k+m,R}\,:\,\mathbb{T}^k\times L^2(\mathbb{R}^3)\to\mathbb{T}^k\times L^2_{m,R}(\mathbb{R}^3),\quad \{\sigma\}\times\{x\}\to\{\sigma\}\times\{P_{m,R}x,\}$$

($(k+m)$ is independent of R) such that the following hold

(i)

$$\|\pi(t,\vartheta_{-t}\omega)\{\sigma_1\}\times\{v_0^1(\vartheta_{-t}\omega)\}-\pi(t,\vartheta_{-t}\omega)\{\sigma_2\}\times\{v_0^2(\vartheta_{-t}\omega)\}\|_{\mathbb{X}}$$
$$\le e^{\int_{-t}^{0}C_3(\vartheta_s\omega)ds}\left(\|v_0^1(\vartheta_{-t}\omega)-v_0^2(\vartheta_{-t}\omega)\|^2+\|\sigma_1-\sigma_2\|_{\mathbb{T}^k}^2\right)^{\frac{1}{2}}, \tag{87}$$

(ii) for $t\ge\frac{2\ln2}{\lambda}$,

$$\|(I_{\mathbb{X}}-\mathbb{P}_{k+m,R})\,\pi(t,\vartheta_{-t}\omega)\{\sigma_1\}\times\{v_0^1(\vartheta_{-t}\omega)\}-(I_{\mathbb{X}}-\mathbb{P}_{k+m,R})\pi(t,\vartheta_{-t}\omega)\{\sigma_2\}\times\{v_0^2(\vartheta_{-t}\omega)\}\|_{\mathbb{X}}$$
$$=\|(I_{L^2(\mathbb{R}^3)}-P_{m,R})[\phi(t,\vartheta_{-t}\omega,\sigma_1,v_0^1(\vartheta_{-t}\omega))-\phi(t,\vartheta_{-t}\omega,\sigma_2,v_0^2(\vartheta_{-t}\omega))]\|$$
$$\le\|y_{2,m,R}(t)\|+\|y_{3,m,R}(t)\|$$
$$\le\left(e^{-\frac{\lambda}{2}t+\int_{-t}^{0}(b|z(\vartheta_s\omega)|+\frac{\varepsilon}{2}e^{-2bz(\vartheta_s\omega)})ds}+\left(H(\varepsilon,2\lambda+\mu_{m+1,R})+\sqrt{c_\lambda I_{\varepsilon,R}}\right)e^{\int_{-t}^{0}C_3(\vartheta_s\omega)ds}\right)$$
$$\times\left(\|v_0^1(\vartheta_{-t}\omega)-v_0^2(\vartheta_{-t}\omega)\|^2+\|\sigma_1-\sigma_2\|_{\mathbb{T}^k}^2\right)^{\frac{1}{2}}, \tag{88}$$

where $C_3(\omega)=b|z(\omega)|+\varepsilon e^{-2bz(\omega)}+\frac{c_{19}}{2\varepsilon}+\frac{1}{2}+\frac{1}{2}K_5^4(\omega)+e^{4bz(\omega)}M_2^8(\omega)+e^{-4bz(\omega)}$.

Proof. Considering Lemma 2.14-2.16, we find (i), (ii) hold. Thus, the lemma holds. □

2.5.3 The boundedness of expectation

We next check the boundedness of expectation, we need the following lemma.

Lemma 2.18 (see [24, 25]). *The Ornstein-Uhlenbeck process $z(\theta_t\omega)$ satisfies*

$$\mathbf{E}\left[e^{\eta\int_{\tau}^{\tau+t}|z(\theta_s\omega)|ds}\right]\le e^{\eta t}\quad \text{for } 0\le\eta^2\le1,\ \tau\in\mathbb{R},\quad t\ge0, \tag{89}$$

$$\mathbf{E}[|z(\theta_t\omega)|^p]=\frac{\Gamma(\frac{1+p}{2})}{\sqrt{\pi}},\ \forall p>0,\ t\in\mathbb{R},\quad \mathbf{E}[e^{\epsilon z(\theta_s\omega)}]\le\frac{4\sqrt{\pi}+3e}{3\sqrt{\pi}},\ s\in\mathbb{R},\ |\epsilon|\le1. \tag{90}$$

where Γ is the Gamma function.

