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# The monotonicity of ratios involving arc tangent function with applications

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**Abstract:** In this paper, we investigate the monotonicity of the functions

$$x \mapsto \frac{1}{x}\left(1-a+\sqrt{\frac{2}{3}ax^2+a^2}\right)\arctan x,$$
$$x \mapsto \frac{1}{x}\left(\frac{4}{\pi^2}+\sqrt{\frac{4}{\pi^2}x^2+a}\right)\arctan x$$

on $(0,\infty)$ for $a>0$, which not only gives relative errors of known bounds with quadratic for $\arctan x$, but also yields some new accurate bounds. Moreover, the known bounds are extended and a more accurate estimate for $\arctan x$ is presented.

**Keywords:** arctangent function, sharp bounds, absolute error, relative error

**MSC:** Primary 33B10; Secondary 26D05

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## 1 Introduction

In [1], Shafer proposed the elementary problem: Show that for all $x>0$ the inequality

$$\frac{3x}{1+2\sqrt{x^2+1}} < \arctan x \tag{1.1}$$

holds. This was proven in [2]. In [3], Qi, Zhang and Guo discussed the monotonicity of the function

$$x \mapsto \frac{a+\sqrt{x^2+1}}{x}\arctan x$$

on $(0,\infty)$, and sharpened and generalized Shafer's inequality (1.1). Chen and Sun [4] further determined the best $b$, $c$ such that the inequalities

$$\frac{bx}{1+a\sqrt{x^2+1}} < \arctan x < \frac{cx}{1+a\sqrt{x^2+1}} \tag{1.2}$$

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hold for $a, x > 0$. More refinements and sharpenings of inequality (1.1) can be seen in [5] and the recent paper [6].

A more general form of Shafer's inequality (1.1) is that

$$B_{a,b,c}(x) = \frac{x}{c + \sqrt{bx^2 + a^2}} < (>) \arctan x$$

for all $x > 0$, where $a, b > 0$ and $c + a > 0$. For this, Shafer [7] established the following analytic inequalities:

$$\frac{8x}{3 + \sqrt{80x^2/3 + 25}} < \arctan x \tag{1.3}$$

for $x > 0$. Zhu [8] found a double inequality

$$\frac{8x}{3 + \sqrt{80x^2/3 + 25}} < \arctan x < \frac{8x}{3 + \sqrt{256x^2/\pi^2 + 25}} \tag{1.4}$$

holds for $x > 0$ with the best constants $80/3$ and $256/\pi^2$. Alirezaei [9] provided other two sharp lower and upper bounds for $\arctan x$, that is, the double inequality

$$\frac{\pi^2 x}{4 + \sqrt{4\pi^2 x^2 + (\pi^2 - 4)^2}} < \arctan x < \frac{\pi^2 x}{\pi^2 - 6 + \sqrt{4\pi^2 x^2 + 36}} \tag{1.5}$$

holds for $x > 0$. Moreover, by observing the graph, he showed that the maximum relative errors of the lower and upper bounds are approximately smaller than $0.27\%$ and $0.23\%$, respectively. Recently, Nishizawa [10] proved that

$$\frac{\pi^2 x}{4 + \sqrt{4\pi^2 x^2 + (\pi^2 - 4)^2}} < \arctan x < \frac{\pi^2 x}{4 + \sqrt{4\pi^2 x^2 + 32}} \tag{1.6}$$

for $x > 0$, where $(\pi^2 - 4)^2$ and 32 are the best constants.

Other approximations for the arctangent function can be found in [11, 12].

To describe the coincidence of an odd function $f(x) = \arctan x$ with its approximation $A(x)$ which is also odd, we use a similar suggestion as presented in [13] by Gasull and Utzet, which states that the function $f$ is equal to $A$ at 0 of order $i \geq 1$ if $f$ and $A$, and their derivatives up to order $(i - 1)$ coincide at 0, that is,

$$\lim_{x \to 0} \frac{f(x) - A(x)}{x^{2k-1}} = 0, \ \ k = 1, 3, \ldots 2i - 1. \tag{1.7}$$

In a similar way, $f$ and $A$ are equal at infinity of order $j \geq 1$ if

$$\lim_{x \to \infty} \frac{f(x) - A(x)}{x^{1-k}} = 0 \text{ for } k = 1, 2, \ldots, j - 1. \tag{1.8}$$

For $i, j \geq 0$, that $f$ and $A$ are equal of order $(i, j)$ if they are equal at 0 of order $i$ and at infinity of order $j$.

Now, expanding in power series gives

$$\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 + O\left(x^7\right), \tag{1.9}$$

$$B_{a,b,c}(x) = \frac{1}{a + c}x - \frac{b}{2a(a + c)^2}x^3 + \frac{b^2(3a + c)}{8a^3(a + c)^3}x^5 + O\left(x^7\right) \tag{1.10}$$

as $x \to 0$, and

$$\arctan x = \frac{\pi}{2} - \arctan\frac{1}{x} = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} + O\left(\frac{1}{x^5}\right), \tag{1.11}$$

$$B_{a,b,c}(x) = \frac{1}{c/x + \sqrt{b + a^2/x^2}} = \frac{1}{\sqrt{b}} - \frac{c}{b}\frac{1}{x} - \frac{a^2 - 2c^2}{2b\sqrt{b}}\frac{1}{x^2} + O\left(\frac{1}{x^3}\right) \tag{1.12}$$

as $1/x \to 0$.

(i) If $\arctan x$ and $B_{a,b,c}(x)$ have a coincidence of order $(2,0)$, that is, the parameters $a, b$ and $c$ satisfy the relation (1.7) for $k = 1, 2$, then comparing respectively the coefficients of $x$ and $x^3$ in the Maclaurin expansions (1.9) and (1.10) we get

$$a + c = 1 \text{ and } \frac{b}{2a(a+c)^2} = \frac{1}{3},$$

which indicates that

$$b = \frac{2a}{3}, c = 1 - a. \tag{1.13}$$

Thus

$$B_{a,b,c}(x) = \frac{x}{1 - a + \sqrt{2ax^2/3 + a^2}} := A_{(2,0)}(x; a). \tag{1.14}$$

Further, if $\arctan x$ and $B_{a,b,c}(x)$ have a coincidence of order $(3,0)$, then comparing respectively the coefficients of $x^5$ in the Maclaurin expansions (1.9) and (1.10) we have

$$\frac{b^2(3a+c)}{8a^3(a+c)^3} = \frac{1}{5},$$

which in combination with (1.13) yields $a = 5/8$. Therefore,

$$A_{(2,0)}\left(x; \frac{5}{8}\right) = \frac{x}{3/8 + \sqrt{5x^2/12 + 25/64}} := A_{(3,0)}(x), \tag{1.15}$$

which is the lower bound given in (1.3). Likewise, if $A_{(2,0)}(x; a)$ satisfies (1.8) for $k = 1$, then

$$\frac{1}{\sqrt{b}} = \frac{\pi}{2},$$

which in combination with (1.13) yields $a = 6/\pi^2$. That is,

$$A_{(2,0)}\left(x; \frac{6}{\pi^2}\right) = \frac{x}{1 - 6/\pi^2 + \sqrt{4x^2/\pi^2 + 36/\pi^4}} := A_{(2,1)}(x), \tag{1.16}$$

which is the upper bound given in (1.5).

(ii) If $\arctan x$ and $B_{a,b,c}(x)$ have a coincidence of order $(1,1)$, that is, the parameters $a, b$ and $c$ satisfy the relations (1.7) for $k = 1$ and (1.8) for $k = 1$, then comparing respectively the coefficients of $x$ in the Maclaurin expansions (1.9) and (1.10), and the constant items in the asymptotic expansions (1.11) and (1.12), we get

$$a + c = 1 \text{ and } \frac{1}{\sqrt{b}} = \frac{\pi}{2},$$

which implies that

$$B_{a,b,c}(x) = \frac{x}{1 - a + \sqrt{4x^2/\pi^2 + a^2}} := A_{(1,1)}(x; a). \tag{1.17}$$

Analogously, we have

$$A_{(1,1)}\left(x; \frac{6}{\pi^2}\right) = \frac{x}{1 - 6/\pi^2 + \sqrt{4x^2/\pi^2 + 36/\pi^4}} := A_{(2,1)}(x),$$

$$A_{(1,1)}\left(x; 1 - \frac{4}{\pi^2}\right) = \frac{x}{4/\pi^2 + \sqrt{4x^2/\pi^2 + \left(1 - 4/\pi^2\right)^2}} := A_{(1,2)}(x).$$

Clearly, $A_{(2,1)}(x)$ and $A_{(1,2)}(x)$ are the upper and lower bounds given in (1.5).

