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Least eigenvalue of the connected graphs whose complements are cacti

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Abstract: Suppose that Γ is a graph of order n and $A(\Gamma) = [a_{i,j}]$ is its adjacency matrix such that $a_{i,j}$ is equal to 1 if v_i is adjacent to v_j and $a_{i,j}$ is zero otherwise, where $1 \leq i, j \leq n$. In a family of graphs, a graph is called minimizing if the least eigenvalue of its adjacency matrix is minimum in the set of the least eigenvalues of all the graphs. Petrović et al. [On the least eigenvalue of cacti, Linear Algebra Appl., 2011, 435, 2357-2364] characterized a minimizing graph in the family of all cacti such that the complement of this minimizing graph is disconnected. In this paper, we characterize the minimizing graphs $G \in \Omega_n^c$, i.e.

$$\lambda_{min}(G) \leq \lambda_{min}(C^c)$$

for each $C^c \in \Omega_n^c$, where Ω_n^c is a collection of connected graphs such that the complement of each graph of order n is a cactus with the condition that either its each block is only an edge or it has at least one block which is an edge and at least one block which is a cycle.

Keywords: adjacency matrix, least eigenvalue, connected graphs, cacti

MSC: 15A18, 05C50, 05C40, 05D05

1 Introduction

Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a graph such that $V(\Gamma) = \{v_i : 1 \leq i \leq n\}$ and $E(\Gamma)$ are set of vertices and edges respectively. Assume that all the considered graphs are simple, finite and undirected. For each i, the degree $d(i)$ is the number of incident edges on v_i. The adjacency matrix of Γ is $A(\Gamma) = [a_{i,j}]$ with $a_{i,j}$ equal to 1 if v_i is linked to v_j and $a_{i,j}$ is zero for the rest case, where $1 \leq i, j \leq n$. The solutions of $\det(A(\Gamma) - \lambda I) = 0$ are eigenvalues of Γ. It is interesting to note that $A(\Gamma)$ is always symmetric and real, all the eigenvalues can be arrange as $\lambda_1(\Gamma) \leq \lambda_2(\Gamma) \leq \ldots \leq \lambda_n(\Gamma)$. The eigenvectors corresponding to the least eigenvalue $\lambda_1(\Gamma)$ and the greatest eigenvalue (spectral radius) $\lambda_n(\Gamma)$ are called first eigenvector (FEv) and Perron-vector respectively.

The spectrum of the adjacency matrix for an undirected graph is first time studied by Collatz and Sinogowitz (1957), see [1]. Later on, many researchers discussed the largest eigenvalue (spectral radius) in the area of spectra of graphs, see [2, 3]. It is observed that the least eigenvalue did not receive the attention of researchers as compare of the largest eigenvalue. From the few of results of the least eigenvalues on the graphs, the bounds related results can be found in [4, 5]. For further study, we refer [6–13]. A graph Γ is said

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to be minimizing in a certain collection of graphs, if the least eigenvalue of $A(\Gamma)$ is minimum in the set of all the least eigenvalues of the other graphs in the same collection.

Let $\mathcal{G}(p,q)$ be a collection of connected graphs in which each graph is of p order and q size such that $0 < q < \frac{p(p-1)}{2}$. The minimizing graph in $\mathcal{G}(p,q)$ characterized by Bell et al. [14] is stated in the below result:

Theorem 1.1. Minimizing graph in $\mathcal{G}(p,q)$ is either a join of two nested split graphs, or a bipartite graph.

It is important to note that the complements of the graphs characterized by Bell contain the cliques such that order of each clique is greater or equal to $\frac{p}{2}$ or these are disconnected. After it, the question is raised to investigate the minimizing graphs in the collection of connected graphs such that the complement of each graph contains the cliques of small sizes. Motivated by it, the minimizing graph in the collection of connected graphs such that the complement of each graph is trees, unicyclic or bicyclic are characterized by Fan, Zhang, Wang, Li and Javaid, see [15–18]. For further study, we refer to [19 – 22]. In this paper, the minimizing graph is characterized in the collection of connected graphs such that the complement of each graph is cactus with the condition that each block of a cactus of order n is only an edge or a cactus of order n has at least one block which is an edge and at least one block which is a cycle. In the rest of paper; Section 2 includes some basic definitions and terminologies, Section 3 contains the proofs of some important lemmas and Section 4 has the main results in which minimizing graph is characterized in the family of connected graphs with the condition that the complement of each graph is cactus.

2 Preliminaries

A connected graph is called cactus if and only if its every block is either a simple cycle or a single edge. A cactus is a tree if and only if its each block is an edge. An edge of a cactus is a cycle edge if it is in some cycle, and tree edge, otherwise. A cactus is said to be a bundle if there is a single common vertex on all of its cycles. Let $B_1(n)$ be the bundle of order $n+1$ obtained from a star $K_{1,n}$ of the same order with central vertex V_0 by adding the edges v_iv_{i+1}, where $i \in \{1,3,5,\dots,n-1\}$ and $n = 0 \mod(2)$. Thus, the central vertex v_0 of the bundle $B_1(n)$ has degree n and each remaining vertex is of degree 2. Similarly, let $B_2(n)$ be a bundle obtained from $K_{1,n}$ by adding the edges v_iv_{i+1}, where $i \in \{1,3,5,\dots,n-2\}$, $n = 1 \mod(2)$ and $|V(B_2(n))| = n+1$. Thus, for the vertices of $B_2(n)$, $d(v_0) = n$, $d(v_n) = 1$ and $d(v_i) = 2$, where $1 \le i \le n-1$.

We define some particular cacti which are obtained from the aforesaid bundles.

Definition 2.1. Assume that $p,q = 0 \mod(2)$ are positive integers. Let $B_1(p)$ and $B_1(q)$ be two bundles. The cactus graph $C_1(p,q)$ is constructed by the join of a vertex of $B_1(p)$ with a vertex $B_1(q)$, where both the vertices are of degree 2. Thus, $V(C_1(p,q)) = \{v_1^j : 1 \le j \le p-2\} \cup \{v_j : 2 \le j \le 7\} \cup \{v_8^j : 1 \le j \le q-2\}$ and $E(C_1(p,q)) = \{v_2v_1^j : 1 \le j \le p-2\} \cup \{v_2v_3, v_2v_4\} \cup \{v_1^jv_1^{j+1} : j = 1,3,5,\dots,p-3\} \cup \{v_jv_{j+1} : 3 \le j \le 5\} \cup \{v_7v_5, v_7v_6\} \cup \{v_7v_8^j : 1 \le j \le q-2\} \cup \{v_8^jv_8^{j+1} : j = 1,3,5,\dots,q-3\}$ with $|C_1(p,q)| = 2+p+q = n$.

Assume that $p,q \equiv 1 \mod(2)$ and $p,q \ge 3$. If a vertex of the bundle $B_2(p)$ is joined with a vertex of the bundle $B_2(q)$ then we obtain the cactus graph $C_1'(p,q)$, where both the chosen vertices are pendent and $n = 2+p+q = |C_1'(p,q)|$. For $p \ge 3$, $q \ge 2$, $p \equiv 1 \mod(2)$ and $q \equiv 0 \mod(2)$, if we join a vertex of the bundle $B_2(p)$ to a vertex of the bundle $B_1(q)$ then we obtain a cactus graph $C_2(p,q)$, where the chosen vertices are of degree 1 and 2 respectively and $n = 2+p+q = |C_2(p,q)|$. Similarly, if we assume $p \ge 2$, $q \ge 3$, $p = 0 \mod(2)$ and $q = 1 \mod(2)$, and choose two vertices of degree 2 and 1 in $B_1(p)$ and $B_2(q)$ respectively. On joining these chosen vertices by an edge, we obtain the cactus graph $C_2'(p,q)$ with $n = 2+p+q = |C_1'(p,q)|$.

We note that $C_1(p,q) \cong C_1(q,p)$ and $C_1'(p,q) \cong C_1'(q,p)$ as p and q both are even in $C_1(p,q)$ and odd in $C_1'(p,q)$. Moreover, as p is odd and q is even in $C_2(p,q)$, and p is even and q is odd in $C_2'(p,q)$ therefore $C_2(p,q) \not\cong C_2(q,p)$, $C_2'(p,q) \not\cong C_2'(q,p)$ and $C_2(q,p) \cong C_2'(p,q)$. The cacti $C_1(p,q)$ and $C_1'(p,q)$ are presented in Figure 1((a) and (b)) and the cacti $C_2(p,q)$ and $C_2'(p,q)$ are presented in Figure 2((a) and (b)).

