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The optimal pebbling of spindle graphs

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Abstract: Given a distribution of pebbles on the vertices of a connected graph G, a *pebbling move* on G consists of taking two pebbles off one vertex and placing one on an adjacent vertex. The *optimal pebbling number* of G, denoted by $\pi_{opt}(G)$, is the smallest number m such that for some distribution of m pebbles on G, one pebble can be moved to any vertex of G by a sequence of pebbling moves. Let P_k be the path on k vertices. Snevily defined the n–k spindle graph as follows: take n copies of P_k and two extra vertices x and y, and then join the left endpoint (respectively, the right endpoint) of each P_k to x (respectively, y), the resulting graph is denoted by $S(n, k)$, and called the *n–k spindle graph*. In this paper, we determine the optimal pebbling number for spindle graphs.

Keywords: pebbling, optimal pebbling, spindle graph

MSC: 05C35

1. Introduction

Graph pebbling was first introduced into the literature by Chung (see [1]). Pebbling has developed its own subfield (see [2]). Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. Let D be a distribution of pebbles on the vertices of G, or a *distribution* on G. For any vertex v of G, $D(v)$ denotes the number of pebbles on v in D. For $S \subseteq V(G)$, we let $D(S) = \sum_{v \in S} D(v)$ and $|D| = \sum_{v \in V(G)} D(v)$. A *pebbling move* consists of removing two pebbles from one vertex and then placing one pebble at an adjacent vertex. For $v \in V(G)$, v is *reachable* under distribution D if v has at least one pebble after some sequence of pebbling moves starting from D. A distribution D is *solvable* if all vertices of G are reachable under D. The *pebbling number* of G, denoted by $\pi(G)$, is the smallest number m such that every distribution of m pebbles on G is solvable. The *optimal pebbling number* of G, denoted by $\pi_{opt}(G)$, is the smallest number m such that some distribution of m pebbles on G is solvable. We say a distribution D is *optimal* if D has $\pi_{opt}(G)$ pebbles and is solvable; that is, it is a solvable distribution of minimum size. We say a distribution D is *smooth* if it has at most two pebbles on each vertex of degree 2. A vertex v is *unoccupied* under a distribution D if $D(v) = 0$.

The optimal pebbling number of G was first introduced by Pachter, Snevily, and Voxman [3]. The optimal pebbling number has been determined for paths [3, 4], cycles [4], m-ary trees [5], caterpillars [6], and ladders [4]. Moews [7] used a probabilistic argument to show that the n-cube Q^n has $\pi_{opt}(Q^n) = (4/3)^{n+O(\log n)}$. In [8], Xue and Yerger investigated the optimal pebbling number of grids and found the optimal pebbling number for the 3 by n grid. For graphs of diameter two (respectively, three), Muntz et al. [9] characterized the classes of graphs having $\pi_{opt}(G)$ equal to a value between 2 and 4 (respectively, between 3 and 8). The lower and upper bounds on the optimal pebbling number were further studied in [4]. Milans and Clark [10] showed that computing optimal pebbling number is NP-hard on arbitrary graphs. Interestingly, exact values for optimal

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pebbling number are known only for paths, cycles, caterpillars, m-ary trees, ladders, and the 3 by n grid. A survey of results of optimal pebbling number can be found in [2].

Let P_k be the path on k vertices. Snevily defined the n–k spindle graph as follows: take n copies of P_k and two extra vertices x and y, and then join the left endpoint (respectively, the right endpoint) of each P_k to x (respectively, y), the resulting graph is denoted by $S(n, k)$, and called the n–k *spindle graph*. In fact, the spindle graph $S(n, k)$ is the graph that the vertices x and y are connected by n internally-disjoint paths of length $k + 1$ (Figure 1 is the graph $S(3, k)$). Snevily and Foster [11] proposed the following Problem 1.1, which appears to be quite difficult.

Problem 1.1 [11] Find $\pi(S(3, k))$.

By $S(1, k) = P_{k+2}$ and $S(2, k) = C_{2k+2}$, we have $\pi(S(1, k)) = 2^{k+1}$ and $\pi(S(2, k)) = 2^{k+1}$(see [4]). Recently, Gao and Yin [12] determined $\pi(S(3, k))$, which solves Problem 1.1.