(iii) If $\arctan x$ and $B_{a,b,c}(x)$ have a coincidence of order $(0,2)$, that is, the parameters $a, b$ and $c$ satisfy the relations (1.8) for $k = 1, 2$, then comparing the constant items and coefficients of $x^{-1}$ in the asymptotic expansions (1.11) and (1.12), we get

$$\frac{1}{\sqrt{b}} = \frac{\pi}{2} \text{ and } \frac{c}{b} = 1.$$

Thus

$$B_{a,b,c}(x)=\frac{x}{(4/\pi^2)+\sqrt{4x^2/\pi^2+a^2}}:=A_{(0,2)}(x;a). \tag{1.18}$$

We also check that

$$A_{(0,2)}\left(x;1-\frac{4}{\pi^2}\right)=\frac{x}{4/\pi^2+\sqrt{(4/\pi^2)x^2+\left(1-4/\pi^2\right)^2}}:=A_{(1,2)}(x), \tag{1.19}$$

$$A_{(0,2)}\left(x;\frac{4\sqrt{2}}{\pi^2}\right)=\frac{x}{4/\pi^2+\sqrt{4x^2/\pi^2+32/\pi^4}}:=A_{(0,3)}(x), \tag{1.20}$$

where $A_{(1,2)}(x)$ and $A_{(0,3)}(x)$ are clearly the lower and upper bounds given in (1.6).

From the inequalities (1.3), (1.5) and (1.6), we see clearly that there are two sharp lower bounds $A_{(3,0)}(x)$, $A_{(1,2)}(x)$ and two sharp upper bounds $A_{(2,1)}(x)$, $A_{(0,3)}(x)$ for $\arctan x$, all of which have the form of $B_{a,b,c}(x)$. Moreover, $A_{(3,0)}(x)$ and $A_{(2,1)}(x)$ are contained in the family of bounds $A_{(2,0)}(x;a)$, while $A_{(1,2)}(x)$ and $A_{(0,3)}(x)$ belong to the family of bounds $A_{(0,2)}(x;a)$. Inspired by these facts, the aim of this paper is to investigate the monotonicity of the ratios

$$R_1(x)=\frac{\arctan x}{A_{(2,0)}(x;a)} \quad \text{and} \quad R_2(x)=\frac{\arctan x}{A_{(0,2)}(x;\sqrt{a})},$$

which gives new proofs of inequalities (1.3), (1.5) and (1.6). Moreover, as we all know, analytic inequality plays an important role in many different blanch of mathematics (See for example, [14–18]). By the obtained monotonicity of $R_1(x)$ and $R_2(x)$ we find some new inequalities, that is, new sharp bounds for $\arctan x$. We show the maximum relative errors and maximum absolute errors estimating for $\arctan x$ by the four known sharp bounds mentioned above, and offer a new sharp bounds in the form of $\lambda B_{a,b,c}(x)$ ($\lambda>0$ with $\lambda\neq 1$).

The main tool dealing with the monotonicity or $R_1$ and $R_2$ is two identities on the derivatives of ratio of two functions $p$ and $q$, where $p$ and $q$ are twice differentiable on $(a,b)$ $(a<b)$ with $q, q'\neq 0$ on $(a,b)$:

$$\left(\frac{p}{q}\right)'=\frac{q'}{q^2}\left(\frac{p'}{q'}q-p\right)=\frac{q'}{q^2}H_{p,q} \tag{1.21}$$

$$H'_{p,q}=\left(\frac{p'}{q'}\right)'q. \tag{1.22}$$

Identities (1.21) and (1.22) (for short, IDR) were introduced in [19] by Yang. We remark that the auxiliary function $H_{p,q}$ and its properties are very helpful to investigate those monotonicity of ratios of two functions, see for example, [20–22]. Similarly, the auxiliary function $H_{p,q}$ together with the IDR (1.21) and (1.22) will be used effectively to prove Theorems 1 and 2.

## 2 Lemmas

The following three lemmas are used to prove Lemma 4.

**Lemma 1.** [23] *For $-\infty\le a<b\le\infty$, let $f$ and $g$ be differentiable functions on $(a,b)$ with $f(a^+)=g(a^+)=0$ or $f(b^-)=g(b^-)=0$. Assume that $g'(x)\neq 0$ for each $x$ in $(a,b)$. If $f'/g'$ is increasing (decreasing) on $(a,b)$ then so is $f/g$.*

**Lemma 2.** [24] *Let $a_n$ and $b_n$ $(n=0,1,2,\ldots)$ be real numbers and let the power series $A(t)=\sum_{n=1}^{\infty}a_nt^n$ and $B(t)=\sum_{n=1}^{\infty}b_nt^n$ be convergent for $|t|<r$. If $b_n>0$ for $n=0,1,2,\ldots$, and $a_n/b_n$ is strictly increasing (or decreasing) for $n=0,1,2,\ldots$, then the function $t\mapsto A(t)/B(t)$ is strictly increasing (or decreasing) on $(0,r)$.*

**Lemma 3.** [25] *The following expansions*

$$\frac{1}{\sin x}=\frac{1}{x}+\sum_{n=1}^{\infty}\frac{2^{2n}-2}{(2n)!}|B_{2n}|x^{2n-1}, \tag{2.1}$$

$$\frac{\cos x}{\sin^2 x} = \frac{1}{x^2} - \sum_{n=1}^{\infty} \frac{(2n-1)\left(2^{2n}-2\right)}{(2n)!} |B_{2n}| x^{2n-2}, \tag{2.2}$$

$$\cot x = \frac{1}{x} - \sum_{n=1}^{\infty} \frac{2^{2n}}{(2n)!} |B_{2n}| x^{2n-1}, \tag{2.3}$$

$$\frac{1}{\sin^2 x} = \frac{1}{x^2} + \sum_{n=1}^{\infty} \frac{(2n-1) 2^{2n}}{(2n)!} |B_{2n}| x^{2n-2}, \ |x| < \pi, \tag{2.4}$$

*hold for $|x| < \pi$, where $B_n$ is the Bernoulli numbers.*

Lemma 4 plays a key role in the proof of Theorems 1 and 2.

**Lemma 4.** *Let*

$$\phi_1(t) = (\sin t - t\cos t)\sin^3 t, \tag{2.5}$$

$$\phi_2(t) = t^2 - 2\cos^2 t \sin^2 t - 2t\cos t \sin^3 t + t\cos t \sin t. \tag{2.6}$$

*Then $\phi_2(t) > 0$ for $t \in (0, \pi/2)$, and $\phi_1(t)/\phi_2(t)$ is strictly decreasing from $(0, \pi/2)$ onto $(4/\pi^2, 15/32)$.*

*Proof.* We write $\phi_2(t)$ as

$$4\phi_2(t) = (2t)^2 + (2t)\sin 2t \cos 2t - 2\sin^2(2t) := \phi_4(2t).$$

Differentiating and expanding in power series by (2.3) and (2.4) yield

$$\frac{\phi_4'(s)}{\sin^2 s} = \frac{3s - 3\cos s \sin s - 2s\sin^2 s}{\sin^2 s} = 3\frac{s}{\sin^2 s} - 3\frac{\cos s}{\sin s} - 2s = \sum_{n=2}^{\infty} \frac{(6n) 2^{2n}}{(2n)!} |B_{2n}| s^{2n-1} > 0, \tag{2.7}$$

where $s = 2t \in (0, \pi)$. Then $\phi_4(2t) = \phi_4(s) > \phi_4(0) = 0$, which implies $\phi_2(t) > 0$ for $t \in (0, \pi/2)$.

Similarly, $\phi_1(t)$ can be written as

$$\phi_1(t) = \frac{1}{8}\left(s\cos s \sin s - 2\sin^2 s - 4\cos s - s\sin s + 4\right),$$

where $s = 2t$. Then $\phi_1(t)/\phi_2(t)$

$$\frac{\phi_1(t)}{\phi_2(t)} = \frac{1}{2}\frac{s\cos s \sin s - 2\sin^2 s - 4\cos s - s\sin s + 4}{s^2 + s\sin s \cos s - 2\sin^2 s} = \frac{1}{2} - \frac{1}{2}\frac{4\cos s + s\sin s + s^2 - 4}{s^2 + s\sin s\cos s - 2\sin^2 s} := \frac{1}{2} - \frac{1}{2}\frac{\phi_3(s)}{\phi_4(s)},$$

where

$$\phi_3(s) = 4\cos s + s\sin s + s^2 - 4.$$

Thus, to prove the desired monotonicity, it suffices to prove that $\phi_3(s)/\phi_4(s)$ is strictly increasing on $(0, \pi)$. Expanding in power series by (2.4), (2.1) and (2.2) leads to

$$\frac{\phi_3'(s)}{\sin^2 s} = \frac{2s - 3\sin s + s\cos s}{\sin^2 s} = 2\frac{s}{\sin^2 s} - \frac{3}{\sin s} + s\frac{\cos s}{\sin^2 s} = \sum_{n=2}^{\infty} \frac{2\left((n-2)2^{2n} + 2n + 2\right)}{(2n)!} |B_{2n}| s^{2n-1}. \tag{2.8}$$

Taking into account (2.7) and (2.8) we get

$$\frac{\phi_3'(s)}{\phi_4'(s)} = \frac{\phi_3'(s)/\sin^2 s}{\phi_4'(s)/\sin^2 s} = \frac{\sum_{n=2}^{\infty} \frac{2\left((n-2)2^{2n}+2n+2\right)}{(2n)!} |B_{2n}| s^{2n-1}}{\sum_{n=2}^{\infty} \frac{(6n)2^{2n}}{(2n)!} |B_{2n}| s^{2n-1}} := \frac{\sum_{n=2}^{\infty} u_n s^{2n}}{\sum_{n=2}^{\infty} v_n s^{2n}}.$$

By Lemmas 1 and 2, it is enough to show that the sequence $\{u_n/v_n\}_{n\geq 2}$ is increasing. A direct verification yields

$$\frac{u_n}{v_n} = \frac{(n-2)2^{2n} + 2n + 2}{(3n)2^{2n}},$$

$$\frac{u_{n+1}}{v_{n+1}}-\frac{u_n}{v_n}=\frac{4^{n+1}-\left(3n^2+6n+4\right)}{6n\left(n+1\right)\times 2^{2n}}.$$

Using the binomial theorem we arrive at

$$4^{n+1}-\left(3n^2+6n+4\right)>1+\left(n+1\right)3+\frac{n\left(n+1\right)}{2}3^2-\left(3n^2+6n+4\right)=\frac{3}{2}n\left(n+1\right)>0,$$

which gives the increasing property of the sequence $\{u_n/v_n\}_{n\geq 2}$.