Let $\Omega_{1,n}$ be the class of cacti other than stars such that each block of a cactus is an edge and $\Omega_{2,n}$ be a class of cacti other than bundles such that at least one block of each cactus is an edge and at least one

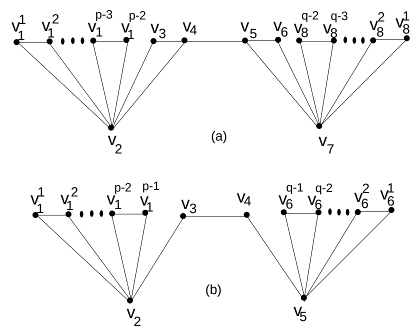

Figure 1: (a)$C_1(p,q)$ and (b)$C_1'(p,q)$.

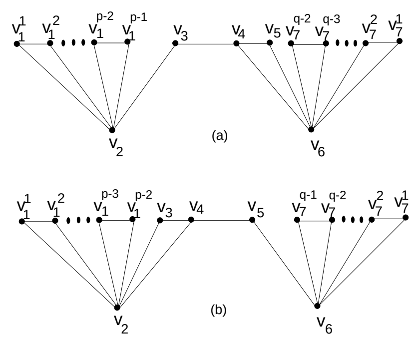

Figure 2: (a)$C_2(p,q)$ and (b)$C_2'(p,q)$.

block is a cycle. Let Ω_n be a class of cacti other than stars and bundles such that either all the blocks of a cactus are edges or a cactus has at least one block which is a cycle and at least one block which is an edge, i.e. $\Omega_n = \Omega_{1,n} \cup \Omega_{2,n}$. Thus, we obatain $\Omega_n^c = \{\Gamma^c : \Gamma^c \text{ is connected}, \quad |\Gamma^c| = n \quad \wedge \quad \Gamma \in \Omega_n\}$. By interlacing theorem, $\lambda_{min}(\Gamma) \leq -1$ if Γ contains at least one edge. Moreover, equality holds if Γ is a complete graph. Another way to achieve this equality is if $\Gamma = \cup_i G_i$, where all G_i are complete graphs and at least one G_i is non-trivial. Thus, for $\Gamma \in \Omega_n$, $\lambda_{min}(\Gamma^c) < -1$.

If $\phi' : V(\Gamma) \to \{X_i : 1 \leq i \leq n\}$ is a 1-1 map such that $\phi'(u_i) = X_i$ for each $u_i \in V(\Gamma)$ then it is said to be defined on the graph Γ. The eigenvector X of $A(\Gamma)$ is naturally defined on $V(\Gamma)$. Thus, we have

$$X^T A X = 2 \sum_{uv \in E(\Gamma)} X_u X_v. \tag{2.1}$$

The eigenequation for each $v \in V(\Gamma)$ is

$$\lambda X_v = \sum_{u \in N_\Gamma(v)} X_u, \tag{2.2}$$

where all adjacent to v are in $N_\Gamma(v)$. If $X \in \mathrm{R}^n$ is an arbitrary unit vector, we have

$$\lambda_{min}(\Gamma) \leq X^T A(\Gamma) X, \tag{2.3}$$

where equality holds iff X is a FEv. If Γ^c is complement of Γ, then $A(\Gamma^c) = J - I - A(\Gamma)$ with J and I as all-ones and identity matrix respectively. Thus, for $X \in \mathrm{R}^n$

$$X^T A(\Gamma^c) X = X^T (J - I) X - X^T A(\Gamma) X. \tag{2.4}$$

Let Y_1 be FEv of $C_1(p,q)^c$ which is defined on it. By (2.2), the vertices v_1^j for $1 \le j \le p-2$, $v_2, v_3, v_4, v_5, v_6, v_7$ and v_8^j for $1 \le j \le q-2$ having values in Y_1, say X_j for $1 \le j \le 8$ respectively. If $\lambda_{min}(C_1(p,q)^c) = \lambda_1$ then

$$\begin{cases} \lambda_1 X_1 = (p-4)X_1 + X_3 + X_4 + X_5 + X_6 + X_7 + (q-2)X_8, \\ \lambda_1 X_2 = X_5 + X_6 + X_7 + (q-2)X_8, \\ \lambda_1 X_3 = (p-2)X_1 + X_5 + X_6 + X_7 + (q-2)X_8, \\ \lambda_1 X_4 = (p-2)X_1 + X_6 + X_7 + (q-2)X_8, \\ \lambda_1 X_5 = (p-2)X_1 + X_2 + X_3 + (q-2)X_8, \\ \lambda_1 X_6 = (p-2)X_1 + X_2 + X_3 + X_4 + (q-2)X_8, \\ \lambda_1 X_7 = (p-2)X_1 + X_2 + X_3 + X_4, \\ \lambda_1 X_8 = (p-2)X_1 + X_2 + X_3 + X_4 + X_5 + X_6 + (q-4)X_8. \end{cases} \tag{2.5}$$

Take $Y_1 = (X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8)^T$ then the matrix equation is $(A - \lambda_1 I)Y_1 = 0$ and

$$\begin{aligned} f_1(\lambda,p,q) = det(A-\lambda I) =& (4+pq-2p-2q) + (16-4p-4q)\lambda + (-4-7pq+12p+12q)\lambda^2 \\ &+ (-48+2pq+14p+14q)\lambda^3 + (-20+7pq-9p-9q)\lambda^4 \\ &+ (24+2pq-16p-16q)\lambda^5 + (24-7p-7q)\lambda^6 + (8-p-q)\lambda^7 + \lambda^8 \ldots \end{aligned} \tag{2.6}$$

with least root λ_1.

Let Y_1' be FEv of $C_1'(p,q)^c$. By (2.2), the vertices v_1^j for $1 \le j \le p-1$, v_2, v_3, v_4, v_5 and v_6^j for $1 \le j \le q-1$ having values in Y_1', say X_j for $1 \le j \le 6$ respectively. If $\lambda_{min}(C_1'(p,q)^c) = \lambda_1'$ then

$$\begin{cases} \lambda_1' X_1 = (p-3)X_1 + X_3 + X_4 + X_5 + (q-1)X_6, \\ \lambda_1' X_2 = X_4 + X_5 + (q-1)X_6, \\ \lambda_1' X_3 = (p-1)X_1 + X_5 + (q-1)X_6, \\ \lambda_1' X_4 = (p-1)X_1 + X_2 + (q-1)X_6, \\ \lambda_1' X_5 = (p-1)X_1 + X_2 + X_3, \\ \lambda_1' X_6 = (p-1)X_1 + X_2 + X_3 + X_4 + (q-1)X_6. \end{cases} \tag{2.7}$$

Take $Y_1' = (X_1, X_2, X_3, X_4, X_5, X_6)^T$ then the matrix equation is $(A - \lambda_1' I)Y_1' = 0$ and

$$\begin{aligned} f_1'(\lambda,p,q) &= det(A-\lambda I) \\ &= (8-2p-2q) + (-8-2pq+7p+7q)\lambda + (-18+3pq+2p+2q)\lambda^2 + (2pq-7p-7q)\lambda^3 \\ &\quad + (11-5p-5q)\lambda^4 + (6-p-q)\lambda^5 + \lambda^6 \ldots \end{aligned} \tag{2.8}$$

with least root λ_1'.

Let Y_2 be FEv of $C_2(p,q)^c$. By (2.2), the vertices v_1^j for $1 \le i \le p-1$, v_2, v_3, v_4, v_5, v_6, and v_7^j for $1 \le j \le q-2$ having values in Y_2, say X_j for $1 \le j \le 7$ respectively. If $\lambda_{min}(C_2(p,q)^c) = \lambda_2$ then

$$\begin{cases} \lambda_2 X_1 = (p-3)X_1 + X_3 + X_4 + X_5 + X_6 + (q-2)X_7, \\ \lambda_2 X_2 = X_4 + X_5 + X_6 + (q-2)X_7, \\ \lambda_2 X_3 = (p-1)X_1 + X_5 + X_6 + (q-2)X_7, \\ \lambda_2 X_4 = (p-1)X_1 + X_2 + (q-2)X_7, \\ \lambda_2 X_5 = (p-1)X_1 + X_2 + X_3 + (q-2)X_7, \\ \lambda_2 X_6 = (p-1)X_1 + X_2 + X_3, \\ \lambda_2 X_7 = (p-1)X_1 + X_2 + X_3 + X_4 + X_5 + (q-4)X_7. \end{cases} \tag{2.9}$$

Take $Y_2 = (X_1, X_2, X_3, X_4, X_5, X_6, X_7)^T$ then the matrix equation is $(A - \lambda_2 I)Y_2 = 0$ and

$$\begin{aligned} f_2(\lambda,p,q) &= det(A-\lambda I) \\ &= (-6-pq+2p+3q) + (-8-2p-2q+3pq)\lambda + (26+pq-10p-13q)\lambda^2 + (23-5pq+3p+q)\lambda^3 \\ &\quad + (-9-2pq+11p+11q)\lambda^4 + (-17+6p+6q)\lambda^5 + (-7+p+q)\lambda^6 - \lambda^7 \ldots \end{aligned} \tag{2.10}$$

with least root λ_2.