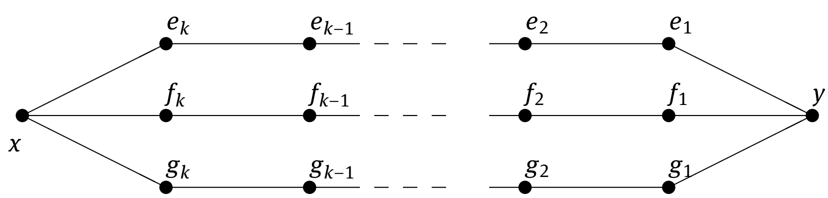

Figure 1: $S(3, k)$.

The focus of this paper is to investigate the optimal pebbling number of $S(n, k)$. By $\pi_{opt}(C_n) = \pi_{opt}(P_n) = \lceil \frac{2n}{3} \rceil$ (see [4]), we have $\pi_{opt}(S(1, k)) = \lceil \frac{2k+4}{3} \rceil$ and $\pi_{opt}(S(2, k)) = \lceil \frac{4k+4}{3} \rceil$. For $n \geq 3$, we further determine $\pi_{opt}(S(n, k))$ in this paper. That is the following Theorem 1.1.

Theorem 1.1 *Let $n \geq 3$ and $p \geq 0$, and denote $\ell = \max\{t | 2^t \leq n\}$.*

(1) If $k < 2\ell$, then $\pi_{opt}(S(n, k)) = 2^{\lfloor k/2 \rfloor} + 2^{\lceil k/2 \rceil}$;

(2) If $k = 2\ell + 3p$, then $\pi_{opt}(S(n, k)) = 2^{\ell+1} + 2np$;

(3) If $k = 2\ell + 3p + 1$, then $\pi_{opt}(S(n, k)) = 2^{\ell+1} + 2^{\ell} + 2np$;

(4) If $k = 2\ell + 3p + 2$ and $2n \geq 2^{\ell+1} + 2^{\ell-1}$, then $\pi_{opt}(S(n, k)) = 2^{\ell+2} + 2np$;

(5) If $k = 2\ell + 3p + 2$ and $2n < 2^{\ell+1} + 2^{\ell-1}$, then $\pi_{opt}(S(n, k)) = 2^{\ell-1} + 2^{\ell} + 2n(p + 1)$.

2. Lemmas

In order to prove Theorem 1.1, we need the following lemmas.

Let D be a distribution on G, and let $H \subseteq G$. The *restriction* of D to H is a pebble distribution D_H which is defined as follows: $D_H(u) = D(u)$ if $u \in V(H)$ and $D_H(u) = 0$ if $u \notin V(H)$. For convenience, we write $A_1, A_2, \ldots, A_n$ for the n copies of P_k in $S(n, k)$. For $1 \leq i \leq n$ and $1 \leq s < t \leq k$, we write $A_i = u_{i,1}u_{i,2} \ldots u_{i,k}$ and $A_i[s, t] = u_{i,s}u_{i,s+1} \ldots u_{i,t}$.

Lemma 2.1. [4] *If G is a connected n-vertex graph, with $n \geq 3$, then G has a smooth optimal distribution with all leaves unoccupied.*

Lemma 2.2. [3, 4] *Let P_k be the path on k vertices. Then $\pi_{opt}(P_k) = \lceil 2k/3 \rceil$.*

The following Lemma gives an upper bound on $\pi_{opt}(S(n, k))$.

Lemma 2.3. *Let $n \geq 3$ and $p \geq 0$, and denote $\ell = \max\{t | 2^t \leq n\}$.*

(1) If $k < 2\ell$, then $\pi_{opt}(S(n, k)) \leq 2^{\lfloor k/2 \rfloor} + 2^{\lceil k/2 \rceil}$;

(2) If $k = 2\ell + 3p$, then $\pi_{opt}(S(n, k)) \le 2^{\ell+1} + 2np$;
(3) If $k = 2\ell + 3p + 1$, then $\pi_{opt}(S(n, k)) \le 2^{\ell+1} + 2^{\ell} + 2np$;
(4) If $k = 2\ell + 3p + 2$ and $2n \ge 2^{\ell+1} + 2^{\ell-1}$, then $\pi_{opt}(S(n, k)) \le 2^{\ell+2} + 2np$;
(5) If $k = 2\ell + 3p + 2$ and $2n < 2^{\ell+1} + 2^{\ell-1}$, then $\pi_{opt}(S(n, k)) \le 2^{\ell-1} + 2^{\ell} + 2n(p + 1)$.

Proof. Clearly, $2^{\ell} \le n < 2^{\ell+1}$ and $\ell \ge 1$.

