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On nearly Hurewicz spaces

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Abstract: In this paper we define and investigate nearly Hurewicz spaces and their star version. It is shown that a nearly Hurewicz space fits between Hurewicz and almost Hurewicz spaces. As a counter example it is shown that a particular point topology which fails to be Lindelöf, Menger, Hurewicz is a nearly Hurewicz space.

Keywords: Hurewicz space, nearly Hurewicz space, nearly star-Hurewicz space, nearly strongly star-Hurewicz space

MSC: 54D20, 54C10, 54C08

1 Introduction

In 1996, Scheepers restructured classical selection principles and started an efficient examination of the selection principles in topology. For selected results on selection principles, see [1–4]. Various topological properties are defined or characterized in terms of these selection principles.

Let N denote the set of positive integers, X be a topological space and $\mathcal{A}$, $\mathcal{B}$ be collections of open covers of X:

Hurewicz space is a topological space which satisfies a particular basic selection principle that generalizes σ–compactness.

A classical Hurewicz covering property $U_{fin}(\mathcal{A}, \mathcal{B})$ is:

For every sequence of open covers $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \ldots, \mathcal{U}_n, \ldots$of the space X by elements of $\mathcal{A}$, there exist finite sets $\mathcal{F}_1 \subset \mathcal{U}_1, \mathcal{F}_2 \subset \mathcal{U}_2, \mathcal{F}_3 \subset \mathcal{U}_3, \ldots, \mathcal{F}_n \subset \mathcal{U}_n, \ldots$ such that for each $x \in X$, x belongs to all but finitely many $\cup\mathcal{F}_1, \cup\mathcal{F}_2, \cup\mathcal{F}_3$, ,...,$\cup\mathcal{F}_n, \ldots$. This property of topological spaces was introduced by Witold Hurewicz [5] in the year 1926. As a consequence Bonanzinga et al. in 2004 in [6] introduced two star versions of the Hurewicz property as follows:

SH: A space X satisfies the star-Hurewicz property $U^{\star}_{fin}(\mathcal{A}, \mathcal{B})$ if for each sequence $(\mathcal{A}_n : n \in N)$ of open covers of X by elements of $\mathcal{A}$, there exists a sequence $(\mathcal{B}_n : n \in N)$ such that for each n, $\mathcal{B}_n$ is a finite subset of $\mathcal{A}_n$ and for each $x \in X$, $x \in St(\cup\mathcal{B}_n, \mathcal{A}_n)$ for all but finitely many n.

SSH: A space X satisfies the strongly star-Hurewicz property $SU^{\star}_{fin}(\mathcal{A}, \mathcal{B})$ if for each sequence $(\mathcal{A}_n : n \in N)$ of open covers of X by elements of $\mathcal{A}$, there exists a sequence $(\mathcal{F}_n : n \in N)$ of finite subsets of X such that for each $x \in X$, $x \in St(\mathcal{F}_n, \mathcal{A}_n)$ for all but finitely many n.

SSM: A space X satisfies the Strongly star-Menger property $SS^{\star}_{fin}(\mathcal{A}, \mathcal{B})$ if for each sequence $(\mathcal{A}_n : n \in \mathbf{N})$ of open covers of X by elements of $\mathcal{A}$, there is a sequence $(\mathcal{F}_n : n \in \mathbf{N})$ such that for all $n \in \mathbf{N}$, F_n is a finite subset of X, and $\cup_{n\in\mathbf{N}}\{\mathbf{St}(F, \mathcal{A}_n) : F \in \mathcal{F}_n\}$ is an element of $\mathcal{B}$.

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Definition 1.1. *[6] A space X is said to be strongly star-Hurewicz (star-Hurewicz) if it satisfies the selection hypothesis SSH (resp., SH).*

On the study of star-Hurewicz spaces, the readers can see the references [1, 6–10].

As a generalization of Hurewicz spaces, the authors [11] defined a space X to be almost Hurewicz if for each sequence $(\mathcal{U}_n : n \in N)$ of open covers of X there exists a sequence $(\mathcal{V}_n : n \in N)$ such that for each $n \in N$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each $x \in X$, $x \in \cup\{cl(V) : V \in \mathcal{V}_n\}$ for all but finitely many n. Kočinac in [12] defined (see also [13]) a space X to be weakly Hurewicz if for each sequence $(\mathcal{U}_n : n \in N)$ of open covers of X, there is a dense subset $Y \subseteq X$ and a sequence $(\mathcal{V}_n : n \in N)$ such that for each n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each $y \in Y$, $y \in \cup\mathcal{V}_n$ for all but finitely many n. We note that every Hurewicz space is almost Hurewicz space and every almost Hurewicz space is weak Hurewicz. In [11] it is shown that every regular almost Hurewicz space is Hurewicz where as a Urysohn almost Hurewicz space fails to be Hurewicz.

We note that in a topological space X:

1) An open cover $\mathcal{U}$ of X is a γ–cover if it is infinite and for every $x \in X$, $\{U \in \mathcal{U} : x \notin U\}$ is finite.

2) An open cover $\mathcal{U}$ of X is an ω–cover if $X \notin \mathcal{U}$ and every finite subset F of X is contained in some $U \in \mathcal{U}$.

3) An $\underline{\omega}$–cover $\mathcal{U}$ is a cover such that X does not belong to the cover $\mathcal{U}$ and every finite subset F of X is such that $F \subseteq scl(U)$ for some $U \in \mathcal{U}$.

We use symbols Γ, Ω, $\underline{\Omega}$ to denote the collection of all γ and ω– and $\underline{\omega}$–covers respectively.