Let Y_2' be the FEv of $C_2'(p,q)^c$. By (2.2), the vertices v_1^j for $1 \le j \le p-2$, v_2, v_3, v_4, v_5, v_6, and v_7^j for $1 \le j \le q-1$ having values in Y_2', say X_j for $1 \le j \le 7$ respectively. If $\lambda_{min}(C_2'(p,q)^c) = \lambda_2$ then

$$\begin{cases} \lambda_2' X_1 = (p-4)X_1 + X_3 + X_4 + X_5 + X_6 + (q-1)X_7, \\ \lambda_2' X_2 = X_5 + X_6 + (q-1)X_7, \\ \lambda_2' X_3 = (p-2)X_1 + X_5 + X_6 + (q-1)X_7, \\ \lambda_2' X_4 = (p-2)X_1 + X_6 + (q-1)X_7, \\ \lambda_2' X_5 = (p-2)X_1 + X_2 + X_3 + (q-1)X_7, \\ \lambda_2' X_6 = (p-2)X_1 + X_2 + X_3 + X_4, \\ \lambda_2' X_7 = (p-2)X_1 + X_2 + X_3 + X_4 + X_5 + (q-3)X_7. \end{cases} \tag{2.11}$$

Take $Y_2' = (X_1, X_2, X_3, X_4, X_5, X_6, X_7)^T$, the matrix equation is $(A - \lambda_2' I)Y_2' = 0$ and

$$\begin{aligned} f_2'(\lambda,p,q) &= det(A - \lambda I) \\ &= (-6 - pq + 2p + 3q) + (-8 - 2p - 2q + 3pq)\lambda + (26 + pq - 13p - 10q)\lambda^2 + (23 - 5pq + p + 3q)\lambda^3 \\ &\quad + (-9 - 2pq + 11p + 11q)\lambda^4 + (-17 + 6p + 6q)\lambda^5 + (-7 + p + q)\lambda^6 - \lambda^7 \ldots \end{aligned} \tag{2.12}$$

with least root λ_2'.

3 Minimizing graphs

Now, we present some important lemmas of the minimizing graph which are frequently used in next section. The classes of cacti which have graphs of even order are discussed from Lemma 3.1 to Lemma 3.6. Moreover, the cacti of odd order are studied from Lemma 3.7 to Lemma 3.10.

Firstly, we discuss the classes of cacti which have graphs of even order.

Lemma 3.1. Suppose that $p, q \ge 4$, $n \ge 12$ are integers with $p, q, n \equiv 0 (\mathrm{mod}\ 2)$. If $p > q + 2$, then

$$\lambda_{min}(C_1(p-2, q+2)^c) < \lambda_{min}(C_1(p,q)^c),$$

where $p + q + 2 = n = |V(C_1(p-2, q+2)^c)| = |V(C_1(p,q)^c)|$.

Proof. From equation (2.5), we have $f_1(-3,p,q) = 325 - 17(p+q) - 35pq$. Since for $p, q \ge 4$ $f_1(-3,p,q) < 0$. Therefore, least root of $f_1(\lambda,p,q)$ is $\lambda_1 < -3$. Moreover, $f_1(\lambda, p-2, q+2) = (pq - 4q) + (16 - 4p - 4q)\lambda + (24 - 2p + 26q - 7pq)\lambda^2 + (-56 + 18p + 10q + 2pq)\lambda^3 + (-48 + 5p - 23q + 7pq)\lambda^4 + (16 - 12p - 20q + 2pq)\lambda^5 + (24 - 7p - 7q)\lambda^6 + (8 - p - q)\lambda^7 + \lambda^8$, and

$$\begin{aligned} f_1(\lambda,p,q) - f_1(\lambda,p-2,q+2) &= -2(p-q-2)(2\lambda^5 + 7\lambda^4 + 2\lambda^3 - 7\lambda^2 + 1) \\ &= -2(p-q-2)(\lambda - \tfrac{1}{2})(\lambda + \tfrac{3+\sqrt{5}}{2})(\lambda + \tfrac{3-\sqrt{5}}{2})(\lambda + \tfrac{1+\sqrt{5}}{2})(\lambda + \tfrac{1-\sqrt{5}}{2}). \end{aligned}$$

As p is greater than $q+2$ and λ is less than -3 therefore $f_1(\lambda,p,q) - f_1(\lambda,p-2,q+2) > 0$. Also, $f_1(-3,p-2,q+2) < 0$ which implies that $\lambda_{min}(C_1(p-2,q+2)^c) < \lambda_{min}(C_1(p,q)^c)$.

Corollary 3.2. Suppose that $p, q \ge 4$, $n \ge 12$ are integers with $p, q, n \equiv 0 (\mathrm{mod}\ 2)$. If $q > p + 2$, then $\lambda_{min}(C_1(p+2, q-2)^c) < \lambda_{min}(C_1(p,q)^c)$, where $p + q + 2 = n$.

Proof. Since, $C_1(p,q)^c \cong C_1(q,p)^c$, therefore proof is same as of Lemma 3.1.

Lemma 3.3. Suppose that $p, q \ge 4$ are integers with $p, q \equiv 0 (\mathrm{mod}\ 2)$ and $p + q + 2 = n = |V(C_1(\frac{n-2}{2}, \frac{n-2}{2})^c)| = |V(C_1(p,q)^c)| = |V(C_1(\frac{n}{2}, \frac{n-4}{2})^c)|$. Then

$$\lambda_{min}(C_1(p,q)^c) \ge \begin{cases} \lambda_{min}(C_1(\frac{n-2}{2}, \frac{n-2}{2})^c) & \text{if } n \equiv 2 (\mathrm{mod}\ 4); \\ \lambda_{min}(C_1(\frac{n}{2}, \frac{n-4}{2})^c) & \text{if } n \equiv 0 (\mathrm{mod}\ 4); \end{cases}$$

where equality holds iff $p = \frac{n-2}{2} = q$ with $n \geq 14$ and $p = \frac{n}{2}$ and $q = \frac{n-4}{2}$ with $n \geq 12$.

Proof. When $n \equiv 2(\text{mod } 4)$, then for $p = \frac{n-2}{2} = q$, the equation (2.5) becomes $f_1(-3, \frac{n-2}{2}, \frac{n-2}{2}) = -(n-7.2)(n+5.1428)$. For $n \geq 14$, we have $f_1(-3, \frac{n-2}{2}, \frac{n-2}{2}) < 0$. (b) When $n \equiv 0(\text{mod } 4)$, then for $p = \frac{n}{2}$ and $q = \frac{n-4}{2}$, the equation (2.5) becomes $f_1(-3, \frac{n}{2}, \frac{n-4}{2}) = -(n-7.5159)(n+5.4588)$. For $n \geq 12$, we have $f_1(-3, \frac{n}{2}, \frac{n-4}{2}) < 0$. Thus, from both the cases least root of $f_1(\lambda, p, q)$ is $\lambda_1 < -3$.

Now, by Lemma 3.1, if $q+2 < p$ and $\lambda < -3$, then $\lambda_{min}(C_1(p-2, q+2)^c) < \lambda_{min}(C_1(p, q)^c)$ and by Corollary 3.1, if $q > p+2$ and $\lambda < -3$, then $\lambda_{min}(C_1(p+2, q-2)^c) < \lambda_{min}(C_1(p, q)^c)$.

Consequently, for $n \geq 14$ and $n \equiv 2(\text{mod } 4)$, we have $\lambda_{min}(C_1(\frac{n-2}{2}, \frac{n-2}{2})^c) \leq \lambda_{min}(C_1(p, q)^c)$ with equality iff $p = \frac{n-2}{2} = q$, and (b) for $n \geq 12$ and $n \equiv 0(\text{mod } 4)$, we have $\lambda_{min}(C_1(\frac{n}{2}, \frac{n-4}{2})^c) \leq \lambda_{min}(C_1(p, q)^c)$ with equality iff $p = \frac{n}{2}$ and $q = \frac{n-4}{2}$.

Lemma 3.4. Suppose that $p \geq 5$, $q \geq 3$, $n \geq 12$ are integers with $p, q \equiv 1(\text{mod } 2)$ and $n \equiv 0(\text{mod } 2)$. If $p > q+2$, then

$$\lambda_{min}(C_1^{'}(p-2, q+2)^c) < \lambda_{min}(C_1^{'}(p, q)^c),$$

where $p+q+2 = n$ is cardinality of both the cacti.