An easy calculation gives

$$\lim_{t\to 0}\frac{\phi_1\left(t\right)}{\phi_2\left(t\right)}=\frac{15}{32}\quad\text{and}\quad\lim_{t\to\pi/2}\frac{\phi_1\left(t\right)}{\phi_2\left(t\right)}=\frac{4}{\pi^2},$$

which completes the proof. □

**Remark 1.** *Using the methods from [26–29], one can directly prove the Lemma 4.*

## 3 The monotonicity of $R_1$ and inequalities

We now state and prove our first main result, which reveals the monotonicity pattern of $R_1$ on $(0,\infty)$.

**Theorem 1.** *(i) If $a\geq 5/8$, then the ratio*

$$R_1\left(x\right)=\frac{1}{x}\left(1-a+\sqrt{\frac{2}{3}ax^2+a^2}\right)\arctan x$$

*is strictly increasing from $(0,\infty)$ onto $\left(1,\pi\sqrt{a/6}\right)$. Therefore, the double inequality*

$$\frac{x}{1-a+\sqrt{2ax^2/3+a^2}}<\arctan x<\sqrt{\frac{a\pi^2}{6}}\frac{x}{1-a+\sqrt{2ax^2/3+a^2}}\tag{3.1}$$

*holds for $x>0$.*

*(ii) If $\left(\sqrt{3\pi^2+2}-\sqrt{2}\right)^2/\left(3\pi^2\right)=a_1<a<5/8$, then there is an $x_0>0$ such that $R_1$ is strictly decreasing on $(0,x_0)$ and increasing on $(x_0,\infty)$. So the double inequality*

$$\frac{R_1\left(x_0\right)\times x}{1-a+\sqrt{2ax^2/3+a^2}}\leq\arctan x<\frac{\max\left(1,\sqrt{a\pi^2/6}\right)\times x}{1-a+\sqrt{2ax^2/3+a^2}}\tag{3.2}$$

*holds for $x>0$. In particular, for $a=6/\pi^2$, we have*

$$\frac{c_{21}x}{1-6/\pi^2+\sqrt{4x^2/\pi^2+36/\pi^4}}\leq\arctan x<\frac{x}{1-6/\pi^2+\sqrt{4x^2/\pi^2+36/\pi^4}},\tag{3.3}$$

*where $c_{21}=R_1\left(x_0\right)=0.9976914...$ is the best possible.*

*(iii) If $0<a\leq a_1=\left(\sqrt{3\pi^2+2}-\sqrt{2}\right)^2/\left(3\pi^2\right)=0.598...$, then $R_1\left(x\right)$ is strictly decreasing from $(0,\infty)$ onto $\left(\pi\sqrt{a/6},1\right)$. Then the double inequality*

$$\frac{\sqrt{a\pi^2/6}\times x}{1-a+\sqrt{2ax^2/3+a^2}}\arctan x<\frac{x}{1-a+\sqrt{2ax^2/3+a^2}}\tag{3.4}$$

*holds for $x>0$.*

*Proof.* Making a change of variable $x = \tan t$ for $t \in (0, \pi/2)$ yields

$$R_1(x) = \frac{1}{x}\left(1 - a + \sqrt{\frac{2}{3}ax^2 + a^2}\right)\arctan x = \frac{1 - a + \sqrt{(2a/3)\tan^2 t + a^2}}{(\tan t)/t} := \frac{p(t)}{q(t)},$$

where

$$p(t) = 1 - a + \sqrt{\frac{2}{3}a\tan^2 t + a^2}, \qquad q(t) = \frac{\tan t}{t}.$$

Differentiation yields

$$\frac{p'(t)}{q'(t)} = \frac{2a}{\sqrt{3}}\frac{t^2 \sin t}{(t - \cos t \sin t)\sqrt{3a^2\cos^2 t + 2a\sin^2 t}},$$

$$\left(\frac{p'(t)}{q'(t)}\right)' = \frac{2a^2}{\sqrt{3}}\frac{(t\cos t) \times [3a\phi_2(t) - 4\phi_1(t)]}{\left(\sqrt{2a\sin^2 t + 3a^2\cos^2 t}\right)^3 (t - \cos t \sin t)^2}, \tag{3.5}$$

where $\phi_1(t)$ and $\phi_2(t)$ are defined by (2.5) and (2.6), respectively. As shown in Lemma 4, $\phi_2(t) > 0$ for $t \in (0, \pi/2)$, $\phi_1(t)/\phi_2(t)$ is strictly decreasing from $(0, \pi/2)$ onto $(4/\pi^2, 15/32)$. Then the relation (3.5) can be written as

$$\left(\frac{p'(t)}{q'(t)}\right)' = 2\sqrt{3}a^2\frac{(t\cos t) \times \phi_2(t)}{\left(\sqrt{2a\sin^2 t + 3a^2\cos^2 t}\right)^3 (t - \cos t \sin t)^2}\left[a - \frac{4}{3}\frac{\phi_1(t)}{\phi_2(t)}\right].$$

Therefore, $\left(p'(t)/q'(t)\right)' > (<) 0$ if and only if

$$a \geq \sup_{t\in(0,\pi/2)}\left[\frac{4}{3}\frac{\phi_1(t)}{\phi_2(t)}\right] = \frac{5}{8} \quad \text{or} \quad a \leq \inf_{t\in(0,\pi/2)}\left[\frac{4}{3}\frac{\phi_1(t)}{\phi_2(t)}\right] = \frac{16}{3\pi^2},$$

while $a \in \left(16/\left(3\pi^2\right), 5/8\right)$, there is a $t_1 \in (0, \pi/2)$ such that $\left(p'(t)/q'(t)\right)' < 0$ for $t \in (0, t_1)$ and $\left(p'(t)/q'(t)\right)' > 0$ for $t \in (t_1, \pi/2)$.

On the other hand, we easily get

$$q'(t) = \left(\frac{\tan t}{t}\right)' = \frac{2t - \sin 2t}{2t^2\cos^2 t} > 0, \tag{3.6}$$

$$\begin{aligned} H_{p,q}(t) = \frac{p'(t)}{q'(t)}q(t) - p(t) &= \frac{2a}{\sqrt{3}}\frac{t^2 \sin t}{(t - \cos t \sin t)\sqrt{3a^2\cos^2 t + 2a\sin^2 t}} \\ &\quad \times\frac{\tan t}{t} - \left(1 - a + \sqrt{\frac{2}{3}a\tan^2 t + a^2}\right) \\ &= \frac{a}{\sqrt{3}}\frac{2\sin^3 t - 3a(t - \cos t \sin t)\cos t}{(t - \cos t \sin t)\sqrt{2a\sin^2 t + 3a^2\cos^2 t}} - (1 - a) \\ &\to \begin{cases} 0 & \text{as } t \to 0, \\ a + \dfrac{2\sqrt{6}}{3\pi}\sqrt{a} - 1 & \text{as } t \to \pi/2. \end{cases} \end{aligned}$$

Now, we distinguish three cases to determine the monotonicity of $p/q$.

**Case 1**: $a \geq 5/8$. By the relation (1.22), we have $H'_{p,q} = \left(p'/q'\right)' q > 0$, and so $H_{p,q}(t) > H_{p,q}(0^+) = 0$ for $t \in (0, \pi/2)$. The relation (1.21) in combination with $q'(t) > 0$ and $H_{p,q}(t) > 0$ leads to $\left(p(t)/q(t)\right)' > 0$ for $t \in (0, \pi/2)$. Hence, for $t \in (0, \pi/2)$,

$$1 = \lim_{t\to 0}\frac{p(t)}{q(t)} < \frac{p(t)}{q(t)} < \lim_{t\to\pi/2}\frac{p(t)}{q(t)} = \sqrt{\frac{a\pi^2}{6}},$$

which is equivalent to (3.1). This proves the first assertion of this theorem.

**Case 2**: $0 < a \le 16/(3\pi^2)$. Likewise, we deduce that $(p(t)/q(t))' < 0$ for $t \in (0, \pi/2)$. So the double inequality (3.4) holds for $x > 0$.