Definition 1.2. *[14] A topological space X is a γ–set if for each sequence $\{\mathcal{U}_n : n \in N\}$ of ω–covers of X there exists a sequence $\{\mathcal{V}_n : n \in N\}$ such that for every $n \in N$, $\mathcal{V}_n \in \mathcal{U}_n$ and $\{\mathcal{V}_n : n \in N\}$ is a γ–cover of X.*

In 1963 Levine in [15] characterized semi open sets in topological spaces. Consequently numerous mathematicians summed up various ideas and examined their properties. A set $S \subset X$ is semi open in a space (X, τ) if and only if $S \subset cl(int(S))$. If S is semi open, then its complement is semi closed [16]. Every open set is always semi open but a semi open set may or may not be an open set. $SO(X)$ denotes the collection of all semi open subsets of X. According to Crossley [16], semi closure and semi interior were defined analogus to closure and interior. A set S is semi open if and only if $S = sInt(S)$, where $sInt(S)$ denotes the semi interior of S in the space X and is the union of all semi open sets contained in S. A set T is semi closed if and only if $T = scl(T)$, where $scl(T)$ denotes the semi closure of T in the space X and is the intersection of all semi closed sets containing T. It is known that for any subset S of X, $Int(S) \subseteq sInt(S) \subseteq S \subseteq scl(S) \subseteq cl(S)$.

Definition 1.3. *A space X is called nearly-compact [17] if for every open cover $\mathcal{U}$ of X has a finite subcollection $\mathcal{V}$ such that $\cup_{V\in\mathcal{V}} int(cl(V)) = X$.*

Definition 1.4. *A space X to be semi-Hurewicz [18] if for each sequence $(\mathcal{U}_n : n \in N)$ of semi open covers of X there exists a sequence $(\mathcal{V}_n : n \in N)$ such that for each $n \in N$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each $x \in X$, $x \in \cup\{V : V \in \mathcal{V}_n\}$ for all but finitely many n.*

The purpose of this paper is to define and investigate topological properties of nearly Hurewicz spaces, nearly star Hurewicz spaces and nearly strongly star Hurewicz spaces.

2 Nearly Hurewicz spaces

Definition 2.1. *A space X is said to have nearly Hurewicz property if for each sequence $(\mathcal{U}_n : n \in N)$ of open covers of X there exists a sequence $(\mathcal{V}_n : n \in N)$ such that for every $n \in N$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each $x \in X$, $x \in \cup\{int\left(cl(V)\right) = scl(V) : V \in \mathcal{V}_n\}$ for all but finitely many n.*

We notice that every Hurewicz space is a nearly Hurewicz space, and every nearly Hurewicz space is almost Hurewicz.

Remark 2.2.

$$\text{Hurewicz} \underset{\not\Leftarrow}{\Rightarrow} \text{nearly Hurewicz} \underset{\not\Leftarrow}{\Rightarrow} \text{almost Hurewicz} \underset{\not\Leftarrow}{\Rightarrow} \text{weakly Hurewicz} \tag{1}$$

Example 2.3. *(1) Real line with the usual Euclidean topology is Hurewicz so is nearly Hurewicz.*

(2) Real line with the cocountable topology is Hurewicz so is nearly Hurewicz.

Example 2.4. *Sorgenfrey line is not almost Menger (see Example 6d, [19]) it can not be almost Hurewicz and hence Sorgenfrey line is not nearly Hurewicz.*

Lemma 2.5. *[20] In a topological space X if O is open, then $scl(O) = int\left(cl(O)\right)$.*

Example 2.6. *Let X be an uncountable set and $p \in X$. Then $\mathcal{T}_p = \{O \subseteq X; p \in O \text{ or } O = \phi\}$ is uncoutable particular point topology on X. Uncountable particular point topology is not Lindelöf [21] so it can not be Menger and can not be Hurewicz because every Menger space is Lindelöf and every Hurewicz space is Menger. To show that X is nearly Hurewicz we will show that for each $x \in X$, $\{O \in \mathcal{T} : x \notin int\left(cl(O)\right)\}$ is a finite subcollection. As for $A \subseteq X$ and $A \in \mathcal{T}_p$, implies $p \in A$. Thus no closed set other than X contains p. Hence closure of any open set except than ϕ is X. This implies $cl(A) = X$. Therefore the collection of interior of closure of open sets is $\{\phi, X\}$ and $x \notin \phi$, for all $x \in X$. Hence X is nearly Hurewicz.*

Example 2.7. *Let R be the set of reals numbers, I the set of irrational numbers and Q the set of rational numbers and for each irrational x we choose a sequence $\{r_i : i \in N\}$ of rational numbers converging to x in the Euclidean topology. The rational sequence topology τ is then defined by declaring both R and ϕ to be open, each rational open and selecting the sets $U_\alpha(x) = \{x_{\alpha,i} : i \in N\} \cup \{x\}$ as a basis for the irrational point x. If $r \in Q$, then the closure of $\{r\}$ with respect to τ is equal $\{r\}$, and for every $x \in I$, the closure of $U_{\alpha(x)}$ is equal $U_{\alpha(x)}$. For every $n \in N$, $\mathcal{U}_n = \{r : r \in Q\} \cup \{U_{n(x)} : x \in I\}$ is an open cover of (R, τ). (R, τ) does not have the almost (nearly) Hurewicz property because (R, τ) is not almost(nearly) Menger [22]. On the other hand, (R, τ) is weakly Hurewicz, because Q is dense in (R, τ) and each $x \in Q$ belongs to $cl(\cup\mathcal{V}_n)$ for all n and $scl(\cup\mathcal{V}_n) = int(cl(\cup\mathcal{V}_n)) = R$.*

Definition 2.8. *A subset B of a topological space X is called s-regular open (resp. s-regular closed) if $B = int(scl(B))(resp.\ B = cl(sInt(B)))$.*

Note that every s-regular open set is open and semi closed. If A is open, then $cl(A)$ is s-regular closed set.