Proof. By (2.8), $f_1^{'}(-3, p, q) = 32 + (p+q) - 21(pq - p - q)$. Since, for $p \geq 5$ and $q \geq 3$, $f_1^{'}(-3, p, q) < 0$. Therefore, least root $f_1^{'}(\lambda, p, q)$ is $\lambda_1 < -3$. Also,

$$\begin{aligned} f_1^{'}(\lambda, p-2, q+2) =& (8-2p-2q) + (-2pq+3p+11q)\lambda + (-30+3pq+8p-4q)\lambda^2 \\ &+ (2pq-3p-11q-8)\lambda^3 + (11-5p-5q)\lambda^4 + (6-p-q)\lambda^5 + \lambda^6 \end{aligned}$$

$$\begin{aligned} f_1^{'}(\lambda, p, q) - f_1^{'}(\lambda, p-2, q+2) =& (8-4p+4q)\lambda^3 + (12-6p+6q)\lambda^2 + (-8+4p-4q) \\ =& -2\lambda(\lambda+2)(\lambda-0.5). \end{aligned}$$

Since p is greater than $q+2$ and λ is less than -3, $f_1^{'}(\lambda, p, q) - f_1^{'}(\lambda, p-2, q+2) > 0$. Also, $f_1^{'}(-3, p-2, q+2) < 0$ which implies that $\lambda_{min}(C_1^{'}(p-2, q+2)^c) < \lambda_{min}(C_1^{'}(p, q)^c)$.

Corollary 3.5. Suppose that $p \geq 3$, $q \geq 5$, $n \geq 12$ are integers with $p, q \equiv 1(\text{mod } 2)$ and $n \equiv 0(\text{mod } 2)$. If $q > p+2$, then $\lambda_{min}(C_1^{'}(p+2, q-2)^c) < \lambda_{min}(C_1^{'}(p, q)^c)$, where $p+q+2 = n$.

Proof. Since, $C_1^{'}(p, q)^c \cong C_1^{'}(q, p)^c$, therefore proof is same as of Lemma 3.4.

Lemma 3.6. Suppose that $p, q \geq 3$ are integers with $p, q \equiv 1(\text{mod } 2)$ and $p+q+2 = n = |V(C_1^{'}(\frac{n-2}{2}, \frac{n-2}{2})^c)| = |V(C_1(p, q)^c)| = |V(C_1(\frac{n}{2}, \frac{n-4}{2})^c)|$. Then,

$$\lambda_{min}(C_1^{'}(p, q)^c) \geq \begin{cases} \lambda_{min}(C_1^{'}(\frac{n}{2}, \frac{n-4}{2})^c) & \text{if } n \equiv 2(\text{mod } 4); \\ \lambda_{min}(C_1(\frac{n}{2}, \frac{n-4}{2})^c) & \text{if } n \equiv 0(\text{mod } 4); \end{cases}$$

where equality holds iff $p = \frac{n}{2}$ and $q = \frac{n-4}{2}$ with $n \geq 14$, and $p = \frac{n-2}{2} = q$ with $n \geq 12$.

Proof. When $n \equiv 2(\text{mod } 4)$, then for $p = \frac{n}{2}$ and $q = \frac{n-4}{2}$ the equation (2.8) becomes $f_1^{'}(-3, \frac{n}{2}, \frac{n-2}{2}) = -\frac{1}{4}(n-6.4842)(n+0.2937)$. For $n \geq 14$, we have $f_1(-3, \frac{n}{2}, \frac{n-2}{2}) < 0$. (b) When $n \equiv 0(\text{mod } 4)$, then for $p = \frac{n-2}{2} = q$ the equation (2.8) becomes $f_1^{'}(-3, \frac{n-2}{2}, \frac{n-2}{2}) = -\frac{1}{4}(n-7.3333)(n-0.8571)$. For $n \geq 12$, we have $f_1^{'}(-3, \frac{n-2}{2}, \frac{n-2}{2}) < 0$. Thus, from both the cases least root of $f_1^{'}(\lambda, p, q)$ is $\lambda_1^{'} < -3$.

Now, by Lemma 3.4, for $p > q+2$ and $\lambda < -3$, $\lambda_{min}(C_1^{'}(p-2, q+2)^c) < \lambda_{min}(C_1^{'}(p, q)^c)$ and by Corollary 3.5, if $q > p+2$ and $\lambda < -3$, then $\lambda_{min}(C_1^{'}(p+2, q-2)^c) < \lambda_{min}(C_1^{'}(p, q)^c)$.

Consequently, for $n \geq 14$ and $n \equiv 2(\text{mod } 4)$, we have $\lambda_{min}(C_1^{'}(\frac{n}{2}, \frac{n-4}{2})^c) \leq \lambda_{min}(C_1^{'}(p, q)^c)$ with equality iff $p = \frac{n}{2}$ and $q = \frac{n-4}{2}$, and (b) for $n \geq 12$ and $n \equiv 0(\text{mod } 4)$, we have $\lambda_{min}(C_1^{'}(\frac{n-2}{2}, \frac{n-2}{2})^c) \leq \lambda_{min}(C_1^{'}(p, q)^c)$ with equality iff $p = \frac{n-2}{2} = q$.

Now, we discuss the classes of graphs having graphs of odd order.

Lemma 3.7. Suppose that $p \geq 5$, $q \geq 2$ and $n \geq 13$ are integers with $p, n \equiv 1 (\mathrm{mod}\ \ 2)$ and $q \equiv 0 (\mathrm{mod}\ \ 2)$. If $p > q + 3$, then

$$\lambda_{min}(C_2(p-2, q+2)^c) < \lambda_{min}(C_2(p, q)^c),$$

where $p + q + 2 = n = |V(C_2(p-2, q+2)^c)| = |V(C_2(p, q)^c)|$.

Proof. From equation (2.10), we have $f_2(-3, p, q) = 117 - (p+q) - 28q(p-1)$. Since, for $p \geq 5$ and $q \geq 2$ $f_2(-3, p, q) < 0$. Therefore, least root of $f_2(\lambda, p, q)$ is $\lambda_2 < -3$. Also, $f_2(\lambda, p-2, q+2) = +(5q - pq) + (-20 + 4p - 8q + 3pq)\lambda + (16 - 8p - 15q + pq)\lambda^2 + (39 - 7p + 11q - 5pq)\lambda^3 + (-1 + 7p + 15q - 2pq)\lambda^4 + (-17 + 6p + 6q)\lambda^5 + (-7 + p + q)\lambda^6 - \lambda^7$, and

$$f_2(\lambda, p, q) - f_2(\lambda, p-2, q+2) = 4(\lambda+1)(\lambda - \frac{1}{2})(\lambda - \frac{\alpha_1 + \alpha_2}{\alpha_3})(\lambda - \frac{\alpha_1 - \alpha_2}{\alpha_3}),$$

where $\alpha_1 = 3 - 2(p-q)$, $\alpha_2 = \sqrt{8(p-q)^2 - 32(p-q) + 33}$ and $\alpha_3 = 2((p-q) - 2)$.

If $p - q \to 4$, then $\frac{\alpha_1+\alpha_2}{\alpha_3} \to 0$ and $\frac{\alpha_1-\alpha_2}{\alpha_3} \to -3$. If $p - q \to +\infty$, then $\frac{\alpha_1+\alpha_2}{\alpha_3} \to -1 + \sqrt{2}$ and $\frac{\alpha_1-\alpha_2}{\alpha_3} \to -1 - \sqrt{2}$. This shows that for $p - q \geq 4$, $\frac{\alpha_1+\alpha_2}{\alpha_3} \in [0, 0.4142[$ and $\frac{\alpha_1-\alpha_2}{\alpha_3} \in [-3, -2.4142[$. Thus, for $p > q + 3$ and $\lambda < -3$, we have $f_2(\lambda, p, q) - f_2(\lambda, p-2, q+2) > 0$. Also, $f_2(-3, p-2, q+2) < 0$ which implies that $\lambda_{min}(C_2(p-2, q+2)^c) < \lambda_{min}(C_2(p, q)^c)$.

Lemma 3.8. Suppose that $p \geq 4$, $q \geq 3$ and $n \geq 13$ are integers with $q, n \equiv 1 (\mathrm{mod}\ \ 2)$ and $p \equiv 0 (\mathrm{mod}\ \ 2)$. If $p > q + 3$, then

$$\lambda_{min}(C_2^{'}(p-2, q+2)^c) < \lambda_{min}(C_2^{'}(p, q)^c),$$

where $p + q + 2 = n$.