**Case 3**: $16/(3\pi^2) < a < 5/8$. As shown previously, by the relation (1.22) it is seen that $H'_{p,q} < 0$ for $t \in (0, t_1)$ and $H'_{p,q} > 0$ for $t \in (t_1, \pi/2)$. Since $H_{p,q}(0^+) = 0$ and

$$H_{p,q}\left(\frac{\pi^-}{2}\right) = a + \frac{2\sqrt{6}}{3\pi}\sqrt{a} - 1 \begin{cases} > 0 \text{ if } a_1 < a < \dfrac{5}{8}, \\ \le 0 \text{ if } \dfrac{16}{3\pi^2} < a \le a_1, \end{cases}$$

where $a_1 = \left(\sqrt{3\pi^2+2} - \sqrt{2}\right)^2 / (3\pi^2) = 0.598\ldots$, we find that

**Subcase 3.1**: $16/(3\pi^2) < a \le a_1$. We have $H_{p,q}(t) < 0$ for $t \in (0, \pi/2)$, so is $(p(t)/q(t))'$, which implies the double inequality (3.4) holds for $x > 0$.

Combining Case 2 and Subcase 3.1 gives the third assertion of this theorem.

**Subcase 3.2**: $a_1 < a < 5/8$. There is a $t_0 \in (t_1, \pi/2)$ such that $H_{p,q}(t) < 0$ for $t \in (0, t_0)$ and $H_{p,q}(t) > 0$ for $t \in (t_0, \pi/2)$, and so is $(p(t)/q(t))'$. That is, the ratio $p/q$ is decreasing on $(0, t_0)$ and increasing on $(t_0, \pi/2)$. This leads to

$$\frac{p(t_0)}{q(t_0)} < \frac{p(t)}{q(t)} < \lim_{t\to 0}\frac{p(t)}{q(t)} = 1 \text{ for } t \in (0, t_0),$$
$$\frac{p(t_0)}{q(t_0)} < \frac{p(t)}{q(t)} < \lim_{t\to \pi/2}\frac{p(t)}{q(t)} = \sqrt{\frac{a\pi^2}{6}} \text{ for } t \in \left(t_0, \frac{\pi}{2}\right),$$

that is,

$$\frac{p(t_0)}{q(t_0)} \le \frac{p(t)}{q(t)} < \max\left(1, \sqrt{\frac{a\pi^2}{6}}\right) \text{ for } x \in (0, \infty),$$

which is equivalent to the double inequality (4.2), where $x_0 = \tan t_0$. In particular, for $a = 6/\pi^2$, solving the equation $H_{p,q}(t) = 0$ gives $t_0 = 1.2900104\ldots$, then $x_0 = \tan t_0 = 3.467\,341\ldots$, so $c_{21} = R_1(x_0) = 0.9976914\ldots$, which proves the second assertion of this theorem.

Thus we completes the proof. □

Taking $a = 5/8$, $a_1 = \left(\sqrt{3\pi^2+2} - \sqrt{2}\right)^2 / (3\pi^2)$ in Theorem 1, we obtain two new sharp double inequalities.

**Corollary 1.** *The following inequalities*

$$\frac{8x}{3 + \sqrt{80x^2/3 + 25}} < \arctan x < \frac{\left(2\pi\sqrt{5/3}\right)x}{3 + \sqrt{80x^2/3 + 25}}, \tag{3.7}$$

$$\frac{\pi^2 x}{4 + \sqrt{4\pi^2x^2 + \left(\sqrt{2(3\pi^2+2)} - 2\right)^2}} < \arctan x < \frac{\left(\sqrt{2(3\pi^2+2)} + 2\right)x}{4 + \sqrt{4\pi^2x^2 + \left(\sqrt{2(3\pi^2+2)} - 2\right)^2}} \tag{3.8}$$

*hold for all $x > 0$. All the bounds are sharp.*

**Remark 2.** *It is easy to verify that*

$$\frac{\partial A_{(2,0)}(x;a)}{\partial a} = -9x\frac{x^2 + 3a - \sqrt{6ax^2+9a^2}}{\left(3 - 3a + \sqrt{6ax^2+9a^2}\right)^2\sqrt{6ax^2+9a^2}} < 0,$$

$$\frac{\partial\left[\sqrt{a}A_{(2,0)}(x;a)\right]}{\partial a} = \frac{9x}{2\sqrt{a}}\frac{(1+a)\sqrt{6ax^2+9a^2} - 3a^2}{\left(3 - 3a + \sqrt{6ax^2+9a^2}\right)^2\sqrt{6ax^2+9a^2}} > 0,$$

*that is,* $a \mapsto A_{(2,0)}(x;a)$ *and* $a \mapsto \sqrt{a}A_{(2,0)}(x;a)$ *are decreasing and increasing on* $(0,\infty)$*, respectively. Then taking* $a = 2/3, \infty$ *in inequalities (3.1) gives*

$$\frac{3x}{x^2+3} < \frac{3x}{1+2\sqrt{x^2+1}} < \arctan x < \frac{\pi x}{1+2\sqrt{x^2+1}} \tag{3.9}$$

*for* $x > 0$*; taking* $a = 1/2, 7/12$ *in (3.4) yields*

$$\frac{\pi x}{\sqrt{3}+\sqrt{4x^2+3}} < \frac{\sqrt{14}\pi x}{5+\sqrt{56x^2+49}} < \arctan x < \frac{12x}{5+\sqrt{56x^2+49}} < \frac{2\sqrt{3}x}{\sqrt{3}+\sqrt{4x^2+3}}$$

*for* $x > 0$.

As a direct consequence of Theorem 1 we immediately obtain the following

**Proposition 1.** *The double inequality*

$$\frac{\beta_1 x}{1-a+\sqrt{2ax^2/3+a^2}} < \arctan x < \frac{\alpha_1 x}{1-a+\sqrt{2ax^2/3+a^2}} \tag{3.10}$$

*holds for* $x > 0$ *with the best constants*

$$\beta_1 = \begin{cases} 1 & \text{if } a \geq \frac{5}{8}, \\ R_1(x_0) & \text{if } a_1 < a < \frac{5}{8}, \\ \sqrt{\frac{a\pi^2}{6}} & \text{if } 0 < a \leq a_1, \end{cases} \quad \text{and} \quad \alpha_1 = \begin{cases} \sqrt{\frac{a\pi^2}{6}} & \text{if } a \geq \frac{6}{\pi^2}, \\ 1 & \text{if } 0 < a \leq \frac{6}{\pi^2}, \end{cases}$$

*where* $a_1 = \left(\sqrt{3\pi^2+2}-\sqrt{2}\right)^2/\left(3\pi^2\right)$ *and* $R_1(x_0)$ *is given in Theorem 1.*

Using Proposition 1 with the decreasing property of $a \mapsto A_{(2,0)}(x;a)$, we deduce the following corollary.

**Corollary 2.** *Let* $a, b > 0$*. The double inequality*

$$\frac{x}{1-a+\sqrt{2ax^2/3+a^2}} < \arctan x < \frac{x}{1-b+\sqrt{2bx^2/3+b^2}} \tag{3.11}$$

*holds for* $x > 0$ *if and only if* $a \geq 5/8 = 0.625$ *and* $0 < b \leq 6/\pi^2 = 0.607\ldots$.

**Remark 3.** *Clearly, the lower and upper bounds given in (3.11) for* $a = 5/8$ *and* $b = 6/\pi^2$*, that are,* $A_{(3,0)}(x)$ *and* $A_{(2,1)}(x)$*, are the sharpest lower and upper bounds of* arctan $x$ *of the form* $B_{a,b,c}(x)$.

**Remark 4.** *The first inequality of (3.11) for* $a = 5/8$ *was first presented in [7], while the second one of (3.11) for* $b = 6/\pi^2$ *appeared in [9]. It is easy to check that*

$$\frac{8x}{3+\sqrt{256x^2/\pi^2+25}} - \frac{\pi^2 x}{\pi^2-6+2\sqrt{\pi^2x^2+9}} > 0$$

*for all* $x > 0$*, the upper bound in (3.11) for* $b = 6/\pi^2$ *is better than one in (1.4).*

Taking into account Proposition 1 and the proof of Theorem 1 we obtain a new double inequality for arctan $x$.

**Corollary 3.** *Let* $a, b > 0$*. The double inequality*

$$\sqrt{\frac{a}{6}}\frac{\pi x}{1-a+\sqrt{2ax^2/3+a^2}} < \arctan x < \sqrt{\frac{b}{6}}\frac{\pi x}{1-b+\sqrt{2bx^2/3+b^2}} \tag{3.12}$$

*holds for* $x > 0$ *if and only if* $0 < a \leq a_1 = \left(\sqrt{3\pi^2+2}-\sqrt{2}\right)^2/\left(3\pi^2\right)$ *and* $b \geq 6/\pi^2$.