Theorem 2.9. *A topological space X is nearly Hurewicz if and only if for each sequence $(\mathcal{U}_n : n \in N)$ of covers of X by s-regular open sets, there exists a sequence $(\mathcal{V}_n : n \in N)$ such that for every $n \in N$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and each $x \in X$, $x \in \cup\{V : V \in \mathcal{V}_n\}$ for all but finitely many n.*

Proof. Let X be a nearly Hurewicz space. Let $(\mathcal{U}_n : n \in N)$ be a sequence of covers of X by s-regular open sets. By assumption, there exists a sequence $(\mathcal{V}_n : n \in N)$ such that for every $n \in N$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and each $x \in X$, $x \in \cup\{V = int\left(cl(V)\right) : V \in \mathcal{V}_n\}$ for all but finitely many n.

Conversely, let $(\mathcal{U}_n : n \in N)$ be a sequence of open covers of X. Let $(\mathcal{U}'_n : n \in N)$ be a sequence defined by $\mathcal{U}'_n = \{Int(scl(U)) : U \in \mathcal{U}_n\}$. Then each $\mathcal{U}'_n$ is a cover of X by s-regular open sets.

By hypothesis there exists a sequence $(\mathcal{V}_n : n \in N)$ such that for every $n \in N$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}'_n$ and each $x \in X$, $x \in \cup\{V : V \in \mathcal{V}_n\}$ for all but finitely many n. By construction, for each $n \in N$ and $V \in \mathcal{V}_n$ there exists $U_V \in \mathcal{U}_n$ such that $V = Int(scl(U_V))$. Since $Int(scl(U_V)) \subseteq scl(U_V) = Int(cl(U_V))$ hence, each $x \in X$, $x \in \cup\{int\left(cl(U_V)\right) : V \in \mathcal{V}_n\}$ for all but finitely many n. This implies that X is a nearly Hurewicz space. □

Theorem 2.10. *For a topological space X the following statements are equivalent:*

(1) *X is nearly Hurewicz;*
(2) *X satisfies* $\mathsf{U}_{\mathrm{fin}}(\mathcal{RO}, \mathcal{RO})$,

where, $\mathcal{RO}$ denotes the collection of regular open sets.

Proof. (1) $\Rightarrow$ (2) Let $(\mathcal{U}_n : n \in N)$ be a sequence of regular open covers of X. Since X is nearly Hurewicz space there exists a sequence $(\mathcal{V}_n : n \in N)$ such that for each n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each $x \in X$ $x \in \bigcup_{n\in N} \bigcup\{Int(Cl(V) : V \in \mathcal{U}_n\}$ for all but finitely many n. Since $Int(Cl(V)) = V$ for each n and each $V \in \mathcal{V}_n$ we conclude that (2) is satisfied.

(2) $\Rightarrow$ (1) Let $(\mathcal{U}_n : n \in N)$ be a sequence of open covers of X. Define for each $n \in N$, $\mathcal{V}_n = \{Int(Cl(U)) : U \in \mathcal{U}_n\}$. Then $(\mathcal{V}_n : n \in N)$ is a sequence of regular open covers of X. By (2) there is a sequence $(\mathcal{W}_n : n \in N)$ such that $\mathcal{W}_n$ is a finite subset of $\mathcal{V}_n$ for each $n \in N$, and each $x \in X$ belongs to $\bigcup_{n\in N} \bigcup\{W : W \in \mathcal{W}_n\}$ for all but finitely many n. Pick for each n and each $W \in \mathcal{W}_n$ a set $U_W \in \mathcal{U}_n$ with $W = Int(Cl(U_W))$ and set $\mathcal{H}_n = \{U_W : W \in W_n\}$, a finite subset of $\mathcal{U}_n$. Since $\bigcup_{n\in N} \bigcup\{W : W \in \mathcal{W}_n\} = \bigcup_{n\in N} \bigcup\{Int(Cl(U_W)) : W \in \mathcal{W}_n\}$ we conclude that X is nearly Hurewicz. □

Definition 2.11. *Let X and Y be topological spaces. A mapping $f : X \longrightarrow Y$ is nearly continuous if for each s-regular open set $B \subset Y$, $f^{-1}(B)$ is open in X. Every continuous mapping is nearly continuous.*

Lemma 2.12. *If $f : X \longrightarrow Y$ is nearly continuous and open mapping, then for every s-regular open set B in Y, $int\left(cl(f^{-1}(B))\right) \subseteq f^{-1}(int\left(cl(B)\right))$.*

Lemma 2.13. *If $f : X \longrightarrow Y$ is nearly continuous and open mapping, then for every open set A in Y, $f\left(int\left(cl((A))\right)\right) \subseteq int\left(cl(f(A))\right)$.*

Theorem 2.14. *Let $f : X \longrightarrow Y$ be a nearly continuous open mapping from a nearlry Hurewicz space X onto Y. Then Y is nearly Hurewicz.*