Lemma 2.19. *Let the coefficient b of the random term in* (19) *be small enough such that*

$$b < \min\left\{\frac{\lambda\sqrt{\pi}}{64}, \frac{\lambda}{768}, \frac{1}{768}\right\} \tag{91}$$

and let ε be small enough, $R > 2$ be large enough such that

$$h_R \leq \varepsilon, \quad \frac{\varepsilon}{2}\cdot\frac{4\sqrt{\pi}+3e}{3\sqrt{\pi}} \leq \frac{\lambda}{64}, \tag{92}$$

then

$$\mathbf{E}[b|z(\omega)| + \frac{\varepsilon}{2}e^{-2bz(\omega)}] \leq \frac{\lambda}{32}, \quad 0 \leq \mathbf{E}[C_3(\omega)],\ \mathbf{E}[C_3^2(\omega)] < \infty.$$

Proof. According to (90), (91) and (92), we have

$$\mathbf{E}[b|z(\omega)| + \frac{\varepsilon}{2}e^{-2bz(\omega)}] \leq \frac{\lambda}{64} + \frac{\lambda}{64} = \frac{\lambda}{32}.$$

Note that

$$C_3^2(\omega) \leq c_{23}[\frac{1}{\varepsilon^2} + z^2(\omega) + e^{4b|z(\omega)|} + e^{8b|z(\omega)|} + e^{16b|z(\omega)|} + K_0^{16}(\omega) + K_6^{16}(\omega) + K_7^{16}(\omega)],$$

where

$$K_6(\omega) = \int_{-\infty}^{0} e^{\int_l^0 (2bz(\theta_s\omega)-\lambda)ds} M_2^2(\theta_l\omega)dl, \tag{93}$$

$$K_7(\omega) = \int_{-\infty}^{0} e^{\int_l^0 (2bz(\theta_s\omega)-\lambda)ds} \left(z^2(\theta_l\omega)M_2^2(\theta_l\omega) + e^{4bz(\theta_l\omega)}M_2^6(\theta_l\omega)\right) dl. \tag{94}$$

By Lemma 3.11 in [19], we have $\mathbf{E}[C_3^2(\omega)] < \infty$, thus $\mathbf{E}[C_3(\omega)] \leq (\mathbf{E}[C_3^2(\omega)])^{\frac{1}{2}} < \infty$. The proof is completed. □

Theorem 2.20. *Assume* $(A1)-(A3)$, (91) *and* (92) *hold, then the NRDS* $\{\phi(t,\omega,\sigma)\}_{t\geq 0,\omega\in\Omega,\sigma\in\mathbb{T}^k}$ *has a* $\mathcal{D}$-*random uniform exponential attractor* $\{\mathcal{M}(\omega)\}_{\omega\in\Omega}$ *with properties:*

(i) $\mathcal{M}$ *is a random compact set;*

(ii) $\dim_f \mathcal{M}(\omega) \leq \frac{2(k+m_1)\ln(\frac{2\sqrt{k+m_1}}{\delta_{\varepsilon_1,R_1,m_1}}+1)}{\ln\frac{4}{3}}, \ \forall\omega\in\Omega.$

(iii) for $\forall\ \omega\in\Omega$, $D\in\mathcal{D}$, *there exist random variables* $\tilde{T}(\omega,\mathbb{D})$ *and* $\tilde{b}(\omega,\mathbb{D})$ *such that*

$$\sup_{\sigma\in\mathbb{T}^k} \operatorname{dist}_{L^2(\mathbb{R}^3)}\left(\phi(t,\vartheta_{-t}\omega,\theta_{-t}\sigma)D(\vartheta_{-t}\omega),\ \mathcal{M}(\omega)\right) \leq \check{b}(\omega,\mathbb{D})e^{-\frac{\lambda\ln\frac{4}{3}}{64\ln 2}t}, \quad t \geq \tilde{T}(\omega,\mathbb{D}),$$

where $\mathbb{D} = \mathbb{T}^k \times D$.