(1) Assume that $k < 2\ell$. Let D be a distribution such that $D(x) = 2^{\lfloor k/2\rfloor}$, $D(y) = 2^{\lceil k/2\rceil}$, and $D(v) = 0$ for each $v \in V(S(n, k)) \setminus \{x, y\}$. Then D is solvable and $\pi_{opt}(S(n, k)) \le 2^{\lfloor k/2\rfloor} + 2^{\lceil k/2\rceil}$.

(2) Assume that $k = 2\ell + 3p$. Let D be a distribution such that $D(x) = D(y) = 2^{\ell}$, $D(u_{i,j}) = 2$ for all $i \in \{1, 2, \dots, n\}$, $j \in \{\ell + 2, \dots, \ell + 3(p - 1) + 2\}$, and $D(v) = 0$ for each $v \in V(S(n, k)) \setminus \{x, y, u_{1,\ell+2}, \dots, u_{1,(\ell+3p-1)}, \dots, u_{n,(\ell+2)}, \dots, u_{n,(\ell+3p-1)}\}$. It is clear to see that D is solvable. Then $\pi_{opt}(S(n, k)) \le 2^{\ell+1} + 2np$.

(3) Assume that $k = 2\ell + 3p + 1$. Let D be a distribution such that $D(x) = 2^{\ell}$, $D(y) = 2^{\ell+1}$, $D(u_{i,j}) = 2$ for all $i \in \{1, 2, \dots, n\}$, $j \in \{\ell + 2, \dots, \ell + 3(p - 1) + 2\}$, and $D(v) = 0$ for each $v \in V(S(n, k)) \setminus \{x, y, u_{1,(\ell+2)}, \dots, u_{1,(\ell+3p-1)}, \dots, u_{n,(\ell+2)}, \dots, u_{n,(\ell+3p-1)}\}$. Then D is solvable and $\pi_{opt}(S(n, k)) \le 2^{\ell+1} + 2^{\ell} + 2np$.

(4) Assume that $k = 2\ell + 3p + 2$ and $2n \ge 2^{\ell+1} + 2^{\ell-1}$. Let D be a distribution such that $D(x) = D(y) = 2^{\ell+1}$, $D(u_{i,j}) = 2$ for all $i \in \{1, 2, \dots, n\}$, $j \in \{\ell + 3, \dots, \ell + 3p\}$, and $D(v) = 0$ for each $v \in V(S(n, k)) \setminus \{x, y, u_{1,(\ell+3)}, \dots, u_{1,(\ell+3p)}, \dots, u_{n,(\ell+3)}, \dots, u_{n,(\ell+3p)}\}$. Then D is solvable and $\pi_{opt}(S(n, k)) \le 2^{\ell+2} + 2np$.

(5) Assume that $k = 2\ell + 3p + 2$ and $2n < 2^{\ell+1} + 2^{\ell-1}$. Let D be a distribution such that $D(x) = 2^{\ell}$, $D(y) = 2^{\ell-1}$, $D(u_{i,j}) = 2$ for all $i \in \{1, 2, \dots, n\}$, $j \in \{\ell + 2, \dots, \ell + 3p + 2\}$, and $D(v) = 0$ for each $v \in V(S(n, k)) \setminus \{x, y, u_{1,(\ell+2)}, \dots, u_{1,(\ell+3p+2)}, \dots, u_{n,(\ell+2)}, \dots, u_{n,(\ell+3p+2)}\}$. Then D is solvable and $\pi_{opt}(S(n, k)) \le 2^{\ell-1} + 2^{\ell} + 2n(p + 1)$. □

Let

$$g(n, k) = \begin{cases} 2^{\lfloor k/2\rfloor} + 2^{\lceil k/2\rceil} & \text{if } k < 2\ell\,, \\ 2^{\ell+1} + 2np & \text{if } k = 2\ell + 3p\,, \\ 2^{\ell+1} + 2^{\ell} + 2np & \text{if } k = 2\ell + 3p + 1, \\ \min\{2^{\ell+2} + 2np, 2^{\ell-1} + 2^{\ell} + 2n(p + 1)\} & \text{if } k = 2\ell + 3p + 2. \end{cases}$$

Then $\pi_{opt}(S(n, k)) \le g(n, k)$.