*Proof.* We only prove that the left hand side inequality of (3.12) holds for $x > 0$ if and only if $0 < a \leq a_1$. The sufficiency follows from Proposition 1 and the increasing property of $a \mapsto \sqrt{a}A_{(2,0)}(x; a)$ on $(0, \infty)$. Suppose that the left hand side inequality of (3.12) holds for all $x > 0$. If $a > a_1$, then $a \geq 5/8$ or $a_1 < a < 5/8$. If $a \geq 5/8$, then by Theorem 1, the second inequality of (3.1) holds for all $x > 0$, which yields a contradiction with the assumption. If $a_1 < a < 5/8$, then from Subcase 3.2 we see that

$$\frac{p(t)}{q(t)} < \lim_{t \to \pi/2} \frac{p(t)}{q(t)} = \sqrt{\frac{a\pi^2}{6}} \text{ for } t \in \left(t_0, \frac{\pi}{2}\right),$$

which is equivalent to

$$\arctan x < \sqrt{\frac{a}{6}} \frac{\pi x}{1 - a + \sqrt{2ax^2/3 + a^2}} \text{ for } x \in (x_0, \infty),$$

where $x_0 = \tan t_0$, which also yields a contradiction with the assumption, and the necessity follows. This completes the proof. □

**Remark 5.** *Corollary 3 offers a new family of lower bounds in the form of* $\lambda B_{a,b,c}(x)$ *(*$\lambda > 0$ *with* $\lambda \neq 1$*) for* arctan $x$. *And, a sharp lower bound is*

$$\sqrt{\frac{a_1}{6}} \frac{\pi x}{1 - a_1 + \sqrt{2a_1x^2/3 + a_1^2}} = \frac{\pi^2 x}{4 + \sqrt{4\pi^2x^2 + \left(\sqrt{2(3\pi^2 + 2)} - 2\right)^2}} := \xi(x). \tag{3.13}$$

## 4 The monotonicity of $R_2$ and inequalities

Our second main result is the following theorem, which exposes the monotonicity pattern of $R_2(x)$ on $(0, \infty)$.

**Theorem 2.** *(i) If* $a \geq a_0 = 2\left(\sqrt{3\pi^2 + 2} - \sqrt{2}\right)^2/\pi^4 = 0.363\ldots$, *then the function*

$$R_2(x) = \frac{1}{x}\left(\frac{4}{\pi^2} + \sqrt{\frac{4}{\pi^2}x^2 + a}\right) \arctan x$$

*is strictly decreasing from* $(0, \infty)$ *onto* $\left(1, 4/\pi^2 + \sqrt{a}\right)$. *Therefore, the double inequality*

$$\frac{x}{4/\pi^2 + \sqrt{4x^2/\pi^2 + a}} < \arctan x < \frac{\left(4/\pi^2 + \sqrt{a}\right)x}{4/\pi^2 + \sqrt{4x^2/\pi^2 + a}} \tag{4.1}$$

*holds for* $x > 0$.

*(ii) If* $32/\pi^4 < a < a_0 = 2\left(\sqrt{3\pi^2 + 2} - \sqrt{2}\right)^2/\pi^4$, *then there is an* $x_0 > 0$ *such that* $R_2$ *is strictly increasing on* $(0, x_0)$ *and decreasing on* $(x_0, \infty)$. *So the inequalities*

$$\frac{\min\left(1, 4/\pi^2 + \sqrt{a}\right) \times x}{4/\pi^2 + \sqrt{4x^2/\pi^2 + a}} < \arctan x \leq \frac{R_2(x_0) \times x}{4/\pi^2 + \sqrt{4x^2/\pi^2 + a}} \tag{4.2}$$

*hold for* $x > 0$, *where* $x_0$ *is the unique solution of the equation* $R_2'(x) = 0$ *on* $(0, \infty)$. *In particular, for* $a = \left(1 - 4/\pi^2\right)^2$, *we have*

$$\frac{x}{4/\pi^2 + \sqrt{4x^2/\pi^2 + \left(1 - 4/\pi^2\right)^2}} < \arctan x \leq \frac{c_{12}x}{4/\pi^2 + \sqrt{4x^2/\pi^2 + \left(1 - 4/\pi^2\right)^2}}, \tag{4.3}$$

*where* $c_{12} = R_2(x_0) = 1.0026766\ldots$ *is the best possible.*

*(iii) If* $0 < a \leq 32/\pi^4 = 0.328\ldots$, *then* $R_2(x)$ *is strictly increasing from* $(0, \infty)$ *onto* $\left(4/\pi^2 + \sqrt{a}, 1\right)$. *Therefore, the double inequality*

$$\frac{\left(4/\pi^2 + \sqrt{a}\right)x}{4/\pi^2 + \sqrt{4x^2/\pi^2 + a}} < \arctan x < \frac{x}{4/\pi^2 + \sqrt{4x^2/\pi^2 + a}} \tag{4.4}$$

*holds for* $x > 0$.

*Proof.* Making a change of variable $x = \tan t$ for $t \in (0, \pi/2)$ yields

$$R_2(x) = \frac{1}{x}\left(\frac{4}{\pi^2} + \sqrt{\frac{4}{\pi^2}x^2 + a}\right)\arctan x = \frac{4/\pi^2 + \sqrt{4(\tan t)^2/\pi^2 + a}}{(\tan t)/t} := \frac{p(t)}{q(t)},$$

where

$$p(t) = \frac{4}{\pi^2} + \sqrt{\frac{4}{\pi^2}(\tan t)^2 + a}, \qquad q(t) = \frac{\tan t}{t}.$$

Differentiation yields

$$\frac{p'(t)}{q'(t)} = \frac{4}{\pi}\frac{t^2 \sin t}{(t - \sin t\cos t)\sqrt{\pi^2 a\cos^2 t + 4\sin^2 t}},$$

$$\left(\frac{p'(t)}{q'(t)}\right)' = \frac{4}{\pi}\frac{(t\cos t)\times\left[a\pi^2\phi_2(t) - 8\phi_1(t)\right]}{\left(\sqrt{4\sin^2 t + \pi^2 a\cos^2 t}\right)^3(t - \cos t\sin t)^2}$$
$$= \frac{4\pi(t\cos t)\times\phi_2(t)}{\left(\sqrt{4\sin^2 t + \pi^2 a\cos^2 t}\right)^3(t - \cos t\sin t)^2}\left[a - \frac{8}{\pi^2}\frac{\phi_1(t)}{\phi_2(t)}\right],$$

where $\phi_1(t)$ and $\phi_2(t)$ are defined by (2.5) and (2.6), respectively. As shown in Lemma 4, $\phi_2(t) > 0$ for $t \in (0, \pi/2)$, $\phi_1(t)/\phi_2(t)$ is strictly decreasing from $(0, \pi/2)$ onto $(4/\pi^2, 15/32)$. Therefore, $(p'(t)/q'(t))' >$ ($<$) $0$ if and only if

$$a \geq \sup_{t\in(0,\pi/2)}\left[\frac{8}{\pi^2}\frac{\phi_1(t)}{\phi_2(t)}\right] = \frac{15}{4\pi^2} \text{ or } a \leq \inf_{t\in(0,\pi/2)}\left[\frac{8}{\pi^2}\frac{\phi_1(t)}{\phi_2(t)}\right] = \frac{32}{\pi^4},$$

while $a \in (32/\pi^4, 15/(4\pi^2))$, there is a $t_1 \in (0, \pi/2)$ such that $(p'(t)/q'(t))' < 0$ for $t \in (0, t_1)$ and $(p'(t)/q'(t))' > 0$ for $t \in (t_1, \pi/2)$.

On the other hand, we easily get

$$H_{p,q}(t) = \frac{p'(t)}{q'(t)}q(t) - p(t) = \frac{4}{\pi}\frac{t^2\sin t}{(t - \sin t\cos t)\sqrt{\pi^2 a\cos^2 t + 4\sin^2 t}}$$
$$\times\frac{\tan t}{t} - \left(\frac{4}{\pi^2} + \frac{1}{\cos t}\sqrt{\frac{4}{\pi^2}\sin^2 t + a\cos^2 t}\right)$$
$$= \frac{4\sin^3 t - \pi^2 a(t - \cos t\sin t)\cos t}{\pi\sqrt{\pi^2 a\cos^2 t + 4\sin^2 t}(t - \cos t\sin t)} - \frac{4}{\pi^2}$$
$$\to \begin{cases} -\dfrac{\pi^2 a + 4\sqrt{a} - 6}{\pi^2\sqrt{a}} & \text{as } t \to 0, \\ 0 & \text{as } t \to \pi/2, \end{cases}$$

Now, we distinguish three cases to determine the monotonicity of $p/q$.

**Case 1:** $a \geq 15/(4\pi^2)$. By the relation (1.22), we have $H'_{p,q} = (p'/q')' q > 0$, and so $H_{p,q}(t) < H_{p,q}(\pi/2) = 0$ for $t \in (0, \pi/2)$. The relation (1.21) in combination with $q'(t) > 0$ and $H_{p,q}(t) < 0$ leads to $(p(t)/q(t))' < 0$ for $t \in (0, \pi/2)$. Therefore, for $t \in (0, \pi/2)$,

$$1 = \lim_{t\to\pi/2}\frac{p(t)}{q(t)} < \frac{p(t)}{q(t)} < \lim_{t\to 0}\frac{p(t)}{q(t)} = \frac{4}{\pi^2} + \sqrt{a},$$

which is equivalent to (4.1).