Proof. From equation (2.12), we have $f_2^{'}(-3, p, q) = 117 - (p+q) - 28p(q-1)$. Since, for $p \geq 4$ and $q \geq 3$, $f_2^{'}(-3, p, q) < 0$. Therefore, least root of $f_2^{'}(\lambda, p, q)$ is $\lambda_2^{'} < -3$. Also,

$$f_2^{'}(\lambda, p-2, q+2) = (-4 + p + 4q - pq) + (-20 + 4p - 8q + 3pq)\lambda + (28 - 11p - 12q + pq)\lambda^2$$

$$+(47 - 9p + 13q - 5pq)\lambda^3 + (-1 + 7p + 15q - 2pq)\lambda^4 + (-17 + 6p + 6q)\lambda^5 + (-7 + p + q)\lambda^6 - \lambda^7,$$

and

$$f_2^{'}(\lambda, p, q) - f_2^{'}(\lambda, p-2, q+2) = 2(\lambda+1)(\lambda - 0.5)(\lambda - 0.1861)(\lambda + 2.6861).$$

Thus, for $p > q$ and $\lambda < -3$, we have $f_2^{'}(\lambda, p, q) - f_2^{'}(\lambda, p-2, q+2) > 0$. Also, $f_2^{'}(-3, p-2, q+2) < 0$ which implies that $\lambda_{min}(C_2^{'}(p-2, q+2)^c) < \lambda_{min}(C_2^{'}(p, q)^c)$.

Corollary 3.9. Suppose that $p \geq 3$, $q \geq 4$ and $n \geq 13$ are integers with $p, n \equiv 1 (\mathrm{mod}\ \ 2)$ and $q \equiv 0 (\mathrm{mod}\ \ 2)$. If $q > p + 3$, then $\lambda_{min}(C_2(p+2, q-2)^c) < \lambda_{min}(C_2(p, q)^c)$, where $p + q + 2 = n$.

Proof. Using Lemma 3.8, if $r > s + 3$ and $r + s + 2 = n$ then

$$\lambda_{min}(C_2^{'}(r-2, s+2)^c) < \lambda_{min}(C_2^{'}(r, s)^c).$$

Therefore, for $n \equiv 1 (\mathrm{mod}\ \ 4)$, we have

$$\begin{aligned}\lambda_{min}(C_2^{'}(n-7, 5)^c) &< \lambda_{min}(C_2^{'}(n-5, 3)^c), \lambda_{min}(C_2^{'}(n-9, 7)^c)\\ &< \lambda_{min}(C_2^{'}(n-7, 5)^c), \ldots, \lambda_{min}(C_2^{'}(\frac{n-5}{2}, \frac{n+1}{2})^c)\\ &< \lambda_{min}(C_2^{'}(\frac{n-1}{2}, \frac{n-3}{2})^c).\end{aligned}$$

Similarly for $n \equiv 3 (\mathrm{mod}\ \ 4)$, we have

$$\begin{aligned}\lambda_{min}(C_2^{'}(n-7, 5)^c) &< \lambda_{min}(C_2^{'}(n-5, 3)^c), \lambda_{min}(C_2^{'}(n-9, 7)^c)\\ &< \lambda_{min}(C_2^{'}(n-7, 5)^c), \ldots, \lambda_{min}(C_2^{'}(\frac{n-3}{2}, \frac{n-1}{2})^c)\end{aligned}$$

$$< \lambda_{min}(C_2^{'}(\frac{n+1}{2}, \frac{n-5}{2})^c).$$

Now, by definition, for $n \equiv 1(\text{mod}\ \ 4)$, $C_2^{'}(n-5,3) \cong C_2(3,n-5)$, $C_2^{'}(n-7,5) \cong C_2(5,n-7)$,..., $C_2^{'}(\frac{n-5}{2}, \frac{n+1}{2}) \cong C_2(\frac{n+1}{2}, \frac{n-5}{2})$ and for $n \equiv 3(\text{mod}\ \ 4)$' $C_2^{'}(n-5,3) \cong C_2(3,n-5)$, $C_2^{'}(n-7,5) \cong C_2(5,n-7)$,...,$C_2^{'}(\frac{n-3}{2}, \frac{n-1}{2}) \cong C_2(\frac{n-1}{2}, \frac{n-3}{2})$.

Consequently, $\lambda_{min}(C_2(p+2,q-2)^c) < \lambda_{min}(C_2(p,q)^c)$ for $q > p$, which complete the proof.

Lemma 3.10. Suppose that $p \geq 3$ and $q \geq 2$ are integers with $p \equiv 1(\text{mod}\ \ 2)$ and $q \equiv 0(\text{mod}\ \ 2)$, where $p+q+2 = n$ is of the cacti. Then

$$\lambda_{min}(C_2(p,q)^c) \geq \begin{cases} \lambda_{min}(C_2(\frac{n+1}{2}, \frac{n-5}{2})^c) & \text{if } n \equiv 1(\text{mod}\ \ 4); \\ \lambda_{min}(C_2(\frac{n-1}{2}, \frac{n-3}{2})^c) & \text{if } n \equiv 3(\text{mod}\ \ 4); \end{cases}$$

where equality holds iff $p = \frac{n+1}{2}$ and $q = \frac{n-5}{2}$ with $n \geq 13$, and $p = \frac{n-1}{2}$ and $q = \frac{n-3}{2}$ with $n \geq 15$.

Proof. When $n \equiv 1(\text{mod}\ \ 4)$, then for $p = \frac{n+1}{2}$ and $q = \frac{n-5}{2}$, the equations (2.10) becomes $f_2(-3, \frac{n+1}{2}, \frac{n-5}{2}) = -(n-7.4647)(n+1.6075)$. Thus, for $n \geq 13$, we have $f_2(-3, \frac{n+1}{2}, \frac{n-5}{2}) < 0$. (b) When $n \equiv 3(\text{mod}\ \ 4)$, then for $p = \frac{n-1}{2}$ and $q = \frac{n-3}{2}$, the equation (2.10) becomes, $f_2(-3, \frac{n-1}{2}, \frac{n-3}{2}) = -(n-5.7569)(n-0.1001)$. So, for $n \geq 15$, we have $f_2(-3, \frac{n-1}{2}, \frac{n-3}{2}) < 0$. Thus, from both the cases least root of $f_2(\lambda, p, q)$ is $\lambda_2 < -3$.

Now, by Lemma 3.7, if p is greater than $q+3$ and λ is less than -3, then $\lambda_{min}(C_2(p-2,q+2)^c) < \lambda_{min}(C_2(p,q)^c)$ and by Corollary 3.9, if $q > p$, $q-p \neq 2$ and $\lambda < -3$, then $\lambda_{min}(C_2(p+2,q-2)^c) < \lambda_{min}(C_2(p,q)^c)$.

Consequently, for $n \geq 13$ and $n \equiv 1(\text{mod}\ \ 4)$, we have $\lambda_{min}(C_2(\frac{n+1}{2}, \frac{n-5}{2})^c) \leq \lambda_{min}(C_2(p,q)^c)$ with equality if $p = \frac{n+1}{2}$ and $p = \frac{n-5}{2}$, and (b) for $n \geq 15$ and $n \equiv 3(\text{mod}\ \ 4)$, we have $\lambda_{min}(C_2(\frac{n-1}{2}, \frac{n-3}{2})^c) \leq \lambda_{min}(C_2(p,q)^c)$ with equality iff $p = \frac{n-1}{2}$ and $q = \frac{n-3}{2}$. This complete the proof.

4 Characterization

This section includes the main results in which minimizing graphs are characterized in the family of connected graphs with the condition that the complement of each graph is a cactus such that either its each block is only an edge or it has at least one block which is an edge and at least one block which is a cycle. In Lemma 4.1 and 4.2, the basic results are developed which are used in the main results. In Lemma 4.3 and Lemma 4.4, minimizing graphs are characterized in $\Omega_{1,n}^c$ and $\Omega_{2,n}^c$ respectively. Finally, in Theorem 4.5 minimizing graphs are characterized in $\Omega_n^c = \Omega_{1,n}^c \cup \Omega_{2,n}^c$.

Lemma 4.1. Let $n \geq 12$ and $X = (X_1, X_2, X_3, \ldots, X_n)^T$ be a real vector defined on $C \in \Omega_n$ such that $|X_1| \geq |X_2| \geq |X_3| \geq \ldots \geq |X_n|$ and all X_i are either non-negative or non-positive. Then

$$\sum_{uv \in E(C)} X_u X_v \leq \begin{cases} \sum\limits_{uv \in E(B_1(n-1))} X_u X_v & \text{if } n \equiv 1(\text{mod } 2); \\ \sum\limits_{uv \in E(B_2(n-1))} X_u X_v & \text{if } n \equiv 0(\text{mod } 2); \end{cases}$$

where equality holds if $C \cong B_1(n-1)$ and $C \cong B_2(n-1)$ respectively.

