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Research Article

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Determinants of two kinds of matrices whose elements involve sine functions

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Abstract: The presented paper is strictly connected, among others, with the paper *On the sum of some alternating series*, Comp. Math. Appl. (2011), written by Wituła and Słota. A problem concerning the form of determinants formulated in the cited paper is solved here. Next, the obtained result is adapted to solve some system of linear equations and the description of the sum of alternating series.

Keywords: determinant, sine matrix, alternating series, Fourier series

MSC: 11C20, 15A06, 42A05, 40A05

1 The main result

The presented paper is strictly connected with paper [1], and also [2, 3]. In paper [1] the following two matrices were considered

$$\Delta_r := \left[\sin\frac{2ij\pi}{r}\right]_{\nu_r\times\nu_r}$$

and

$$\Delta_{r,n} := \left[\Delta_{r,n}^{ij}\right]_{\nu_r\times\nu_r}, \qquad \text{with} \qquad \Delta_{r,n}^{ij} := \begin{cases} \sin\dfrac{2ij\pi}{r}, & j \neq n, \\ \dfrac{\pi}{2r}(r-2i), & j = n, \end{cases}$$

where

$$\nu_r := \left\lfloor \frac{r-1}{2} \right\rfloor$$

for every $r \in \{3, 4, \ldots\}$ and $n \in \{1, 2, \ldots, \nu_r\}$. Also, the formula for determinant Δ_r (see Theorem 1 below) was given, but verified only for prime $r < 1051$, using *Mathematica* software. In this paper we prove that formula is valid for all natural $r \geq 3$. Moreover, we also give the compact formula for the determinant of $\Delta_{r,n}$. The formulas are given in the theorem below.

Theorem 1. *For all natural $r \geq 3$ and $n \in \{1, 2, \ldots, \nu_r\}$ we have*

$$\det \Delta_r = (-1)^{\nu_r(\nu_r-1)/2}\left(\frac{r}{4}\right)^{\nu_r/2} \tag{1}$$

and

$$\det \Delta_{r,n} = \frac{\pi}{r}\det\Delta_r \cot\frac{n\pi}{r}. \tag{2}$$

Remark 1. *Let us also notice that in formula (1) the multiplier $(-1)^{\nu_r(\nu_r-1)/2}$ vanishes if we reverse the order of columns (or rows) of Δ_r.*

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Remark 2. *The hypothesis on the formula of* $\det \Delta_r$ *were sent by the authors of [1] to The American - Mathematical Monthly, Section: Problems and Solutions. However, it was not published since it was considered insufficient and neither authors nor editors had an idea how to complete it.*

Remark 3. *The determinant related to our determinant, although having some other structure, is discussed in papers [4, 5].*

Example 1. *We give the explicit form of formulas (1) and (2) for* $r = 7$ ($\nu_7 = 3$, *whence* $n \in \{1, 2, 3\}$). *Note that if we expand the determinants we get some nontrivial trigonometric equalities.*

$$\det \Delta_7 = \begin{vmatrix} \sin \frac{2\pi}{7} & \sin \frac{4\pi}{7} & \sin \frac{6\pi}{7} \\ \sin \frac{4\pi}{7} & \sin \frac{8\pi}{7} & \sin \frac{12\pi}{7} \\ \sin \frac{6\pi}{7} & \sin \frac{12\pi}{7} & \sin \frac{18\pi}{7} \end{vmatrix} = -\frac{7\sqrt{7}}{8}$$

$$\det \Delta_{7,1} = \begin{vmatrix} \frac{5\pi}{14} & \sin \frac{4\pi}{7} & \sin \frac{6\pi}{7} \\ \frac{3\pi}{14} & \sin \frac{8\pi}{7} & \sin \frac{12\pi}{7} \\ \frac{\pi}{14} & \sin \frac{12\pi}{7} & \sin \frac{18\pi}{7} \end{vmatrix} = -\frac{\sqrt{7}}{8}\pi \cot \frac{\pi}{7}$$

$$\det \Delta_{7,2} = \begin{vmatrix} \sin \frac{2\pi}{7} & \frac{5\pi}{14} & \sin \frac{6\pi}{7} \\ \sin \frac{4\pi}{7} & \frac{3\pi}{14} & \sin \frac{12\pi}{7} \\ \sin \frac{6\pi}{7} & \frac{\pi}{14} & \sin \frac{18\pi}{7} \end{vmatrix} = -\frac{\sqrt{7}}{8}\pi \cot \frac{2\pi}{7}$$