Proof. Let $(\mathcal{U}_n : n \in N)$ be a sequence of covers of Y by s-regular open sets. Let $\mathcal{U}'_n = \{f^{-1}(U) : U \in \mathcal{U}_n\}$ for each $n \in N$. Then $(\mathcal{U}'_n : n \in N)$ is a sequence of open covers of X. By hypothesis, there exists a sequence $(\mathcal{V}_n : n \in N)$ such that for every $n \in N$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}'_n$ and each $x \in X$, $x \in \cup\{int\left(cl(V)\right) : V \in \mathcal{V}_n\}$ for all but finitely many n. For each $n \in N$ and $V \in \mathcal{V}_n$ we can choose $U_V \in \mathcal{U}_n$ such that $V = f^{-1}(U_V)$. Let $\mathcal{W}_n = \{U_V : V \in \mathcal{V}_n\}$. If $y = f(x) \in Y$, then there exists $n \in N$ and $V \in \mathcal{V}_n$ such that $x \in scl(V)$. Therefore $x \in int\left(cl(f^{-1}(U_V))\right) \subset f^{-1}(int\left(cl(U_V)\right)) = f^{-1}(scl(U_V)) = f^{-1}(U_V)$. Hence $y = f(x) \in U_V \in \mathcal{W}_n$ for all but finitely many n. □

Definition 2.15. *Let X and Y be topological spaces. A mapping $f : X \longrightarrow Y$ is nearly open if $f^{-1}(int\left(cl(B)\right)) \subseteq int\left(cl(f^{-1}(B))\right)$ for any subset B of Y.*

Theorem 2.16. *If $f : X \to Y$ is nearly open and perfect continuous mapping and Y is a nearly Hurewicz space, then X is a nearly Hurewicz space.*

Proof. Let $(\mathcal{U}_n : n \in N)$ be a sequence of open covers of X. Then due to perfect continuity of f, for each $y \in Y$ and every $n \in N$, there is a finite subfamily $\mathcal{U}_{ny}$ of $\mathcal{U}_n$ such that $f^{-1}(y) \subset \cup\mathcal{U}_{n_y}$. Let $U_{n_y} = \cup\mathcal{U}_{n_y}$. Then $V_{n_y} = Y - f(X \setminus U_{n_y})$ is an open neighborhood of y, since f is closed. For every $n \in N$, $\mathcal{V}_n = \{V_{n_y} : y \in Y\}$, is an open cover of Y. Y is nearly Hurewics so there exist a sequence $(\mathcal{V}'_n : n \in N)$ such that for every $n \in N$, $\mathcal{V}'_n$ is a finite subset of $\mathcal{V}_n$ and for each $y \in Y$, $y \in \cup\{int\left(cl(V)\right) : V \in \mathcal{V}'_n\}$ for all but finitely many n. We may assume $\mathcal{V}'_n = \{V_{n_{y_i}} : i \le n'\}$ for each $n \in N$. For each $n \in N$, let $\mathcal{U}'_n = \cup_{i\le n'}\mathcal{U}_{n_{y_i}}$. Then $\mathcal{U}'_n$ is a finite subset of $\mathcal{U}_n$. Since f is nearly open. Then

$$x \in f^{-1}(\cup_{n>n_o} \cup \{int\left(cl(V_{n_{y_i}})\right) : i \le n'\}) = \cup_{n>n_o} \cup \{f^{-1}(int\left(cl(V_{n_{y_i}})\right)) : i \le n'\} \tag{2}$$

$$\subset \cup_{n>n_o} \cup \{int\left(cl(f^{-1}(V_{n_{y_i}}))\right) : i \le n'\} \subset \cup_{n>n_o} \cup \{int\left(cl(U_{n_{y_i}})\right) : i \le n'\} \tag{3}$$

$$= \cup_{n>n_o} \cup \{int\left(cl(\cup\mathcal{U}_{n_{y_i}})\right) : i \le n'\} = \cup_{n>n_o} \cup \{int\left(cl(U)\right) : U \in \mathcal{U}'_n\}. \tag{4}$$

Hence X is nearly Hurewicz. □

Corollary 2.17. *A continuous open surjective image of a nearly Hurewicz space is nearly Hurewicz.*

Definition 2.18. *Let X and Y be topological spaces. A mapping $f : X \longrightarrow Y$ is sr–cotinuous if the inverse image of each open set V is s-regular open.*

Theorem 2.19. *An sr–continuous surjective image of a nearly Hurewicz space X is Hurewicz.*

Proof. Let $(\mathcal{V}_n : n \in N)$ be a sequence of open covers of Y. Since f is sr–continuous, for each $n \in N$ and each $V \in \mathcal{V}_n$ the set $f^{-1}(V)$ is s-regular open. Therefore, for each n, the set $\mathcal{U}_n = \{f^{-1}(V) : V \in \mathcal{V}_n\}$ is a cover of X by s-regular open sets. As X is a nearly Hurewicz there exists a sequence $(\mathcal{G}_n : n \in N)$ such that for each n, $\mathcal{G}_n$ is a finite subset of $\mathcal{U}_n$ and each $x \in X$, $x \in \cup\{G : G \in \mathcal{G}_n\}$ for all but finitely many n. Then $\mathcal{W}_n = \{f(G) : G \in \mathcal{G}_n\}$ is a finite subset of $\mathcal{V}_n$ for each $n \in N$ and $y \in \mathcal{W}_n$ for $n > n_o \in N$. This means that Y is a Hurewicz space. □

Definition 2.20. *A mapping $f : X \longrightarrow Y$ is strongly $s - \theta$ continuous if for each $x \in X$ and each open set V in Y containing $f(x)$ there is an open set U in X containing x such that $f(int(cl(U)))$ is a subset of V.*

Theorem 2.21. *Let $f : X \longrightarrow Y$ be strongly s-θ continuous surjection and X be a nearly Hurewicz space, then Y is Hurewicz.*