**Case 2:** $0 < a \leq 32/\pi^4$. Similarly, we deduce that $(p(t)/q(t))' > 0$ for $t \in (0, \pi/2)$. So the inequalities (4.4) hold for $x > 0$, which proves the third assertion of this theorem.

**Case 3:** $32/\pi^4 < a < 15/(4\pi^2)$. As shown previously, by the relation (1.22) it is seen that $H'_{p,q} < 0$ for $t \in (0, t_1)$ and $H'_{p,q} > 0$ for $t \in (t_1, \pi/2)$. Since $H_{p,q}(\pi/2) = 0$ and

$$H_{p,q}(0) = -\frac{\pi^2 a + 4\sqrt{a} - 6}{\pi^2\sqrt{a}}\begin{cases} > 0 \text{ if } \dfrac{32}{\pi^4} < a < a_0, \\ \leq 0 \text{ if } a_0 \leq a < \dfrac{15}{4\pi^2}, \end{cases}$$

where $a_0 = 2\left(\sqrt{3\pi^2+2}-\sqrt{2}\right)^2/\pi^4 = 0.363...$, we find that

**Subcase 3.1**: $a_0 \le a < 15/\left(4\pi^2\right)$. We have $H_{p,q}(t) < 0$ for $t \in (0, \pi/2)$, so is $\left(p(t)/q(t)\right)$. So the double inequality (4.1) also holds for $x > 0$. This in combination with Case 1 proves the first assertion of theorem.

**Subcase 3.2**: $32/\pi^4 < a < a_0$. There is a $t_0 \in (0, t_1)$ such that $H_{p,q}(t) > 0$ for $t \in (0, t_0)$ and $H_{p,q}(t) < 0$ for $t \in (t_0, \pi/2)$, and so is $\left(p(t)/q(t)\right)'$. This yields

$$\frac{4}{\pi^2}+\sqrt{a} = \lim_{t\to 0}\frac{p(t)}{q(t)} < \frac{p(t)}{q(t)} < \frac{p(t_0)}{q(t_0)} \text{ for } t \in (0, t_0),$$
$$1 = \lim_{t\to \pi/2}\frac{p(t)}{q(t)} < \frac{p(t)}{q(t)} < \frac{p(t_0)}{q(t_0)} \text{ for } t \in \left(t_0, \frac{\pi}{2}\right),$$

that is,

$$\min\left(1, \frac{4}{\pi^2}+\sqrt{a}\right) < \frac{p(t)}{q(t)} \le \frac{p(t_0)}{q(t_0)} \text{ for } x \in (0, \infty),$$

which is equivalent to the double inequality (4.2), where $x_0 = \tan t_0$. In particular, for $a = \left(1-4/\pi^2\right)^2$, solving the equation $H_{p,q}(t) = 0$ gives $t_0 = 0.9081516...$, then $x_0 = \tan t_0 = 1.2814739...$, so $c_{12} = R_2(x_0) = 1.0026766......$, which proves the second assertion of this theorem.

The proof is finished. □

Taking $a = a_0 = 2\left(\sqrt{3\pi^2+2}-\sqrt{2}\right)^2/\pi^4, 32/\pi^4$ in Theorem 2, we obtain two new sharp double inequalities.

**Corollary 4.** *The following inequalities*

$$\frac{\pi^2 x}{4+\sqrt{4\pi^2x^2+\left(\sqrt{2\left(3\pi^2+2\right)}-2\right)^2}} < \arctan x < \frac{\left(\sqrt{2\left(3\pi^2+2\right)}+2\right)x}{4+\sqrt{4\pi^2x^2+\left(\sqrt{2\left(3\pi^2+2\right)}-2\right)^2}}, \tag{4.5}$$

$$\frac{2\left(\sqrt{2}+1\right)x}{2+\sqrt{\pi^2x^2+8}} < \arctan x < \frac{\left(\pi^2/2\right)x}{2+\sqrt{\pi^2x^2+8}} \tag{4.6}$$

*for $x > 0$. All the bounds are sharp.*

**Remark 6.** *It is interesting that the double inequality (4.5) is the same as (3.8).*

**Remark 7.** *Obviously, $a \mapsto A_{(0,2)}\left(x; \sqrt{a}\right)$ is decreasing on $(0, \infty)$. And, $a \mapsto \left(4/\pi^2+\sqrt{a}\right)A_{(0,2)}\left(x;\sqrt{a}\right)$ is increasing on $(0, \infty)$ due to*

$$\frac{\partial\left[\left(4/\pi^2+\sqrt{a}\right)A_{(0,2)}\left(x;\sqrt{a}\right)\right]}{\partial a} = \frac{2x}{\sqrt{a}}\frac{\pi x^2+\sqrt{4x^2+a\pi^2}-\sqrt{a}\pi}{\sqrt{4x^2+a\pi^2}\left(\sqrt{4x^2+a\pi^2}+4/\pi\right)^2} > 0$$

*for $x, a > 0$. Then taking $a = 36/\pi^4, 4/\pi^2, 4/9$ in (4.1) we obtain that for $x > 0$,*

$$\frac{\left(3\pi^2/2\right)x}{6+\pi\sqrt{9x^2+\pi^2}} < \frac{\pi^2x/2}{2+\pi\sqrt{x^2+1}} < \frac{\pi^2x/2}{2+\sqrt{\pi^2x^2+9}} < \arctan x$$
$$< \frac{5x}{2+\sqrt{\pi^2x^2+9}} < \frac{(\pi+2)x}{2+\pi\sqrt{x^2+1}} < \frac{\left(\pi^2+6\right)x}{6+\pi\sqrt{9x^2+\pi^2}}.$$

*Taking $a = 32/\pi^4, 3/\pi^2, 0$ in (4.4) we have that for $x > 0$,*

$$\frac{2x}{\pi x+2} < \frac{\left(\sqrt{3}\pi+4\right)x}{4+\pi\sqrt{4x^2+3}} < \frac{2\left(\sqrt{2}+1\right)x}{2+\sqrt{\pi^2x^2+8}} < \arctan x$$
$$< \frac{\left(\pi^2/2\right)x}{2+\sqrt{\pi^2x^2+8}} < \frac{\pi^2 x}{4+\pi\sqrt{4x^2+3}} < \frac{\pi^2 x}{2\pi x+4}. \tag{4.7}$$

As a direct consequence of Theorem 2, we have the following proposition.

**Proposition 2.** *The double inequality*

$$\frac{\beta_2 x}{4/\pi^2+\sqrt{4x^2/\pi^2+a}} < \arctan x < \frac{\alpha_2 x}{4/\pi^2+\sqrt{4x^2/\pi^2+a}} \tag{4.8}$$

*holds for $x>0$ with the best constants*

$$\beta_2=\begin{cases}1 & \text{if } a \ge \left(1-\frac{4}{\pi^2}\right)^2,\\ \frac{4}{\pi^2}+\sqrt{a} & \text{if } 0<a\le\left(1-\frac{4}{\pi^2}\right)^2,\end{cases} \quad \text{and } \alpha_2=\begin{cases}\frac{4}{\pi^2}+\sqrt{a} & \text{if } a\ge a_0,\\ R_2(x_0) & \text{if } \frac{32}{\pi^4}<a<a_0,\\ 1 & \text{if } 0<a\le\frac{32}{\pi^4},\end{cases}$$

*where $a_0=2\left(\sqrt{3\pi^2+2}-\sqrt{2}\right)^2/\pi^4$ and $R_2(x_0)$ is given in Theorem 2.*

Proposition 2 with the decreasing property of $a\mapsto A_{(0,2)}\left(x;\sqrt{a}\right)$ implies the following corollary.

**Corollary 5.** *The double inequality*

$$\frac{x}{4/\pi^2+\sqrt{4x^2/\pi^2+a}} < \arctan x < \frac{x}{4/\pi^2+\sqrt{4x^2/\pi^2+b}} \tag{4.9}$$

*holds for $x>0$ if and only if $a\ge\left(1-4/\pi^2\right)^2=0.353\ldots$ and $0<b\le 32/\pi^4=0.328\ldots$.*

**Remark 8.** *Obviously, the lower and upper bounds given in (4.9) for $a=\left(1-4/\pi^2\right)^2$ and $b=32/\pi^4$, that are, $A_{(1,2)}(x)$ and $A_{(0,3)}(x)$, are also the sharpest lower and upper bounds of* arctan $x$ *of the form $B_{a,b,c}(x)$.*

**Remark 9.** *Inequalities (4.9) for $a=\left(1-4/\pi^2\right)^2$ and $b=32/\pi^4$ were first proven in [10]. Clearly, we here give a new proof.*

By Corollaries 2 and 5 with the increasing property of $a\mapsto A_{(1,1)}(x;a)$ we easily deduce the relation between arctan $x$ and $A_{(1,1)}(x;a)$.