$$\det \Delta_{7,3} = \begin{vmatrix} \sin \frac{2\pi}{7} & \sin \frac{4\pi}{7} & \frac{5\pi}{14} \\ \sin \frac{4\pi}{7} & \sin \frac{8\pi}{7} & \frac{3\pi}{14} \\ \sin \frac{6\pi}{7} & \sin \frac{12\pi}{7} & \frac{\pi}{14} \end{vmatrix} = -\frac{\sqrt{7}}{8}\pi \cot \frac{3\pi}{7}$$

The paper is organized as follows. In Section 2 we prove four auxiliary lemmas which we use in Section 3 to prove Theorem 1. In section 4 we discuss two applications of the main result.

2 Auxiliary facts

The following lemmas are necessary for proving the main results. We will use the well-known trigonometric identities (see [6–9])

$$\prod_{k=1}^{n-1} \sin \frac{k\pi}{n} = \frac{n}{2^{n-1}}, \tag{3}$$

$$\prod_{k=1}^{n-1} \cos \frac{k\pi}{n} = \frac{\sin \frac{n\pi}{2}}{2^{n-1}} = \begin{cases} 0, & n \in 2\mathbb{N}, \\ \frac{(-1)^{(n-1)/2}}{2^{n-1}}, & n \in 2\mathbb{N}+1, \end{cases} \tag{4}$$

where $n \geq 2$ and

$$\sum_{k=1}^{n} \sin k\theta = \frac{1}{2}\cot\frac{\theta}{2} - \frac{\cos\left(n+\frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}} = \frac{\sin\frac{n\theta}{2}\sin\frac{(n+1)\theta}{2}}{\sin\frac{\theta}{2}}, \tag{5}$$

$$\sum_{k=1}^{n} \cos k\theta = -\frac{1}{2} + \frac{\sin\left(n+\frac{1}{2}\right)\theta}{2\sin\frac{\theta}{2}} = \frac{\sin\frac{n\theta}{2}\cos\frac{(n+1)\theta}{2}}{\sin\frac{\theta}{2}}, \tag{6}$$

where $n \geq 1$ and $\theta \notin 2\pi\mathbb{Z}$.

Note that identities (3)–(6) were used in [10] to describe the Cartan matrix in complex simple Lie algebra.

Lemma 1. *Let*

$$D_m := [\sin jx_i]_{m\times m},$$

where $x_i \in \mathbb{R}$ *for* $1 \le i \le m$. *Then*

$$\det D_m = 2^{m(m-1)/2} \prod_{k=1}^{m} \sin x_k \prod_{1\le i<j\le m} (\cos x_j - \cos x_i).$$

Proof. The proof can be found in [2, 3]. It is based on transforming D_m to Vandermonde matrix by means of elementary operations. □

Remark 4. *Notice that matrix* Δ_r *is a special case of* D_m *for* $m = \nu_r$ *and* $x_i = 2i\pi/r$.

Lemma 2. *For each natural number* $n \ge 3$ *the equality*

$$\prod_{i=1}^{\nu_n} \cos\frac{i\pi}{n} \prod_{j=1}^{\nu_n}\prod_{k=1}^{2j-1} \sin\frac{k\pi}{n} = \frac{1}{2^{\nu_n^2}} \left(\frac{n}{4}\right)^{\nu_n/2}$$

is valid, where $\nu_n := \left\lfloor \frac{n-1}{2} \right\rfloor$.