Proof. Let $(\mathcal{V}_n : n \in N)$ be a sequence of open covers of Y. Let $x \in X$. For each $n \in N$ there is a set $V_{x,n} \in \mathcal{V}_n$ containing $f(x)$. Since f is strongly s-θ-continuous there is an open set $U_{x,n}$ in X containing x such that $f\left(int\left(cl(U_{x,n})\right)\right)$ is a subset of $V_{x,n}$. Therefore for each n, the set $\mathcal{U}_n = \{U_{x,n} : x \in X\}$ is an open cover of X. As X is nearly Hurewicz space so there exists a sequence $(\mathcal{F}_n)_{n\in N}$ of finite sets such that for each n, $\mathcal{F}_n$ is a subset of $\mathcal{U}_n$ and each $x \in X$, $x \in \cup\{int\left(cl(F)\right) : F \in \mathcal{F}_n\}$ for $n > n_o \in N$. To each $F \in \mathcal{F}_n$ assign a set $W_F \in \mathcal{V}_n$ with $f\left(int\left(cl(F)\right)\right)$ a subset of W_F and put $\mathcal{W}_n = \{W_F : F \in \mathcal{F}_n\}$. We obtain the sequence $(\mathcal{W}_n)_{n\in N}$ of finite subsets of $\mathcal{V}_n$ $n \in N$, which witnesses for $(\mathcal{V}_n : n \in N)$ that Y is a Hurewicz space. □

Definition 2.22. *[23] A mapping $f : X \longrightarrow Y$ is called weakly continuous, if for each open set U in X containing x and there exists an open set V in Y containing $f(x)$ such that $f(U) \subseteq cl(V)$.*

Definition 2.23. *A mapping $f : X \longrightarrow Y$ is called s-weakly continuous, if for each open set U in X containing x there exists an open set V in Y containing $f(x)$ such that $f(U) \subseteq int(cl(V))$.*

Theorem 2.24. *Let $f : X \longrightarrow Y$ be an s-weakly continuous surjection and X be a Hurewicz space the Y is nearly Hurewicz.*

Proof. Let $(\mathcal{V}_n : n \in N)$ be a sequence of open covers of Y. Let $x \in X$. For each $n \in N$ and each open set $U_{x,n}$ containing x there is a set $V_{x,n} \in \mathcal{V}_n$ containing $f(x)$ such that $f(U_{x,n}) \subseteq int\left(cl(V_{x,n})\right)$. The set $\mathcal{U}_n = \{U_{x,n} : x \in X\}$ is an open cover of X. Apply the fact that X is a Hurewicz space to the sequence $(\mathcal{U}_n)_{n\in N}$ and find a sequence $(\mathcal{F}_n)_{n\in N}$ such that for each n, $\mathcal{F}_n$ is a finite subset of $\mathcal{U}_n$ and each $x \in X$, x$\in \cup(\mathcal{F}_n)_{n>n_o\in N}$. To each n and each $F \in \mathcal{F}_n$ assign a set $V_F \in \mathcal{V}_n$ such that $f(F)$ is a subset $int\left(cl(V_F)\right)$ and put $\mathcal{W}_n = \{V_F : F \in \mathcal{F}_n\}$. Then

$$y = f(x) \in f(\cup_{n>n_o} \cup \{F : F \in \mathcal{F}_n\}) \tag{5}$$

$$\subset \cup_{n>n_o} \cup \{int\left(cl(V_F)\right) : F \in \mathcal{F}_n\}, \tag{6}$$

that is Y is a nearly Hurewicz space. □

Definition 2.25. *An open cover $\mathcal{U}$ of a space X is a nearly γ–cover if it is infinite and for every $x \in X$, $\{U \in \mathcal{U}: x \notin int\left(cl(U)\right)\}$ is finite.*

Definition 2.26. *A topological space X is a nearly γ–set if for each sequence $(\mathcal{U}_n : n \in N)$ of ω–covers of X there exists a sequence $(\mathcal{V}_n : n \in N)$ such that for every $n \in N$, $\mathcal{V}_n \in \mathcal{U}_n$ and $\{\mathcal{V}_n : n \in N\}$ is a nearly γ–cover of X.*

Definition 2.27. *A mapping $f : X \longrightarrow Y$ is s-θ-continuous if for each $x \in X$, and each open set V in Y containing $f(x)$ there is an open set U containing x such that $f(int\left(cl(U)\right)) \subseteq int\left(cl(V)\right)$*

Theorem 2.28. *Let $f : X \longrightarrow Y$ be an s-θ-continuous surjection and let X be a nearly γ–set. Then Y is a nearly Hurewicz space.*

Proof. Let $(\mathcal{V}_n : n \in N)$ be a sequence of open covers of Y and $x \in X$. For each $n \in N$ there is a set $V_{x,n} \in \mathcal{V}_n$ containing $f(x)$. By assumption there is an open set $U_{x,n}$ in X containing x and $f(int\left(cl(U_{x,n})\right))$ is a subset of $int\left(cl(V_{x,n})\right)$. For each n let $\mathcal{U}_n$ be the set of all finite unions of sets $\{U_{x,n} : x \in X\}$. Evidently each $\mathcal{U}_n$ is an $\omega - cover$ of X. X is a nearly γ–set so there exists a sequence $(U_n)_{n\in N}$ such that for each n, $U_n \in \mathcal{U}_n$ and for each $x \in X$, the set $\{n \in N : x \notin int\left(cl(U_n)\right)\}$ is finite.