Proof. By $(A1)-(A3)$, we take $\frac{2\ln 2}{\lambda} \leq t = t_0 = \frac{8\ln 2}{\frac{\lambda}{2}}$ in (88) and the right-hand side of (87). From Lemma 2.19, it follows that the number $\nu = t_0^2\left(2\mathbf{E}[C_3^2(\omega)] + \frac{\lambda}{2}\mathbf{E}[C_3(\omega)]\right) < \infty$. Write

$$\eta = \min\left\{\frac{1}{16},\ e^{-\frac{2}{\ln\frac{3}{2}}\nu}\right\}.$$

Evidently, we can choose $\varepsilon = \varepsilon_1$ small enough and $R = R_1 \geq 2$ big enough such that

$$2\sqrt{c_\lambda I_{\varepsilon_1,R_1}} \leq \frac{\eta}{2}, \quad h_{R_1} \leq \varepsilon_1.$$

For fixed ε_1, R_1, by $\mu_{m+1,R_1}\to\infty$, there exists a $m = m_1$ big enough such that

$$2H(\varepsilon_1, 2\lambda + \mu_{m_1+1,R_1}) \leq \frac{\eta}{2}.$$

Thus

$$\delta_{\varepsilon_1,R_1,m_1} = 2H(\varepsilon_1, 2\lambda + \mu_{m_1+1,R_1}) + 2\sqrt{c_\lambda I_{\varepsilon_1,R_1}} \le \eta.$$

Considering **(a-11)**–**(a-13)**, Lemma 2.17, Lemma 2.19, by Theorem 2.8, we know the skew-product cocycle π generated by ϕ and θ has a $\mathbb{D}_{\mathbb{X}}$-random exponential attractor $\{\mathcal{O}(\omega)\}_{\omega\in\Omega}$ with

$$\dim_f \mathcal{O}(\omega) \le \frac{2(k+m_1)\ln(\frac{2\sqrt{k+m_1}}{\delta_{\varepsilon_1,R_1,m_1}}+1)}{\ln\frac{4}{3}}, \quad \forall\omega\in\Omega.$$

Consequently, ϕ has a $\mathcal{D}$-random uniform exponential attractor $\mathcal{M} = P_{L^2(\mathbb{R}^3)}\mathcal{O}$ with $\dim_f \mathcal{M}(\omega) \le \dim_f \mathcal{O}(\omega)$, $\forall\omega\in\Omega$. Moreover, by (15), for $\forall\,\omega\in\Omega$, $D\in\mathcal{D}$,

$$\sup_{\sigma\in\mathbb{T}^k} \operatorname{dist}_{L^2(\mathbb{R}^3)}\big(\phi(t,\vartheta_{-t}\omega,\theta_{-t}\sigma)D(\vartheta_{-t}\omega),\ \mathcal{M}(\omega)\big) \le \check{b}(\omega,\mathbb{D})e^{-\frac{\lambda\ln\frac{4}{3}}{64\ln 2}t}, \quad t\ge\tilde{T}(\omega,\mathbb{D}),$$

where $\mathbb{D} = \mathbb{T}^k\times D$. The proof is complete. □

Remark 2.21. *Similarly, we can obtain the existence of a random uniform exponential attractor for the following reaction-diffusion equation with quasi-periodic external force and additive white noise in $\mathbb{R}^3$:*

$$\begin{cases} du + (\lambda u - \Delta u)dt = (f(x,u) + g(x,\widetilde{\sigma}(t)))dt + q(x)dW(t), & t>0 \\ u(x,0) = u_0(x), \quad x\in\mathbb{R}^3, \end{cases} \tag{95}$$

*where $q\in H^1(\mathbb{R}^3)\cap H^2(\mathbb{R}^3)$ and g, f satisfy **(H1)**-**(H2)**, the random term is understood in the Itô sense.*

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