Proof. By substituting first $j = \nu_n - i + 1$ and next $k = n - l$ we get

$$\prod_{j=1}^{\nu_n}\prod_{k=1}^{2j-1} \sin\frac{k\pi}{n} = \prod_{i=1}^{\nu_n}\prod_{k=1}^{2\nu_n-2i+1} \sin\frac{k\pi}{n} = \prod_{i=1}^{\nu_n}\prod_{l=n-2\nu_n+2i-1}^{n-1} \sin\frac{(n-l)\pi}{n} = \prod_{i=1}^{\nu_n}\prod_{l=n-2\nu_n+2i-1}^{n-1} \sin\frac{l\pi}{n}. \tag{7}$$

Let us assume first that n is even. Then $n - 2\nu_n = 2$ and from (7) we obtain the equality

$$\prod_{j=1}^{\nu_n}\prod_{k=1}^{2j-1} \sin\frac{k\pi}{n} = \prod_{j=1}^{\nu_n}\prod_{k=2i+1}^{n-1} \sin\frac{k\pi}{n}.$$

Substituting $i = \frac{n}{2} - m$ yields

$$\prod_{i=1}^{\nu_n} \cos\frac{i\pi}{n} = \prod_{m=1}^{\nu_n} \cos\frac{(\frac{n}{2}-m)\pi}{n} = \prod_{m=1}^{\nu_n} \sin\frac{m\pi}{n}.$$

Hence, by the successive algebraic operations we have

$$\begin{aligned}\left(\prod_{i=1}^{\nu_n} \cos\frac{i\pi}{n} \prod_{j=1}^{\nu_n}\prod_{k=1}^{2j-1} \sin\frac{k\pi}{n}\right)^2 &= \prod_{i=1}^{\nu_n} \cos\frac{i\pi}{n} \prod_{i=1}^{\nu_n} \sin\frac{i\pi}{n} \prod_{j=1}^{\nu_n}\prod_{k=1}^{2j-1} \sin\frac{k\pi}{n} \prod_{j=1}^{\nu_n}\prod_{k=2j+1}^{n-1} \sin\frac{k\pi}{n} \\ &= \left(\frac{1}{2}\right)^{\nu_n} \prod_{i=1}^{\nu_n} \sin\frac{2i\pi}{n} \prod_{j=1}^{\nu_n}\prod_{k=1}^{2j-1} \sin\frac{k\pi}{n} \prod_{j=1}^{\nu_n}\prod_{k=2j+1}^{n-1} \sin\frac{k\pi}{n} \\ &= \left(\frac{1}{2}\prod_{k=1}^{n-1} \sin\frac{k\pi}{n}\right)^{\nu_n},\end{aligned}$$

and from (3) we finally get

$$\left(\prod_{i=1}^{\nu_n} \cos\frac{i\pi}{n} \prod_{j=1}^{\nu_n}\prod_{k=1}^{2j-1} \sin\frac{k\pi}{n}\right)^2 = \left(\frac{n}{2^n}\right)^{\nu_n} = \frac{1}{2^{2\nu_n^2}} \left(\frac{n}{4}\right)^{\nu_n}.$$

Now, if we assume that n is odd, then $n - 2\nu_n = 1$ and (7) yields

$$\prod_{j=1}^{\nu_n}\prod_{k=1}^{2j-1} \sin\frac{k\pi}{n} = \prod_{j=1}^{\nu_n}\prod_{k=2i}^{n-1} \sin\frac{k\pi}{n}.$$

Moreover, by substituting $i = n - m$ we obtain

$$\prod_{i=1}^{\nu_n} \cos\frac{i\pi}{n} = \prod_{m=\nu_n+1}^{n-1} \cos\frac{(n-m)\pi}{n} = (-1)^{(n-1)/2} \prod_{m=\nu_n+1}^{n-1} \cos\frac{m\pi}{n},$$

whence

$$\begin{aligned}\left(\prod_{i=1}^{\nu_n} \cos\frac{i\pi}{n} \prod_{j=1}^{\nu_n}\prod_{k=1}^{2j-1} \sin\frac{k\pi}{n}\right)^2 &= (-1)^{(n-1)/2} \prod_{i=1}^{\nu_n} \cos\frac{i\pi}{n} \prod_{i=\nu_n+1}^{n-1} \cos\frac{i\pi}{n} \prod_{j=1}^{\nu_n}\prod_{k=1}^{2j-1} \sin\frac{k\pi}{n} \prod_{j=1}^{\nu_n}\prod_{k=2j}^{n-1} \sin\frac{k\pi}{n}\\ &= (-1)^{(n-1)/2} \prod_{i=1}^{n-1} \cos\frac{i\pi}{n} \prod_{j=1}^{\nu_n}\prod_{k=1}^{n-1} \sin\frac{k\pi}{n}\\ &= (-1)^{(n-1)/2} \prod_{i=1}^{n-1} \cos\frac{i\pi}{n} \left(\prod_{k=1}^{n-1} \sin\frac{k\pi}{n}\right)^{\nu_n}.\end{aligned}$$