Let $U_n = U_{x_1,n} \cup U_{x_2,n} \cup \ldots \cup U_{x_{i(n)},n}$. To each $U_{x_j,n}$, $j \le i(n)$, assign a set $V_{x_j,n} \in \mathcal{V}_n$ with $f(int\left(cl(U_{x_j,n})\right)) \subset int\left(cl(V_{x_j,n})\right)$. Let $y = f(x) \in Y$. Then from $x \in int(cl(U_n))$ for all n$\ge$ n_o for some $n_o \in N$, we get $x \in int\left(cl(U_{x_p,n})\right)$ for some $1 \le p \le i(n)$ which implies $y \in f(int\left(cl(U_{x_p,n})\right)) \subseteq int\left(cl(V_{x_p,n})\right)$. If we put $\mathcal{W}_n = \{V_{x_j,n} : j = 1, 2, ..i(n)\}$, we obtain the sequence $(\mathcal{W}_n; n \in N)$ of finite subsets of $\mathcal{V}_n$, $n \in N$, such that each $y \in Y$ belongs to all but finitely many sets of $\cup\{int\left(cl(W)\right) : W \in \mathcal{W}_n\}$, that is Y is a nearly Hurewicz space. □

3 Nearly Star-Hurewicz spaces

Definition 3.1. *A topological space X is a nearly star-Hurewicz if for each sequence $\{\mathcal{U}_n : n \in N\}$ of open covers of X there exists a sequence $\{\mathcal{V}_n : n \in N\}$ such that for every $n \in N$, $\mathcal{V}_n$ is finite subset of $\mathcal{U}_n$ and each $x \in X$, $x \in \cup\{int\left(cl\left(St(\cup\mathcal{V}_n, \mathcal{U}_n)\right)\right) : n \in N\}$ for all but finitely many n.*

Theorem 3.2. *A topological space X is nearly star-Hurewicz if and only if for each sequence $(\mathcal{U}_n : n \in N)$ of covers of X by s-regular open sets, there exists a sequence $(\mathcal{V}_n : n \in N)$ such that for every $n \in N$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and each $x \in X$, $x \in \cup\{St(\cup\mathcal{V}_n, \mathcal{U}_n) : n \in N\}$ for all but finitely many n.*

Proof. Since every s-regular open set is open so direct part is obvious.

Converse: We will prove that X is a nearly star-Hurewicz space. Let $(\mathcal{U}_n : n \in N)$ be a sequence of open covers of X . Let $(\mathcal{U}'_n : n \in N)$ be a sequence defined by $\mathcal{U}'_n = \{int(scl(U)) : U \in \mathcal{U}_n\}$. Then each $\mathcal{U}'_n$ is a cover of X by s-regular open sets. By assumption**,** there exists a sequence $(\mathcal{V}_n : n \in N)$ such that for every $n \in N$**,** $\mathcal{V}_n$ is a finite subset of $\mathcal{U}'_n$ and and each $x \in X$, $x \in \cup\{int\left(cl\left(St(\cup\mathcal{V}_n, \mathcal{U}'_n)\right)\right) : n \in N\}$ for all but finitely many n.

Claim 1: $St(U, \mathcal{U}_n) = St(int(scl(U)), \mathcal{U}_n)$, for all $U \in \mathcal{U}_n$.

Since $U \subset int(scl(U))$, it is obvious that $St(U, \mathcal{U}_n) \subset St(int(scl(U)), \mathcal{U}_n)$. Now let $x \in St(int(scl(U)), \mathcal{U}_n)$ then there exists $V \in \mathcal{U}_n$ such that $x \in V$ and $V \cap int(scl(U)) \neq \emptyset$. Then we have $V \cap U \neq \emptyset$ which implies $x \in St(U, \mathcal{U}_n)$ so $St(Int(scl(U)), \mathcal{U}_n) \subset St(U, \mathcal{U}_n)$. Now for every $V \in \mathcal{V}_n$ we can choose $U_V \in \mathcal{U}_n$ such that $V = Int(scl(U_v))$, by construction. Let $\mathcal{W}_n = \{U_V : V \in \mathcal{V}_n\}$. We shall prove that $x \in \cup\{int\left(cl(St(\cup\mathcal{W}_n, \mathcal{U}_n))\right) : n \in N\}$ for all $n \ge n_0$.

Let $x \in X$ then there exists $n \in N$ such that $x \in int\left(cl\left(St(\cup\mathcal{V}_n, \mathcal{U}'_n)\right)\right)$. For every neighbourhood V of X, we have $V\cap$ $St(\cup\mathcal{V}_n, \mathcal{U}'_n) \neq \phi$. Then there exist $U \in \mathcal{U}_n$ such that $(V \cap int(scl(U) \neq \phi)$ and $(\cup\mathcal{V}_n \cap Int(scl(U)) \neq \phi)$ this implies $(V \cap U \neq \phi)$ and $(\cup\mathcal{V}_n \cap U \neq \emptyset)$. By claim 1 we have that $\cup\mathcal{W}_n \cap U \neq \phi$, so $x \in int(cl(St(\cup\mathcal{W}_n, \mathcal{U}_n)))$ for all but finitely many n. □

Theorem 3.3. *The product $X \times Y$ of a nearly star-Hurewicz space X and a nearly compact space Y is nearly star Hurewicz.*

Theorem 3.4. *The nearly continuous open surjective image of a nearly star Hurewicz space is nearly star-Hurewicz.*

Proof. Let $f : X \to Y$ be nearly continuous open surjection and X is nearly star Hurewicz. Let $(\mathcal{U}_n : n \in N)$ be a sequence of covers of Y by s-regular open sets. Let $\mathcal{U}'_n = \{f^{-1}(U) : U \in \mathcal{U}_n\}$ for each $n \in N$. Then $(\mathcal{U}'_n : n \in N)$ is a sequence of open covers of X, by hypothesis, there exists a sequence $(\mathcal{V}'_n : n \in N)$ such that for every $n \in N$, $\mathcal{V}'_n$ is a finite subset of $\mathcal{U}'_n$ and each $x \in X$, $x \in \cup\{int\left(cl(St(\cup\mathcal{V}'_n, \mathcal{U}'_n))\right) : n \in N\}$ for all but finitely many n.