Proof. Suppose that X is non-negative and discuss the following two cases:

Case-1. Suppose that $C \in \Omega_n$ is a cactus graph such that its each block is an edge i.e. $C \in \Omega_{1,n}$. Let X_1 be the value of the vertex $v \in C$ assigned by X. As $n \geq 10$ and is not a bundle, therefore there exist a vertex in C say u which is not adjacent to v. Thus, a vertex w adjacent to u ($w \sim u$) exists on a path from v to u as C is connected. A new cactus $\tilde{C}$ having each block as an edge can be found on the deletion of the edge wu and addition of vu in C. We find a star $K_{1,n-1}$ with center v by repeating the same process for the non-neighbor of v in the cactus $\tilde{C}$ and so on. Thus, we have

$$\sum_{uv \in E(C)} X_u X_v \leq \sum_{uv \in E(\tilde{C})} X_u X_v \leq \sum_{i=2}^{n} X_1 X_i = \sum_{uv \in E(K_{1,n-1})} X_u X_v \tag{4.1}$$

Case-2. Suppose that $C \in \Omega_n$ is a cactus graph such that it has at least one block which is an edge and at least one block which is a cycle i.e. $C \in \Omega_{2,n}$. Assume that k is number of cycles in C, then $1 \le k \le \lceil \frac{n-2}{2} \rceil$, where n is order of C. Now, we delete an edge in each cycle under the conditions, (i) no two deleted edges of any two cycles have common vertex and (ii) the deleted edge in a cycle which has the vertex with label X_1 is not incident on this vertex. Thus, we have a cactus graph such that its each block is an edge, say C' i.e. $C' \in \Omega_{1,n}$. Then by Case 1, we have $\sum\limits_{uv\in E(C')} X_uX_v \le \sum\limits_{uv\in E(K_{1,n-1})} X_uX_v$. Now, we add the deleted edges, then

$$\begin{aligned}\sum_{uv\in E(C)} X_uX_v &= \sum_{uv\in E(C')} X_uX_v + \underbrace{(X_{v_{r_1}}X_{v_{r_1+1}} + X_{v_{r_2}}X_{v_{r_2+1}}+, \dots, +X_{v_{r_k}}X_{v_{r_k+1}})}_{k \text{ terms}} \\ &\le \sum_{uv\in E(K_{1,n-1})} X_uX_v + (X_{v_{r_1}}X_{v_{r_1+1}} + X_{v_{r_2}}X_{v_{r_2+1}}+, \dots, +X_{v_{r_k}}X_{v_{r_k+1}}),\end{aligned}$$

where $v_r, v_{r+1}, v_s, v_{s+1}, \dots, v_t, v_{t+1}$ are $2k$ distinct vertices of C. Moreover, the inequality in (4.1) does not disturb, if we add k terms $(X_{v_{r_1}}X_{v_{r_1+1}} + X_{v_{r_2}}X_{v_{r_2+1}}+, \dots, +X_{v_{r_k}}X_{v_{r_k+1}})$ in its right hand side. Consequently, from both the cases, we have

$$\sum_{uv\in E(C)} X_uX_v \le \sum_{uv\in E(K_{1,n-1})} X_uX_v + (X_{v_{r_1}}X_{v_{r_1+1}} + X_{v_{r_2}}X_{v_{r_2+1}}+, \dots, +X_{v_{r_k}}X_{v_{r_k+1}}). \tag{4.2}$$

(a) Assume $n \equiv 1 (\mathrm{mod}\ 2)$. Since, in $K_{1,n-1}$, one vertex, say v_1 with $d(v_1) = n-1$ has value X_1 and remaining $n-1$ pendent vertices have values X_i for $2 \le i \le n$. Pairing the vertices of degree 1 and joining each pair by an edge, we have $\frac{n-1}{2}$ edges v_iv_{i+1}, where $i \in I = \{2, 4, \dots, n-1\}$. Thus, $(X_{v_r}X_{v_{r+1}} + X_{v_s}X_{v_{s+1}}+, \dots, +X_{v_t}X_{v_{t+1}}) \le \sum\limits_{i\in I} X_iX_{i+1}$. Since, $\sum\limits_{uv\in E(K_{1,n-1})} X_uX_v + \sum\limits_{i\in I} X_iX_{i+1} = \sum\limits_{uv\in E(B_1(n-1))} X_uX_v$. Consequently (4.2) becomes, $\sum\limits_{uv\in E(C)} X_uX_v \le \sum\limits_{uv\in E(B_1(n-1))} X_uX_v$, where equality holds if C has one vertex of degree $n-1$ and remaining $n-1$ vertices are of degree two, i.e. $C \cong B_1(n-1)$.

(b) Assume $n \equiv 0 (\mathrm{mod}\ 2)$. Since, in $K_{1,n-1}$, one vertex, say v_1 with $d(v_1) = n-1$ has value X_1 and remaining $n-1$ pendent vertices have values X_i for $2 \le i \le n$. Pairing the $n-2$ vertices of degree 1 and joining each pair by an edge, we have $\frac{n-2}{2}$ edges v_iv_{i+1}, where $i \in I = \{2, 4, \dots, n-2\}$. Thus, $(X_{v_{r_1}}X_{v_{r_1+1}} + X_{v_{r_2}}X_{v_{r_2+1}}+, \dots, +X_{v_{r_k}}X_{v_{r_k+1}}) \le \sum\limits_{i\in I} X_iX_{i+1}$. Since, $\sum\limits_{uv\in E(K_{1,n-1})} X_uX_v + \sum\limits_{i\in I} X_iX_{i+1} = \sum\limits_{uv\in E(B_2(n-1))} X_uX_v$. So, (4.2) takes form $\sum\limits_{uv\in E(C)} X_uX_v \le \sum\limits_{uv\in E(B_2(n-1))} X_uX_v$, where equality holds if C has one pendent vertex, one of degree $n-1$ and $n-2$ with degree 2, i.e. $C = B_2(n-1)$.

Lemma 4.2. For $n \ge 12$ and $C^c \in \Omega_n^c$, the first eigenvector X of C^c has at least 2 negative and 2 positive entries.

Proof. Assume that there is a unique vertex $v \in C^c$ having positive value labeled by X. The degree of v in C^c is non-zero, i.e $d_{C^c}(v) \ne 0$. As, if $d_{C^c}(v) = 0$, then C is a bundle, which is a contradiction to the construction of Ω_n^c. Consequently, $1 \le d_{C^c}(v) \le n-1$. Let u be another vertex in C^c. Since all the vertices are supposed to have negative values except v, we claim $u \sim v$, otherwise (2.2) does not holds for $u \in C^c$ as $\lambda X_u > 0$ and $\sum\limits_{w\in N_G(u)} X_w < 0 \Rightarrow \lambda X_u \ne \sum\limits_{w\in N_G(u)} X_w$. Consequently, $d_{C^c}(v) = n-1$ for each $v \in C^c$. It shows that C is disconnected, which is a contradiction to the construction of Ω_n^c. Similarly, we can prove, if $v \in C^c$ is a unique vertex with negative value assigned by X.

Lemma 4.3. Suppose that C is a cactus graph such that $C^c \in \Omega_{2,n}^c$ and $|V(C)| = n = p + q + 2 \ge 12$.

(a) If $n \equiv 0 (\mathrm{mod}\ 2)$, then either $\lambda_{min}(C_1(p,q)^c) \le \lambda_{min}(C^c)$, or $\lambda_{min}(C_1'(p,q)^c) \le \lambda_{min}(C^c)$, where equalities hold if $C \cong C_1(p,q)$ or $C \cong C_1'(p,q)$, respectively.

(b) If $n \equiv 1 (\mathrm{mod}\ 2)$, then $\lambda_{min}(C_2(p,q)^c) \le \lambda_{min}(C^c)$ with equality if $C \cong C_2(p,q)$.

Proof. Assume X is a unit first eigenvector of C^c. Define $V_+ = \{v : X_v \ge 0, v \in V(C^c)\}$ and $V_- = \{v : X_v < 0, v \in V(C^c)\}$ such that $|V_+|, |V_-| \ge 2$ by Lemma 4.2. Suppose that the subgraphs C_+ and C_- of C are induced by the vertex sets V_+ and V_- respectively and $E^{'} \neq \Phi$ is subset of $E(C)$ with one end in C_+ and other in C_-. Thus, we have

$$\sum_{uv \in E(C)} X_u X_v = \sum_{uv \in E(C_+)} X_u X_v + \sum_{uv \in E(C_-)} X_u X_v + \sum_{uv \in E'} X_u X_v. \tag{4.3}$$

(a) Assume that $n \equiv 0 (\text{mod } 2)$, where $n = p + q + 2$. Let $\bar{C}$ be a graph obtained from C by some possible addition or deletion of edges in C_+ and C_- such that the subgraph $\bar{C}_+$ and $\bar{C}_-$ of $\bar{C}$ induced by C_+ and C_- are cactus graphs satisfying one of the following possibilities, (i) each block of one of the subgraphs $\bar{C}_+$ and $\bar{C}_-$ is an edge and other has at least one block which is a cycle and at least one block which is an edge, (ii) all the blocks of both the subgraphs $\bar{C}_+$ and $\bar{C}_-$ are cycles, (iii) all the blocks of one subgraph are edges and of other are cycles, (iv) each block of one of the subgraphs $\bar{C}_+$ and $\bar{C}_-$ is a cycle and other has at least one block which is a cycle and at least one block which is an edge, and (v) both the subgraphs $\bar{C}_+$ and $\bar{C}_-$ have at least one block which is a cycle and at least one block which is an edge.