**Corollary 6.** *Let $a,b>0$. The double inequality*

$$\frac{x}{1-a+\sqrt{4x^2/\pi^2+a^2}} < \arctan x < \frac{x}{1-b+\sqrt{4x^2/\pi^2+b^2}} \tag{4.10}$$

*holds if and only if $0<a\le 1-4/\pi^2$ and $b\ge 6/\pi^2$.*

*Proof.* The necessity follows from

$$\lim_{x\to\infty} x\left(\arctan x-\frac{x}{1-a+\sqrt{4x^2/\pi^2+a^2}}\right)=-\frac{\pi^2}{4}\left[a-\left(1-\frac{4}{\pi^2}\right)\right]\ge 0,$$

$$\lim_{x\to 0+}\frac{1}{x^3}\left(\arctan x-\frac{x}{1-b+\sqrt{4x^2/\pi^2+b^2}}\right)=-\frac{1}{3b}\left(b-\frac{6}{\pi^2}\right)\le 0.$$

The sufficiency follows from the first inequality of (4.9) for $a=\left(1-4/\pi^2\right)^2$ and the second inequality of (3.11) for $b=6/\pi^2$ with the facts that

$$\frac{\partial A_{(1,1)}(x;a)}{\partial a}=x\frac{\sqrt{4x^2/\pi^2+a^2}-a}{\sqrt{4x^2/\pi^2+a^2}\left(1-a+\sqrt{4x^2/\pi^2+a^2}\right)^2}>0$$

for $x,a>0$. □

**Remark 10.** *Corollary 6 shows that $A_{(1,2)}(x)$ and $A_{(2,1)}(x)$ are the sharpest lower and upper bounds of* arctan $x$ *of the form $B_{a,b,c}(x)$ in the sense that $A_{(1,1)}(x;a) < \arctan x < A_{(1,1)}(x;b)$.*

Proposition 2 also contains another new double inequality.

**Corollary 7.** *Let $a, b > 0$. The double inequality*

$$\frac{\left(4/\pi^2+\sqrt{a}\right)x}{4/\pi^2+\sqrt{4x^2/\pi^2+a}} < \arctan x < \frac{\left(4/\pi^2+\sqrt{b}\right)x}{4/\pi^2+\sqrt{4x^2/\pi^2+b}} \tag{4.11}$$

*holds for $x > 0$ if and only if $0 < a \le \left(1-4/\pi^2\right)^2 = 0.353\ldots$ and $b \ge a_0 = 2\left(\sqrt{3\pi^2+2}-\sqrt{2}\right)^2/\pi^4 = 0.363\ldots$.*

**Remark 11.** *Corollary 7 provides a new family of upper bounds in the form of $\lambda B_{a,b,c}(x)$ ($\lambda > 0$ with $\lambda \ne 1$) for* arctan $x$. *And, a sharp upper bound is*

$$\frac{\left(4/\pi^2+\sqrt{a_0}\right)x}{4/\pi^2+\sqrt{4x^2/\pi^2+a_0}} = \frac{\left(\sqrt{2\left(3\pi^2+2\right)}+2\right)x}{4+\sqrt{4\pi^2x^2+\left(\sqrt{2\left(3\pi^2+2\right)}-2\right)^2}} := \eta(x). \tag{4.12}$$

## 5 Concluding remarks

For $i, j \in \{0,1,2,3\}$ such that $i + j = 3$, let $A_{(i,j)}(x)$ be given by (1.15), (1.16), (1.19) and (1.20), respectively.

**Remark 12.** *From Corollaries 2, 5 and 6 we see that $A_{(3,0)}(x)$ and $A_{(1,2)}(x)$ are the sharpest lower bounds of* arctan $x$ *of the form $B_{a,b,c}(x)$, while $A_{(2,1)}(x)$ and $A_{(0,3)}(x)$ are the sharpest upper bounds of* arctan $x$ *of the form $B_{a,b,c}(x)$. Then we have*

$$\max\left(A_{(3,0)}(x), A_{(1,2)}(x)\right) < \arctan x < \min\left(A_{(2,1)}(x), A_{(0,3)}(x)\right). \tag{5.1}$$

*Numeric computations show that the two sharpest lower bounds $A_{(3,0)}(x)$ and $A_{(1,2)}(x)$ are not comparable for all $x > 0$, so are the two sharpest upper bounds $A_{(2,1)}(x)$ and $A_{(0,3)}(x)$.*

**Remark 13.** *From the inequalities (3.3), (3.7), (4.3) and (4.6) we obtain that for all $x > 0$,*

$$-0.0023139\ldots = 1-\frac{1}{c_{21}} \le 1-\frac{A_{(2,1)}(x)}{\arctan x} < 0,$$

$$0 < 1-\frac{A_{(3,0)}(x)}{\arctan x} < 1-\frac{12}{\sqrt{15}\pi} = 0.013\,752\ldots,$$

$$0 < 1-\frac{A_{(1,2)}(x)}{\arctan x} \le 1-\frac{1}{c_{12}} = 0.0026694\ldots,$$

$$-0.022030\ldots = 1-\frac{\pi^2}{4\left(\sqrt{2}+1\right)} \le 1-\frac{A_{(0,3)}(x)}{\arctan x} < 0.$$

*These indicate that the maximum relative errors of the four sharpest bounds of* arctan $x$ *equal* $0.0023139\ldots$, $0.013\,752\ldots$, $0.0026694\ldots$ *and* $0.022030\ldots$.

**Remark 14.** *It has been shown in [9] that $D_{(1,2)}(x) = \arctan x - A_{(1,2)}(x)$ is strictly increasing on $(0, x_{12})$ and decreasing on $(x_{12}, \infty)$, while $D_{(2,1)}(x) = \arctan x - A_{(2,1)}(x)$ is strictly decreasing on $(0, x_{21})$ and increasing on $(x_{21}, \infty)$, where*

$$x_{12} = \frac{\left(\pi^2-4\right)\sqrt{2\left(10-\pi^2\right)\left(\pi^2-8\right)}}{\pi^4-8\pi^2-16} = 1.67133\ldots,$$

$$x_{21} = \frac{\sqrt{\left(5\pi^2 - 48\right)\left(12 - \pi^2\right)}}{\pi\left(10 - \pi^2\right)} = 4.36812\ldots.$$

*These actually reveal the maximum absolute errors estimating for* $\arctan x$ *by* $A_{(1,2)}(x)$ *and* $A_{(2,1)}(x)$*. More precisely, we have*

$$0 < \arctan x - A_{(1,2)}(x) \le D_{(1,2)}(x_{12}) = 0.0025995\ldots,$$

$$-0.0030381\ldots = D_{(2,1)}(x_{21}) \le \arctan x - A_{(2,1)}(x) < 0$$

*for all* $x > 0$*, which means that the maximum absolute errors of bounds* $A_{(1,2)}(x)$ *and* $A_{(2,1)}(x)$ *equal* $0.0025995\ldots$ *and* $0.0030381\ldots$*, respectively.*

To show the maximum absolute errors of bounds $A_{(3,0)}(x)$ and $A_{(0,3)}(x)$, we need to prove Theorem 3.

**Theorem 3.** *(i) The difference* $D_{(3,0)}(x) = \arctan x - A_{(3,0)}(x)$ *is strictly increasing on* $(0, \infty)$*, and therefore, the double inequality*

$$0 < \arctan x - A_{(3,0)}(x) < \frac{\pi}{2} - \frac{2}{5}\sqrt{15} = 0.021602\ldots$$

*holds for all* $x > 0$*. (ii) The difference* $D_{(0,3)}(x) = \arctan x - A_{(0,3)}(x)$ *is strictly decreasing on* $(0, x_{03})$ *and increasing on* $(x_{03}, \infty)$*, where*

$$x_{03} = \frac{\sqrt{8\left(\pi^4 - 8\pi^2 - 16\right)}}{\pi\left(12 - \pi^2\right)} = 0.66178\ldots.$$

*Consequently, the double inequality*

$$-0.0080482\ldots = D_{(0,3)}(x_{03}) \le \arctan x - A_{(0,3)}(x) < 0$$

*holds for all* $x > 0$.