From (3) and (4) we have

$$\left(\prod_{i=1}^{\nu_n} \cos\frac{i\pi}{n} \prod_{j=1}^{\nu_n}\prod_{k=1}^{2j-1} \sin\frac{k\pi}{n}\right)^2 = \frac{1}{2^{n-1}}\left(\frac{n}{2^{n-1}}\right)^{\nu_n} = \frac{1}{2^{2\nu_n^2}}\left(\frac{n}{4}\right)^{\nu_n},$$

which finally proves Lemma 2. □

Lemma 3. *For each natural number $n \geq 3$ and each $i, j \in \{1, 2, \dots, \nu_n\}$ we have*

$$\sum_{k=1}^{\nu_n} \sin\frac{2ik\pi}{n} \sin\frac{2kj\pi}{n} = \frac{n\delta_{ij}}{4}, \tag{8}$$

where δ_{ij} denotes the Kronecker delta.

Proof. From the formula for the product of sines, we get

$$\sum_{k=1}^{\nu_n} \sin\frac{2ik\pi}{n} \sin\frac{2kj\pi}{n} = \frac{1}{2}\sum_{k=1}^{\nu_n} \cos\frac{2(i-j)k\pi}{n} - \frac{1}{2}\sum_{k=1}^{\nu_n} \cos\frac{2(i+j)k\pi}{n}. \tag{9}$$

Assume first that $i \neq j$. We have to consider three cases: when n is odd, when n is even and $i+j$ is odd and when both n and $i+j$ are even. We will investigate the second case, the other ones should be discussed analogically. Namely, let n be even and $i+j$ odd. From (6) and (9) we obtain

$$\sum_{k=1}^{\nu_n} \sin\frac{2ik\pi}{n} \sin\frac{2kj\pi}{n} = \frac{\sin\frac{(n-2)(i-j)\pi}{2n}\cos\frac{(i-j)\pi}{2}}{2\sin\frac{(i-j)\pi}{n}} - \frac{\sin\frac{(n-2)(i+j)\pi}{2n}\cos\frac{(i+j)\pi}{2}}{2\sin\frac{(i+j)\pi}{n}} = 0.$$

Now, assume that $i = j$. Then equality (9) takes the form

$$\sum_{k=1}^{\nu_n} \sin\frac{2ik\pi}{n} \sin\frac{2kj\pi}{n} = \frac{\nu_n}{2} - \frac{1}{2}\sum_{k=1}^{\nu_n} \cos\frac{4ik\pi}{n}. \tag{10}$$

There are only two cases to consider: n is either odd or even. If n is odd, then from (6) and (10) we have

$$\sum_{k=1}^{\nu_n} \sin\frac{2ik\pi}{n} \sin\frac{2kj\pi}{n} = \frac{n-1}{4} + \frac{1}{4} - \frac{\sin 2i\pi}{4\sin\frac{2i\pi}{n}} = \frac{n}{4},$$

which finishes the proof. □

Remark 5. *Identity (8) in less general form can be found in [9], in the chapter devoted to the finite sums.*

Lemma 4. *For each natural number $n \geq 3$ and $j \in \{1, 2, \dots, \nu_n\}$ it is true that*

$$\sum_{k=1}^{\nu_n} (n-2k) \sin \frac{2jk\pi}{n} = \frac{n}{2} \cot \frac{j\pi}{n}.$$

Proof. Let us notice that, by introducing the auxiliary variable x, we can write the left-hand side of the examined equality in the following way

$$\sum_{k=1}^{\nu_n} (n-2k) \sin \frac{2ik\pi}{n} = n \sum_{k=1}^{\nu_n} \sin \frac{2ik\pi}{n} + \frac{n}{i\pi} \left(\sum_{k=1}^{\nu_n} \cos \frac{2ik\pi x}{n} \right)' \Big|_{x=1}.$$