For (i), suppose $\bar{C}_+$ is a cactus such that its each block is an edge, otherwise we take $-X$ as a first eigenvector. Let $u^{'}$ be a vertex of $\bar{C}_+$ with maximum modulus among all the vertices, then by discussion of equation (4.1) in Lemma 4.1, we obtain a cactus with each block as an edge which is infect a star $K_{1,p}$. Similarly, suppose that $v^{'}$ is a vertex with maximum modulus among all the vertices of $\bar{C}_-$. Firstly, we delete an edge in each block such that no two deleted edges of any two blocks have a common vertex in $\bar{C}_-$ and the deleted edge in a block which has $v^{'}$ is not incident on this vertex. Thus, we obtain a subgraph of $\bar{C}_-$ such that its each block is an edge. Then by the same discussion as of equation (4.2) in Lemma 4.1, we obtain a cactus with at least one block as a cycle and at least one block as an edge which is infect a star $K_{1,q}$ with edges among the pendent vertices having different end points.

Since $n \equiv 0 (\text{mod } 2)$ and $n = p + q + 2$, where $n = |V_+ \cup V_-| = |\bar{C}_+ \cup \bar{C}_-|$, $p + 1 = |V_+| = |\bar{C}_+|$ and $q + 1 = |V_-| = |\bar{C}_-|$. Therefore, either both p and q are even or odd.

Suppose p and q both are even. By pairing the pendent vertices of the star $K_{1,p}$ which is obtained from $\bar{C}_+$ and joining them by edges, we have a bundle $B_1(p)$ with center $u^{'}$ having maximum modulus value, where $p + 1 = |V_+| \geq 3$ is odd. Similarly, pair the remaining possible pendent vertices of the subgraph obtained from $\bar{C}_-$ and join them by edges. Thus, we obtain a bundle $B_1(q)$ with center $v^{'}$ having maximum modulus value, where $q + 1 = |V_+| \geq 3$ is odd. Thus, by Lemma 4.1 (a), we have

$$\sum_{uv \in E(C_+)} X_u X_v \leq \sum_{uv \in E(\bar{C}_+)} X_u X_v \leq \sum_{uv \in E(B_1(p))} X_u X_v \tag{4.4}$$

and

$$\sum_{uv \in E(C_-)} X_u X_v \leq \sum_{uv \in E(\bar{C}_-)} X_u X_v \leq \sum_{uv \in E(B_1(q))} X_u X_v. \tag{4.5}$$

Suppose p and q both are odd. By pairing the pendent vertices of the star which is obtained from $\bar{C}_+$ and joining them by edges, we have a bundle $B_2(p)$ with center $u^{'}$ having maximum modulus value, where $p + 1 = |V_+| \geq 4$ is even. Similarly, pair the remaining possible pendent vertices of the subgraph obtained from $\bar{C}_-$ and join them by edges. Thus, we obtain a bundle $B_2(q)$ with center $v^{'}$ having maximum modulus value, where $q + 1 = |V_+| \geq 4$ is even. Thus, by Lemma 4.1 (b), we have

$$\sum_{uv \in E(C_+)} X_u X_v \leq \sum_{uv \in E(\bar{C}_+)} X_u X_v \leq \sum_{uv \in E(B_2(p))} X_u X_v, \tag{4.6}$$

$$\sum_{uv \in E(C_-)} X_u X_v \leq \sum_{uv \in E(\bar{C}_-)} X_u X_v \leq \sum_{uv \in E(B_2(q))} X_u X_v. \tag{4.7}$$

Assume that $u^{''} \in \bar{C}_+$ and $v^{''} \in \bar{C}_-$ have minimum modulus values. Then

$$\sum_{uv \in E'} X_u X_v \leq X_{u''} X_{v''}. \tag{4.8}$$

Using (4.4), (4.5), (4.8) and (4.6), (4.7), (4.8) in (4.3) respectively, we have

$$\sum_{uv\in E(C)} X_uX_v \le \sum_{uv\in E(B_1(p))} X_uX_v + \sum_{uv\in E(B_1(q))} X_uX_v + X_{u''}X_{v''}, \tag{4.9}$$

$$\sum_{uv\in E(C)} X_uX_v \le \sum_{uv\in E(B_2(p))} X_uX_v + \sum_{uv\in E(B_2(q))} X_uX_v + X_{u''}X_{v''}. \tag{4.10}$$

Since $p \ge q \ge 2$. Therefore, if we take $u'' \in B_1(p)$, $v'' \in B_1(q)$ of degree 2 and $u'' \in B_2(p)$, $v'' \in B_2(q)$ of degree 1. Then (4.9) and (4.10) becomes,

$$\sum_{uv\in E(C)} X_uX_v \le \sum_{uv\in E(C_1(p,q))} X_uX_v, \tag{4.11}$$

$$\sum_{uv\in E(C)} X_uX_v \le \sum_{uv\in E(C_1'(p,q))} X_uX_v. \tag{4.12}$$

Now by the equations (2.1)-(2.4) and (4.11), we have $\lambda_{min}(C^c) = X^TA(C^c)X = X^T(J - I - A(C))X = X^T(J - I)X - X^TA(C)X \ge X^T(J - I)X - X^TA(C_1(p, q))X = X^TA(C_1(p, q)^c)X \ge \lambda_{min}(C_1(p, q)^c) \Rightarrow \lambda_{min}(C_1(p, q)^c) \le \lambda_{min}(C^c)$. Similarly by equations (2.1)-(2.4) and (4.12), we have $\lambda_{min}(C_1'(p, q)^c) \le \lambda_{min}(C^c)$.

Thus, for $n \equiv 0 (\mathrm{mod}\ 2)$, either $\lambda_{min}(C_1(p, q)^c) \le \lambda_{min}(C^c)$ or $\lambda_{min}(C_1'(p, q)^c) \le \lambda_{min}(C^c)$, where $p + q + 2 = n$. On the same way, it can be prove that the results are also true for all the possibilities (ii)-(v).

(b) Assume that $n \equiv 1 (\mathrm{mod}\ 2)$, where $n = p + q + 2$. Let $\bar{C}$ be a graph obtained from C with subgraph $\bar{C}_+$ and $\bar{C}_-$ induced from C_+ and C_- are cactus graphs satisfying any one of the possibilities which are stated in (a). For (i), we proceed same as in (a) and find the cactus graphs which are infect stars with some possible edges among the pendent vertices having different end points from both the subgraphs $\bar{C}_+$ and $\bar{C}_-$ after the deletion and addition of some edges. Since $n \equiv 1 (\mathrm{mod}\ 2)$ and $n = p + q + 2$, where $n = |V_+ \cup V_-| = |\bar{C}_+ \cup \bar{C}_-|$, $p + 1 = |V_+| = |\bar{C}_+|$ and $q + 1 = |V_-| = |\bar{C}_-|$. Therefore, either p is even and q is odd or vice versa. Without loss of generality, we assume p as odd and q even.

Suppose that u' and u'' in $\bar{C}_+$, and v' and v'' in $\bar{C}_-$ have have maximum and minimum modulus values respectively. Then by the same discussion as in (a) with the help of Lemma 4.1, we have

$$\sum_{uv\in E(C_+)} X_uX_v \le \sum_{uv\in E(\bar{C}_+)} X_uX_v \le \sum_{uv\in E(B_2(p))} X_uX_v, \tag{4.13}$$

$$\sum_{uv\in E(C_-)} X_uX_v \le \sum_{uv\in E(\bar{C}_-)} X_uX_v \le \sum_{uv\in E(B_1(q))} X_uX_v, \tag{4.14}$$

and

$$\sum_{uv\in E'} X_uX_v \le X_{u''}X_{v''}, \tag{4.15}$$

where the bundle $B_2(p)$ has center u' with maximum modulus value and $p + 1 = |V_+| \ge 4$ is even. Similarly, the bundle $B_1(q)$ has center v', with maximum modulus value and $q + 1 = |V_+| \ge 3$ is odd. Now, using (4.13), (4.14) and (4.15) in (4.3), we have

$$\sum_{uv\in E(C)} X_uX_v \le \sum_{uv\in E(B_2(p))} X_uX_v + \sum_{uv\in E(B_1(q))} X_uX_v + X_{u''}X_{v''}. \tag{4.16}$$

Since $p, q \ge 2$. Therefore, if we take $u'' \in B_2(p)$ and $v'' \in B_1(p)$ such that degree of u'' in $B_2(p)$ is 1 and degree of v'' in $B_1(q)$ is 2. Thus, (4.16) becomes

$$\sum_{uv\in E(C)} X_uX_v \le \sum_{uv\in E(C_2(p,q))} X_uX_v. \tag{4.17}$$

Now by the equations (2.1)-(2.4) and (4.17), $\lambda_{min}(C_2(p,q)^c) \leq \lambda_{min}(C^c)$, where $n = p + q + 2 > 10$ and $n \equiv 1 (\mathrm{mod}\ 2)$. Similarly, it also can be prove for all other possibilities.