*Proof.* (i) Differentiation yields

$$D'_{(3,0)}(x) = \frac{1}{x^2+1} - \frac{8}{\sqrt{80x^2/3+25}+3} + \frac{640x^2/3}{\sqrt{80x^2/3+25}\left(\sqrt{80x^2/3+25}+3\right)^2}$$
$$= \frac{\left(8x^2/3+10\right)\sqrt{80x^2/3+25} - 10\left(4x^2+5\right)}{\left(x^2+1\right)\sqrt{80x^2/3+25}\left(\sqrt{80x^2/3+25}+3\right)^2} > 0,$$

where the inequality holds due to

$$\left[\left(\frac{8}{3}x^2+10\right)\sqrt{\frac{80}{3}x^2+25}\right]^2 - \left[10\left(4x^2+5\right)\right]^2 = \frac{5120}{27}x^6 > 0.$$

This leads to

$$0 = D_{(3,0)}(0) < D_{(3,0)}(x) < D_{(3,0)}(\infty) = \frac{\pi}{2} - \frac{2}{5}\sqrt{15} = 0.021602\ldots.$$

(ii) Analogously, we have

$$D'_{(0,3)}(x) = \frac{1}{x^2+1} - \frac{\pi^2}{2\sqrt{\pi^2x^2+8}+4} + \frac{2\pi^4x^2}{\sqrt{\pi^2x^2+8}\left(2\sqrt{\pi^2x^2+8}+4\right)^2}$$
$$= 4\frac{\left(12-\pi^2\right)\sqrt{\pi^2x^2+8} - 4\left(\pi^2-8\right)}{\left(x^2+1\right)\sqrt{\pi^2x^2+8}\left(2\sqrt{\pi^2x^2+8}+4\right)^2}.$$

Then

$$\operatorname{sgn} D'_{(0,3)}(x) = \operatorname{sgn}\left[\left(12-\pi^2\right)\sqrt{\pi^2x^2+8} - 4\left(\pi^2-8\right)\right]$$

$$= \operatorname{sgn}\left[\left(12-\pi^2\right)^2\left(\pi^2x^2+8\right)-16\left(\pi^2-8\right)^2\right] = \operatorname{sgn}\left(x^2-x_{03}^2\right).$$

It is therefore deduced that $D'_{(0,3)}(x) < 0$ for $x \in (0, x_{03})$ and $D'_{(0,3)}(x) > 0$ for $x \in (x_{03}, \infty)$, so we arrive at

$$-0.0080482\ldots = D_{(0,3)}(x_{03}) \le D_{(0,3)}(x) < \max\left(D_{(0,3)}(0), D_{(0,3)}(\infty)\right) = 0,$$

which completes the proof. □

**Remark 15.** *Taking into account Remarks 13, 14 and Theorem 3, we have the following table:*

| Sharp bounds | Type | Maximum absolute errors | Maximum relative errors |
|---|---|---|---|
| $A_{(3,0)}(x)$ | lower | 0.021602... | 0.01352... |
| $A_{(2,1)}(x)$ | upper | 0.0030381... | 0.0023139... |
| $A_{(1,2)}(x)$ | lower | 0.0025995 .. | 0.0026694... |
| $A_{(0,3)}(x)$ | upper | 0.0080482... | 0.022030... |

*From the table, we see that $A_{(1,2)}(x)$ and $A_{(2,1)}(x)$ are respectively better than $A_{(3,0)}(x)$ and $A_{(0,3)}(x)$ in the sense that both of the maximum absolute errors and maximum relative errors are minimum on* $(0, \infty)$.

**Remark 16.** *As mentioned in Remarks 5 and 11, inequalities (3.12) and (4.11) offer a new family of bounds in the form of $\lambda B_{a,b,c}(x)$ ($\lambda > 0$ with $\lambda \ne 1$) for* arctan $x$. *As sharp bounds, by (3.8) or (4.5), we have $\xi(x) < \arctan x < \eta(x)$ for all $x > 0$. Similar to Theorem 3 we can prove that*

$$0 < \arctan x - \xi(x) < 0.0055530\ldots$$

*for all $x > 0$; also, by (3.8), we have that for all $x > 0$,*

$$0 < 1 - \frac{\xi(x)}{\arctan x} < 1 - \frac{\pi^2}{\sqrt{2\left(3\pi^2+2\right)}+2} = 0.0081747\ldots.$$

*Thus it can be seen that the maximum absolute error and maximum relative error estimating* arctan $x$ *by* $\xi(x)$ *are less than* 0.0056 *and* 0.82%.

**Remark 17.** *The families of bounds $A_{(2,0)}(x;a)$ and $A_{(0,2)}(x;a)$ for* arctan $x$ *also contain several rational bounds listing in (3.9) and (4.7), that are, for $x > 0$,*

$$\frac{3x}{x^2+3} < \arctan x,$$

$$\frac{2x}{\pi x+2} < \arctan x < \frac{\pi^2 x}{2\pi x+4}.$$

**Remark 18.** *Due to the identity*

$$\arctan x = \frac{\pi}{2} - \arctan\frac{1}{x} \tag{5.2}$$

*for $x > 0$, if* $\arctan x < (>) A(x)$ *for all $x > 0$, then there must be*

$$\arctan x > (<) \frac{\pi}{2} - A\left(\frac{1}{x}\right)$$

*for all $x > 0$. Thus the inequalities (5.1) can be extended as*

$$\max\left(A_{(3,0)}(x), A_{(1,2)}(x), \frac{\pi}{2} - A_{(2,1)}\left(\frac{1}{x}\right), \frac{\pi}{2} - A_{(0,3)}\left(\frac{1}{x}\right)\right)$$
$$< \arctan x < \min\left(A_{(2,1)}(x), A_{(0,3)}(x), \frac{\pi}{2} - A_{(3,0)}\left(\frac{1}{x}\right), \frac{\pi}{2} - A_{(1,2)}\left(\frac{1}{x}\right)\right).$$

*And, all four lower (upper) bounds given in the above inequalities are not comparable.*

**Remark 19.** *Similarly, inequalities (3.8) can be extended as*

$$\max\left(\xi(x), \frac{\pi}{2} - \xi\left(\frac{1}{x}\right)\right) < \arctan x < \min\left(\eta(x), \frac{\pi}{2} - \eta\left(\frac{1}{x}\right)\right),$$

*where $\xi(x)$ and $\eta(x)$ are defined by (3.13) and (4.12), respectively.*

**Remark 20.** *Although the bound $A_{(3,0)}(x)$ for $\arctan x$ is inferior to others in the sense that both of the maximum absolute errors and maximum relative errors are minimal on $(0,\infty)$, but it is a priority selection in the applications of engineering because that for $0 < x < 1$,*

$$0 < \arctan x - A_{(3,0)}(x) < \frac{\pi}{4} - \frac{\sqrt{465}-9}{16} = 0.000156\ldots,$$
$$0 < \frac{\arctan x - A_{(3,0)}(x)}{\arctan x} < 1 - \frac{\sqrt{465}-9}{4\pi} = 0.000199\ldots,$$

*and for $x > 1$,*

$$-0.000156\ldots = \frac{\sqrt{465}-9}{16} - \frac{\pi}{4} < \arctan x - \frac{\pi}{2} + A_{(3,0)}\left(\frac{1}{x}\right) < 0,$$
$$-0.000199\ldots = -1 + \frac{\sqrt{465}-9}{4\pi} < \frac{\arctan x - \pi/2 + A_{(3,0)}(1/x)}{\arctan x} < 0,$$

*where the first and third inequalities follow from Theorem 3, the second follows from Theorem 1, and the last follows from Theorem 4.*

**Theorem 4.** *The ratio*

$$R_0(x) = \frac{\pi/2 - 8/\left(3x + \sqrt{80/3 + 25x^2}\right)}{\arctan x}$$

*is strictly decreasing from $(0,\infty)$ onto $(1,\infty)$. In particular, the double inequality*

$$1 < \frac{\pi/2 - 8/\left(3x + \sqrt{80/3 + 25x^2}\right)}{\arctan x} < 2 - \frac{\sqrt{465}-9}{4\pi} = 1.000\,199\ldots$$

*holds for $x > 1$.*

*Proof.* Making a change of variable $x = (4\tan t)/\sqrt{15}$ for $t \in (0, \pi/2)$ yields

$$R_0(x) = \frac{\pi/2 - 8/\left(3x + \sqrt{80/3 + 25x^2}\right)}{\arctan x} = \frac{\pi/2 - 2\sqrt{15}(\cos t)/(3\sin t + 5)}{\arctan\left((4\tan t)/\sqrt{15}\right)} := \frac{p(t)}{q(t)},$$

where

$$p(t) = \frac{\pi}{2} - 2\sqrt{15}\frac{\cos t}{3\sin t + 5}, \qquad q(t) = \arctan\left[(4\tan t)/\sqrt{15}\right].$$

Differentiation yields

$$\frac{p'(t)}{q'(t)} = \frac{1}{2}\frac{(3 + 5\sin t)\left(15 + \sin^2 t\right)}{(5 + 3\sin t)^2},$$
$$\left(\frac{p'(t)}{q'(t)}\right)' = \frac{15}{2}\frac{(7 + \sin t)(1 - \sin t)^2\cos t}{(5 + 3\sin t)^3} > 0.$$

This together with $q > 0$ yields $H'_{p,q} = (p'/q')' q > 0$. Since

$$H_{p,q}(t) = \frac{p'(t)}{q'(t)}q(t) - p(t) = \frac{1}{2}\frac{(3 + 5\sin t)\left(15 + \sin^2 t\right)}{(5 + 3\sin t)^2}$$
$$\times \arctan\left((4\tan t)/\sqrt{15}\right) - \left(\frac{\pi}{2} - 2\sqrt{15}\frac{\cos t}{3\sin t + 5}\right) \to 0$$

as $t \to \pi/2$, we achieve that $H_{p,q}(t) < \lim_{t\to(\pi/2)^-} H_{p,q}(t) = 0$ for $t \in (0, \pi/2)$, which in combination with $q' > 0$ gives $(p/q)' = (q'/q^2) H_{p,q} < 0$ for $t \in (0, \pi/2)$. That is, $R_0$ is strictly decreasing on $(0,\infty)$. Clearly, $R_0(0^+) = \infty$, $R_0(1) = 2 - \left(\sqrt{465} - 9\right)/(4\pi)$, $R_0(\infty) = 1$, which completes the proof. □

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