From (5) and (6) we get

$$\sum_{k=1}^{\nu_n} (n-2k) \sin \frac{2ik\pi}{n} = \frac{n}{2} \cot \frac{i\pi}{n} - \frac{n \cos \alpha i\pi}{2 \sin \frac{i\pi}{n}} + \frac{n}{i\pi} \left(-\frac{1}{2} + \frac{\sin \alpha i\pi x}{2 \sin \frac{i\pi x}{n}} \right)' \Big|_{x=1}, \tag{11}$$

where $\alpha := (2\nu_n + 1)/n$. Since

$$\left(\frac{\sin \alpha i\pi x}{2 \sin \frac{i\pi x}{n}} \right)' = \frac{\alpha i\pi \cos \alpha i\pi x}{2 \sin \frac{i\pi x}{n}} - \frac{i\pi \cos \frac{i\pi x}{n} \sin \alpha i\pi x}{2n \sin^2 \frac{i\pi x}{n}},$$

we obtain

$$\frac{n}{i\pi} \left(-\frac{1}{2} + \frac{\sin \alpha i\pi x}{2 \sin \frac{i\pi x}{n}} \right)' \Big|_{x=1} = \frac{\alpha n \cos \alpha i\pi}{2 \sin \frac{i\pi}{n}} - \frac{\cos \frac{i\pi}{n} \sin \alpha i\pi}{2 \sin^2 \frac{i\pi}{n}}.$$

Now, if n is even, then we have

$$\begin{aligned} \frac{n}{i\pi} \left(-\frac{1}{2} + \frac{\sin \alpha i\pi x}{2 \sin \frac{i\pi x}{n}} \right)' \Big|_{x=1} &= \frac{n \cos \alpha i\pi}{2 \sin \frac{i\pi}{n}} - \frac{\cos \alpha i\pi}{2 \sin \frac{i\pi}{n}} - \frac{\cos \frac{i\pi}{n} \sin \alpha i\pi}{2 \sin^2 \frac{i\pi}{n}} \\ &= \frac{n \cos \alpha i\pi}{2 \sin \frac{i\pi}{n}} - \frac{\sin \left(\alpha i\pi + \frac{i\pi}{n} \right)}{2 \sin^2 \frac{i\pi}{n}} = \frac{n \cos \alpha i\pi}{2 \sin \frac{i\pi}{n}} - \frac{\sin i\pi}{2 \sin^2 \frac{i\pi}{n}} = \frac{n \cos \alpha i\pi}{2 \sin \frac{i\pi}{n}}. \end{aligned}$$

Similarly, if n is odd, then we get

$$\frac{n}{i\pi} \left(-\frac{1}{2} + \frac{\sin \alpha i\pi x}{2 \sin \frac{i\pi x}{n}} \right)' \Big|_{x=1} = \frac{n \cos \alpha i\pi}{2 \sin \frac{i\pi}{n}} - \frac{\cos \frac{i\pi}{n} \sin i\pi}{2 \sin^2 \frac{i\pi}{n}} = \frac{n \cos \alpha i\pi}{2 \sin \frac{i\pi}{n}},$$

which, on the grounds of (11), gives the desired identity. □

3 Proof of main result

Proof of Theorem 1, formula (1). From Lemma 1 we have

$$\begin{aligned} \det \Delta_r &= 2^{\nu_r(\nu_r-1)/2} \prod_{k=1}^{\nu_r} \sin \frac{2k\pi}{r} \prod_{1 \leq i < j \leq \nu_r} \left(\cos \frac{2j\pi}{r} - \cos \frac{2i\pi}{r} \right) \\ &= 2^{\nu_r(\nu_r+1)/2} \prod_{k=1}^{\nu_r} \cos \frac{k\pi}{r} \prod_{k=1}^{\nu_r} \sin \frac{k\pi}{r} \prod_{1 \leq i < j \leq \nu_r} \left(\cos \frac{2j\pi}{r} - \cos \frac{2i\pi}{r} \right). \end{aligned}$$

From the formula for the difference of cosines we get

$$\begin{aligned} &\prod_{k=1}^{\nu_r} \sin \frac{k\pi}{r} \prod_{1 \leq i < j \leq \nu_r} \left(\cos \frac{2j\pi}{r} - \cos \frac{2i\pi}{r} \right) \\ &= (-2)^{\nu_r(\nu_r-1)/2} \prod_{k=1}^{\nu_r} \sin \frac{k\pi}{r} \prod_{1 \leq i < j \leq \nu_r} \sin \frac{(i+j)\pi}{r} \prod_{1 \leq i < j \leq \nu_r} \sin \frac{(j-i)\pi}{r}. \end{aligned} \tag{12}$$