Let $\mathcal{V}_n = \{U : f^{-1}(U) \in \mathcal{V}'_n\}$. $f^{-1}(\cup\mathcal{V}_n) = \cup\mathcal{V}'_n$, and let $x \in X$, then there is $n \in N$ such that $x \in int\left(cl(St(f^{-1}(\cup\mathcal{V}_n), \mathcal{U}'_n)\right)$ for all but finitely many n. If $y = f(x) \in Y$, then $y \in f(int\left(cl(St(f^{-1}(\cup\mathcal{V}_n), \mathcal{U}'_n))\right) \subseteq int\left(cl(f(St(f^{-1}(\cup\mathcal{V}_n), \mathcal{U}'_n))\right) \subseteq int\left(cl(St(\cup\mathcal{V}_n, \mathcal{U}_n))\right)$ for all but finitely many n. We will prove the last inclusion:

Suppose that $f^{-1}(\cup\mathcal{V}_n) \cap f^{-1}(U) \neq \phi$. Then also $f(f^{-1}(\cup\mathcal{V}_n)) \cap f(f^{-1}(U)) \neq \phi$, so $\cup\mathcal{V}_n \cap U \neq \phi$.

So, the sequence $(\mathcal{V}_n : n \in N)$ witnesses that X is a nearly star-Hurewicz. □

Theorem 3.5. *If each finite power of a space X is nearly star-Hurewicz, then X satisfies $U^{\star}_{fin}(\mathcal{O}, \underline{\Omega})$.*

Proof. Let $(\mathcal{U}_n : n \in N)$ be a sequence of open covers of X and consider $N = N_1 \cup N_2 \cup \ldots$ be a partition of N into infinitely many pairwise disjoint sets. for every $k \in N$ and every $j \in N_k$. Let $\mathcal{W}_j = \{U_1 \times U_2 \times \ldots \times U_k : U_1, U_2, \ldots, U_k \in \mathcal{U}_j\} = \mathcal{U}^k_j$. Then $(\mathcal{W}_j : j \in N_k)$ is a sequence of open covers of X^k. Since X^k is nearly star-Hurewicz so we can choose a sequence $(\mathcal{H}_j : j \in N_k)$ such that for each j, $\mathcal{H}_j$ is finite subset of $\mathcal{W}_j$ and each $x \in X^k$, $x \in \cup\{int\left(cl\left(St(\cup H_j, \mathcal{W}_j)\right)\right) : j \in N_k\}$ for all but finitely many j. For every $j \in N_k$ and every $H \in \mathcal{H}_j$ we have $H = U_1(H) \times U_2(H) \times \ldots \times U_k(H)$, where $U_i(H) \in \mathcal{U}_j$ for every $i \leq k$. Now consider $\mathcal{V}_j = \{U_i(H) : i \leq k, H \in \mathcal{H}_j\}$. Then for each $j \in N_k$, $\mathcal{V}_j$ is finite subset of $\mathcal{U}_j$.

We claim that $\{int\left(cl\left(St(\cup\mathcal{V}_n, \mathcal{U}_n)\right)\right) : n \in N\}$ is an ω–cover of X. Let $F = \{x_1, \ldots, x_p\}$ be a finite subset of X. Then $y = (x_1, \ldots, x_p) \in X^p$ so that there is an $n \in N_p$ such that $y \in \{int\left(cl\left(St(\cup H, \mathcal{U}_n)\right)\right); H \in \mathcal{H}_n\}$. But $H = U_1(H) \times U_2(H) \times \ldots \times U_p(H)$, where $U_1(H), U_2(H), \ldots, U_p(H) \in \mathcal{V}_n$. The point y belongs to some $W \in \mathcal{W}_n$ of the form $V_1 \times V_2 \times \ldots \times V_p$, $\mathcal{V}_i \in \mathcal{U}_n$ for each $i \leq p$, which meets $U_1(H) \times U_2(H) \times \ldots \times U_p(H)$. This implies that for each $i \leq p$, we have $x_i \in int\left(cl\left(St(U_i(H), \mathcal{U}_n)\right)\right) \subset int\left(cl\left(St(\cup\mathcal{V}_n, \mathcal{U}_n)\right)\right)$ for all but finitely many n, that is, $F \subset int\left(cl\left(St(\cup\mathcal{V}_n, \mathcal{U}_n)\right)\right)$. Hence X satisfy $U^{\star}_{fin}(\mathcal{O}, \underline{\Omega})$. □

Definition 3.6. *A space X is nearly strongly star-Hurewicz if for each sequence $(\mathcal{U}_n : n \in N)$ of open covers of X there is a sequence $(F_n : n \in N)$ of finite subsets of X such that each $x \in X$, $x \in \cup\left\{int\left(cl\left(St(F_n, \mathcal{U}_n)\right)\right) : n \in N\right\}$ for all but finitely many n.*

Definition 3.7. *[24] A space X is meta compact if every open cover $\mathcal{U}$ of X has a point-finite open refinement $\mathcal{V}$ (that is, every point of X belongs to at most finitely many members of $\mathcal{V}$).*

Theorem 3.8. *Every nearly strongly star-Hurewicz meta compact space is a Hurewicz space.*

Proof. Let X be nearly strongly star Hurewicz meta compact space. Let $(\mathcal{U}_n : n \in N)$ be a sequence of open covers of X. For each $n \in N$, let $\mathcal{V}_n$ be a point-finite open refinement of $\mathcal{U}_n$. Since X is nearly strongly star-Hurewicz, there is a sequence $(F_n : n \in N)$ of finite subsets of X such that each $x \in \cup int\left(cl\left(St(F_n, \mathcal{V}_n)\right)\right)$ for all but finitely many n.