Now finally we prove that there does not exist any vertex in V_+ such that its value given by X is zero and $E^{'}$ has exactly one edge. Firstly, among the vertices of $C_1(p,q)$, we prove that $v_2 = u^{'}$ and $v_4 = u^{''}$ are unique ones in C_+ and, $v_5 = v^{''}$ and $v_7 = v^{'}$ are unique ones in C_- with maximum and minimum modulus, respectively. For this, we will show $0 \leq X_4 < X_3 < X_1 < X_2$ and $X_7 < X_8 < X_6 < X_5 < 0$. By Lemma 4.2, we have X_1, X_2, X_3 non negative and X_4, X_5, X_6, X_7, X_8 negative values in the first eigenvector X of $C_1(p,q)^c$. By (2.5), $\lambda_1(X_2 - X_1) = -(p-4)X_1 - X_3 - X_4 < 0$, $(\lambda_1 + 1)(X_1 - X_3) = -(X_1 - X_4) < 0$, and $\lambda_1(X_3 - X_4) = X_5 < 0 \Rightarrow X_2 - X_1 > 0$, $X_1 - X_3 > 0$ and $X_3 - X_4 > 0$. Thus

$$0 \leq X_4 < X_3 < X_1 < X_2. \tag{4.18}$$

Similarly, $\lambda_1(X_8 - X_7) = X_5 + X_6 + (q-4)X_8 < 0$, $(\lambda_1 + 1)(X_6 - X_8) = -X_5 + X_8 < 0$ and $\lambda_1(X_5 - X_6) = -X_4 < 0 \Rightarrow X_8 - X_7 > 0$, $X_5 - X_6 > 0$ and $X_6 - X_8 > 0$. Thus

$$X_7 < X_8 < X_6 < X_5 < 0. \tag{4.19}$$

If any one of the vertices v_1, v_2 and v_3 has value zero assigned by X, then by (4.18) $X_3 = 0 = X_4$. Moreover, by (2.5), we have $X_5 = 0 = X_6$, which is a contradiction to the construction of V_- and C_-. If the value of the vertex v_4 labeled by X is zero, then by (2.5), $\lambda_1(X_5 - X_6) = 0 \Rightarrow X_5 = X_6$ which is a contradiction to (4.19) (i.e. X_5 is a unique one in C_-). Consequently, X_1, X_2, X_3 and X_4 are non zero positive values of X. Thus, $v \notin V_+$ such that $X_v = 0$. By (4.4), (4.5), (4.8), (4.9) and the above discussion, we have $C_+ = \bar{C}_+ = B_1(p)$, $C_- = \bar{C}_- = B_1(q)$ and $E^{'}$ has only one edge $u^{''}v^{''} = v_4v_5$ in $B_1(p,q)$. Similarly, we can prove for $B_1^{'}(p,q)$ and $B_2(p,q)$. This complete the proof.

Similarly, we can prove the following result:

Lemma 4.4. Suppose that C is a cactus graph of order $n = p + q + 2 \geq 10$ such that $C^c \in \Omega_{1,n}^c$.
(a) If $n \equiv 0 (\mathrm{mod}\ 2)$, then either $\lambda_{min}(C_1(p,q)^c) < \lambda_{min}(C^c)$ or $\lambda_{min}(C_1^{'}(p,q)^c) < \lambda_{min}(C^c)$.
(b) If $n \equiv 1 (\mathrm{mod}\ 2)$, then $\lambda_{min}(C_2(p,q)^c) < \lambda_{min}(C^c)$.

Theorem 4.5. Suppose that C is a cactus graph of order n such that $C^c \in \Omega_n^c = \Omega_{1,n}^c \cup \Omega_{2,n}^c$.
(1)Assume that $n \equiv 0 (\mathrm{mod}\ 2)$, $p, q \geq 4$ and $n = p + q + 2$:

(a)For $p, q \equiv 0 (\mathrm{mod}\ 2)$;

(i)If $n \geq 14$ and $n \equiv 2 (\mathrm{mod}\ 4)$, then $\lambda_{min}(C_1(\frac{n-2}{2}, \frac{n-2}{2})^c) \leq \lambda_{min}(C_1(p,q)^c) \leq \lambda_{min}(C^c)$, where equalities hold if $C \cong C_1(\frac{n-2}{2}, \frac{n-2}{2}) \cong \lambda_{min}(C_1(p,q))$,

(ii)If $n \geq 12$ and $n \equiv 0 (\mathrm{mod}\ 4)$, then $\lambda_{min}(C_1(\frac{n}{2}, \frac{n-4}{2})^c) \leq \lambda_{min}(C_1(p,q)^c) \leq \lambda_{min}(C^c)$, where equalities hold if $C \cong C_1(\frac{n}{2}, \frac{n-4}{2}) \cong \lambda_{min}(C_1(p,q))$,

(b)For $p, q \equiv 1 (\mathrm{mod}\ 2)$;

(i)If $n \geq 14$ and $n \equiv 2 (\mathrm{mod}\ 4)$, then $\lambda_{min}(C_1^{'}(\frac{n}{2}, \frac{n-4}{2})^c) \leq \lambda_{min}(C_1^{'}(p,q)^c) \leq \lambda_{min}(C^c)$, where equalities hold if $C \cong C_1^{'}(\frac{n}{2}, \frac{n-4}{2}) \cong \lambda_{min}(C_1^{'}(p,q))$,

(ii)If $n \geq 12$ and $n \equiv 0 (\mathrm{mod}\ 4)$, then $\lambda_{min}(C_1^{'}(\frac{n-2}{2}, \frac{n-2}{2})^c) \leq \lambda_{min}(C_1^{'}(p,q)^c) \leq \lambda_{min}(C^c)$, where equalities hold if $C \cong C_1^{'}(\frac{n-2}{2}, \frac{n-2}{2}) \cong \lambda_{min}(C_1^{'}(p,q))$.

(2) Assume that $n \equiv 1 (\mathrm{mod}\ 2)$, $p \equiv 1 (\mathrm{mod}\ 2)$ and $q \equiv 0 (\mathrm{mod}\ 2)$:

(i)If $n \geq 13$ and $n \equiv 1 (\mathrm{mod}\ 4)$, then $\lambda_{min}(C_2(\frac{n+1}{2}, \frac{n-5}{2})^c) \leq \lambda_{min}(C_2(p,q)^c) \leq \lambda_{min}(C^c)$, where equalities hold if $C \cong C_2(\frac{n+1}{2}, \frac{n-5}{2}) \cong \lambda_{min}(C_2(p,q))$,

(ii)If $n \geq 15$ and $n \equiv 3 (\mathrm{mod}\ 4)$, then $\lambda_{min}(C_2(\frac{n-1}{2}, \frac{n-3}{2})^c) \leq \lambda_{min}(C_2(p,q)^c) \leq \lambda_{min}(C^c)$, with equality if $C \cong C_2(\frac{n-1}{2}, \frac{n-3}{2}) \cong \lambda_{min}(C_2(p,q))$.

Proof. The result follows from Lemma 3.3-Lemma 3.5 and Lemma 4.3-Lemma 4.4.

5 Conclusions

Petrović et al. [23] explored a unique cactus as a minimizing graph from the class of cacti such that the order of each cactus is n. But, it is noted that the complement of the proposed minimizing graph is disconnected. In this paper, we characterize the minimizing graphs in a collection of connected graphs such that the complement of each graph of order n is a cactus with the condition that either its each block is only an edge or it has at least one block which is an edge and at least one block which is a cycle. However, the problem is still open to characterize the minimizing graphs in a collection of connected graphs whose complements are in the complete class of cacti (each block of a cactus is only an edge, at least one block is an edge and at least one block is a cycle, or each block is a cycle).

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