Next, changing the indices gives

$$\prod_{1\le i<j\le \nu_r} \sin\frac{(i+j)\pi}{r} = \prod_{j=2}^{\nu_r}\prod_{i=1}^{j-1} \sin\frac{(i+j)\pi}{r} = \prod_{j=2}^{\nu_r}\prod_{k=j+1}^{2j-1} \sin\frac{k\pi}{r}$$

and

$$\prod_{1\le i<j\le \nu_r} \sin\frac{(j-i)\pi}{r} = \prod_{j=2}^{\nu_r}\prod_{i=1}^{j-1} \sin\frac{(j-i)\pi}{r} = \prod_{j=2}^{\nu_r}\prod_{k=1}^{j-1} \sin\frac{k\pi}{r}.$$

Hence (12) can be written in the following way

$$\prod_{k=1}^{\nu_r} \sin\frac{k\pi}{r} \prod_{1\le i<j\le \nu_r}\left(\cos\frac{2j\pi}{r} - \cos\frac{2i\pi}{r}\right) = (-2)^{\nu_r(\nu_r-1)/2}\prod_{j=1}^{\nu_r}\prod_{k=1}^{2j-1} \sin\frac{k\pi}{r}.$$

It gives us

$$\det \Delta_r = (-1)^{\nu_r(\nu_r-1)/2} 2^{\nu_r^2} \prod_{k=1}^{\nu_r} \cos\frac{k\pi}{r} \prod_{j=1}^{\nu_r}\prod_{k=1}^{2j-1} \sin\frac{k\pi}{r}.$$

From Lemma 2 we obtain (1). □

Proof of Theorem 1, formula (2). Firstly, we show that

$$\Delta_r^{-1} = \frac{4}{r}\Delta_r.$$

For this purpose let us notice that

$$\frac{4}{r}\Delta_r \cdot \Delta_r = \left[\frac{4}{r}\sin\frac{2ij\pi}{r}\right]_{\nu_r\times\nu_r} \cdot \left[\sin\frac{2ij\pi}{r}\right]_{\nu_r\times\nu_r} = \left[\frac{4}{r}\sum_{k=1}^{\nu_r} \sin\frac{2ik\pi}{r}\sin\frac{2kj\pi}{r}\right]_{\nu_r\times\nu_r} = \left[\delta_{ij}\right]_{\nu_r\times\nu_r},$$

where the last equality results from Lemma 3.
Next, we can write the matrix Δ_r^{-1} in the form

$$\Delta_r^{-1} = \left[\frac{(-1)^{i+j}M_{ji}(\Delta_r)}{\det \Delta_r}\right]_{\nu_r\times\nu_r},$$

where $M_{ij}(A)$ is the (i,j) minor of a matrix A. By comparing it with the values of elements of the matrix Δ_r^{-1} we get

$$M_{ij}(\Delta_r) = (-1)^{i+j}\frac{4}{r}\det \Delta_r \sin\frac{2ij\pi}{r}.$$

In order to compute the determinant of $\Delta_{r,n}$ let us notice that the following equality holds

$$M_{in}(\Delta_{r,n}) = M_{in}(\Delta_r) = (-1)^{i+n}\frac{4}{r}\det \Delta_r \sin\frac{2in\pi}{r}.$$

Hence, by applying the cofactor expansion along the n-th column we have

$$\det \Delta_{r,n} = \sum_{i=1}^{\nu_r}(-1)^{i+n}\Delta_{r,n}^{in}M_{in}(\Delta_{r,n}) = \frac{2\pi}{r^2}\det \Delta_r \sum_{i=1}^{\nu_r}(r-2i)\sin\frac{2in\pi}{r}.$$

At the end, by Lemma 4, the above formula takes the form given in the theorem. □

4 Applications of main result

In this section we present two applications of Theorem 1. The first is algebraic, the second – analytic.