As $\mathcal{V}_n$ is point-finite open refinement and each F_n is finite, elements of each F_n belongs to finitely many members of $\mathcal{V}_n$ say $V_{n1}, V_{n2}, V_{n3}, \ldots V_{nk}$. Let $\mathcal{V}'_n = \{V_{n1}, V_{n2}, V_{n3}, \ldots, V_{nk}\}$. Then $int\left(cl\left(St(F_n, \mathcal{V}_n)\right)\right) = \cup\mathcal{V}'_n$ for each $n \in N$ so we have that each x belongs to X belongs to $\cup\mathcal{V}'_n$ for all but finitely many n. For every $V \in \mathcal{V}'_n$ choose $U_V \in \mathcal{U}_n$ such that $V \subset U_V$ then for every n, $\mathcal{W}_n = \{U_V : V \in \mathcal{V}'_n\}$ is a finite subfamily of $\mathcal{U}_n$ and each x belongs to X belongs to $\cup\mathcal{W}_n$ for all but finitely many n, that is X is a Hurewicz space. □

Definition 3.9. *[25] A space X is said to be meta Lindelöf if every open cover $\mathcal{U}$ of X has a point-countable open refinement $\mathcal{V}$.*

Theorem 3.10. *Every nearly strongly star Hurewicz meta Lindelöf space is a Lindelöf space.*

Proof. Let X be nearly strongly star Hurewicz meta $Lindel$öf space. Let $\mathcal{U}$ be an open covers of X and $\mathcal{V}$ a point-countable open refinement of $\mathcal{U}$ by hypothesis, there is a sequence $(F_n : n \in N)$ of finite subsets of X

such that each x belongs to x belongs to $\cup int\left(cl\left(St(F_n, \mathcal{V}_n)\right)\right)$ for all but finitely many n.

For every $n \in N$, denote by $\mathcal{W}_n$ the collection of all members of $\mathcal{V}$ which intersect F_n. Since $\mathcal{V}$ is point-countable and F_n is finite so $\mathcal{W}_n$ is countable. Therefor the collection $\mathcal{W} = \cup_{n\in N}\mathcal{W}_n$ is a countable subfamily of $\mathcal{V}$ and is a cover of X. For every $W \in \mathcal{W}$ pick a member $U_W \in \mathcal{U}$ such that $W \in U_W$. Then $\{U_W : W \in \mathcal{W}\}$ is a countable subcover of $\mathcal{U}$. Hence X is a $Lindel$öf space. □

Theorem 3.11. *A continuous image of a nearly strongly star-Hurewicz space is nearly strongly star-Hurewicz space.*

Proof. Let $f : X \to Y$ be a continuous mapping from a nearly strongly star-Hurewicz space X onto a space Y. Let $(\mathcal{U}_n : n \in N)$ be a sequence of open covers of Y. For each $n \in N$, let $\mathcal{V}_n = \{f^{-1}(U) : U \in \mathcal{U}_n\}$. Then $(\mathcal{V}_n : n \in N)$ is a sequence of open covers of X. Since X is nearly strongly star-Hurewicz space, there exists a sequence $(A_n : n \in N)$ of finite subsets of X such that for each $x \in X$, $x \in \cup int\left(cl(St(A_n, \mathcal{U}_n))\right)$ for all but finitely many n. Thus $(f(A_n) : n \in N)$ is a sequence of finite subsets of Y such that for each $y \in Y$, $y \in int\left(cl(St(f(A_n), \mathcal{U}_n))\right)$ for all but finitely many n. In fact, let $y \in Y$. Then there is $x \in X$ such that $f(x) = y$. Hence $x \in \cup int\left(cl(St(A_n, \mathcal{V}_n))\right)$ for all but finitely many n, which shows that Y is nearly strongly star-Hurewicz space. □

Theorem 3.12. *If each finite power of a space X is nearly strongly star-Hurewicz space, then X satisfies $SS^{\star}_{fin}(\mathcal{O}, \underline{\Omega})$.*

Proof. Let $(\mathcal{U}_n : n \in N)$ be a sequence of open covers of X and consider $N = N_1 \cup N_2 \cup \ldots$ be a partition of N into infinite pairwise disjoint sets. For every $k \in N$ and every $j \in N_k$. Let $\mathcal{W}_j = \{U_1 \times U_2 \times \ldots \times U_k : U_1, U_2, \ldots, U_k \in \mathcal{U}_j\} = \mathcal{U}_j^k$. Then $(\mathcal{W}_j : j \in N_k)$ is a sequence of open covers of X^k. Since X^k is nearly strongly star-Hurewicz space so we can choose a sequence $(\mathcal{V}_j : j \in N_k)$ such that for each j, $\mathcal{V}_j$ is finite subset of X and each $x \in X^k$, $x \in \cup\{int\left(cl\left(St(V, \mathcal{W}_j)\right)\right) : V \in \mathcal{V}_j\}$ for all but finitely many j. For every $j \in N_k$ consider A_j a finite suset of X such that $V_j \subset A_t^k$.

We show that $\{int\left(cl\left(St(A_n, \mathcal{U}_n)\right)\right) : n \in N\}$ is an ω–cover of X. Let $F = \{x_1, \ldots, x_p\}$ be a finite subset of X. Then $(x_1, \ldots, x_p) \in X^p$ so that there is an $n \in N_p$ such that $(x_1, \ldots, x_p) \in int\left(cl\left(St(V_n, \mathcal{W}_n)\right)\right) \subset int\left(cl\left(St(A_n^p, \mathcal{W}_n)\right)\right)$ that is, $F \subset int\left(cl\left(St(A_n, \mathcal{U}_n)\right)\right)$. □

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