1. Let us consider the following system of linear equations

$$\begin{cases} x_{r,1}\sin\frac{2\pi}{r} + x_{r,2}\sin\frac{4\pi}{r} + \ldots + x_{r,\nu_r}\sin\frac{2\nu_r\pi}{r} = \frac{\pi}{2r}(r-2) \\ x_{r,1}\sin\frac{4\pi}{r} + x_{r,2}\sin\frac{6\pi}{r} + \ldots + x_{r,\nu_r}\sin\frac{4\nu_r\pi}{r} = \frac{\pi}{2r}(r-4) \\ \ldots \\ x_{r,1}\sin\frac{2\nu_r\pi}{r} + x_{r,2}\sin\frac{4\nu_r\pi}{r} + \ldots + x_{r,\nu_r}\sin\frac{2\nu_r^2\pi}{r} = \frac{\pi}{2r}(r-2\nu_r) \end{cases} \tag{13}$$

where $\nu_r := \left\lfloor\frac{r-1}{2}\right\rfloor$ and $r \geq 3$ is natural. Then from Cramer's rule and Theorem 1 we get

$$x_{r,n} = \frac{\det \Delta_{r,n}}{\det \Delta_r} = \frac{\frac{\pi}{r}\det\Delta_r \cot\frac{n\pi}{r}}{\det\Delta_r} = \frac{\pi}{r}\cot\frac{n\pi}{r}.$$

2. Let us consider the sum of the alternating series of the following form

$$S_{r,n} = \sum_{k=0}^{\infty}\left(\frac{1}{kr+n} - \frac{1}{(k+1)r-n}\right),$$

where $r \geq 2$ is natural and $n \in \{1, 2, \ldots, \nu_r\}$.

Let $\gamma \in (0, \pi)$. Applying the theory of Fourier series one can prove that the expansion of the 2π-periodic function

$$f(x) = \begin{cases} 1, & \pi - \gamma < |x| \leq \pi, \\ \frac{1}{2}, & |x| = \pi - \gamma, \\ 0, & |x| < \pi - \gamma, \end{cases}$$

into the cosine series is given by the formula

$$f(x) = \frac{\gamma}{\pi} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin((\pi-\gamma)n)}{n}\cos(nx).$$

In the special case, by substituting $x = \pi - \gamma$ we get

$$\frac{\pi}{2} - \gamma = \sum_{n=1}^{\infty}\frac{1}{n}\sin(2n\gamma),$$

whence by taking $\gamma = \frac{k\pi}{r}$, we obtain the identity

$$\frac{\pi}{2r}(r-2k) = \sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{2kn\pi}{r}. \tag{14}$$

Let us notice that we have

$$\sum_{n=1}^{\nu_r} S_{r,n}\sin\frac{2kn\pi}{r} = \sum_{n=1}^{\lfloor r/2\rfloor} S_{r,n}\sin\frac{2kn\pi}{r},$$

since $\sin\frac{2k\lfloor r/2\rfloor\pi}{r} = 0$ when r is even. Hence

$$\begin{aligned} \sum_{n=1}^{\nu_r} S_{r,n}\sin\frac{2kn\pi}{r} &= \sum_{n=1}^{\lfloor r/2\rfloor}\sin\frac{2kn\pi}{r}\sum_{j=0}^{\infty}\left(\frac{1}{jr+n} - \frac{1}{(j+1)r-n}\right) \\ &= \sum_{n=1}^{\lfloor r/2\rfloor}\sum_{j=0}^{\infty}\left(\frac{\sin\frac{2k(jr+n)\pi}{r}}{jr+n} + \frac{\sin\frac{2k((j+1)r-n)\pi}{r}}{(j+1)r-n}\right) = \sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{2kn\pi}{r} \overset{(14)}{=} \frac{\pi}{2r}(r-2k). \end{aligned}$$

Therefore $S_{r,n}$ are the solutions to the system (13), so we have

$$S_{r,n} = \frac{\pi}{r} \cot \frac{n\pi}{r}, \tag{15}$$

for natural $r \geq 2$ and $n \in \{1, 2, \ldots, \nu_r\}$.

Remark 6. *Formula (15) can be also derived in a purely analytical way, from the residue theory, which will be omitted here.*

Remark 7. *In [11], series of similar type as above are considered. But the Author focused on increasing the speed of their convergence, and not on the form of sum of these series.*

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