Lemma 2.4. *Assume that $n \ge 3$, $k \ge 2$, and $\alpha_1 \ge 0, \alpha_2 \ge 0$, $\alpha_1 + \alpha_2 \le k$. Let D be a smooth solvable distribution on $S(n, k)$. If there are at most $2^{\alpha_1} + \beta_1$ pebbles can be put on x by a sequence of pebbling moves starting from D, and there are at most $2^{\alpha_2} + \beta_2$ pebbles can be put on y by a sequence of pebbling moves, then $|D| \ge 2^{\alpha_1} + 2^{\alpha_2} + n\lceil 2(k - \alpha_1 - \alpha_2)/3\rceil$, where $0 \le \beta_i \le 2^{\alpha_i} - 1$ for $i \in \{1, 2\}$.*

Proof. Clearly, for $i \in \{1, 2, \dots, n\}$, we can move at most one pebble to x from A_i as D is a smooth distribution. Similarly, we can move at most one pebble to y from A_i. Consider a smooth solvable distribution D_0 with $D_0(x) = 2^{\alpha_1}$, $D_0(y) = 2^{\alpha_2}$.

If $|D_0| \le 2^{\alpha_1} + 2^{\alpha_2} + n\lceil 2(k - \alpha_1 - \alpha_2)/3\rceil - 1$, then there are at most

$$|D_0| - D_0(x) - D_0(y) \le n\lceil 2(k - \alpha_1 - \alpha_2)/3\rceil - 1$$

pebbles on $A_1[1 + \alpha_1, k - \alpha_2], \dots, A_n[1 + \alpha_1, k - \alpha_2]$. For $i \in \{2, \dots, n\}$, by Lemma 2.2, we need at least $\lceil 2(k - \alpha_1 - \alpha_2)/3\rceil$ pebbles that ensure each vertex in $A_i[1 + \alpha_1, k - \alpha_2]$ to be reachable. Thus, there is some vertex in $A_i[1 + \alpha_1, k - \alpha_2]$ which is not reachable. Hence, $|D_0| \ge 2^{\alpha_1} + 2^{\alpha_2} + n\lceil 2(k - \alpha_1 - \alpha_2)/3\rceil$. Note that each pebbling move reduces the size of D. Therefore, $|D| \ge |D_0| \ge 2^{\alpha_1} + 2^{\alpha_2} + n\lceil 2(k - \alpha_1 - \alpha_2)/3\rceil$. □

Lemma 2.5. *Let a be a nonnegative integer, $a = 3p + s$, $s \in \{0, 1, 2\}$. Then $2^{\lceil a/2\rceil} + 2^{\lfloor a/2\rfloor} \ge 4p + s + 2$.*

Proof. Firstly, assume that $a = 2b$. Then $p = (2b - s)/3$, and $2^{\lceil a/2\rceil} + 2^{\lfloor a/2\rfloor} = 2^{b+1}$. Let $f(t) = 2^{t+1} - 2 - s - (8t - 4s)/3$. If $a = 0$, then $b = p = s = 0$, and $f(0) \ge 0$. If $a = 2$, then $b = 1$, $p = 0$, $s = 2$, and $f(1) \ge 0$. We have that $f'(t) = 2^{t+1}\ln 2 - 8/3 > 0$ for $t \ge 1$. Hence, $f(t) > f(1) \ge 0$.

Now, we assume that $a = 2b + 1$. Then $p = (2b + 1 - s)/3$, and $2^{\lceil a/2\rceil} + 2^{\lfloor a/2\rfloor} = 3 \times 2^b$. If $a = 1$, then $b = p = 0, s = 1$, and $2^{\lceil a/2\rceil} + 2^{\lfloor a/2\rfloor} \geq 4p + s + 2$. Let $f(t) = 3 \times 2^t - 2 - s - (8t + 4 - 4s)/3$. If $a = 3$, then $b = p = 1, s = 0$, and $f(1) \geq 0$. We get $f'(t) = 3 \times 2^t \ln 2 - 8/3 > 0$ for $t \geq 1$. Thus, $f(t) > f(1) \geq 0$. Therefore, $2^{\lceil a/2\rceil} + 2^{\lfloor a/2\rfloor} \geq 4p + s + 2$. □

3. Proof of Theorem 1.1

Proof of Theorem 1.1. Clearly, $S(n, 1)$ is isomorphic to $K_{2,n}$, and $\pi_{opt}(K_{2,n}) = 3$. We assume $k \geq 2$. By Lemma 2.3, we have that $\pi_{opt}(S(n, k)) \leq g(n, k)$. We now show that $\pi_{opt}(S(n, k)) \geq g(n, k)$. By Lemma 2.1, we can assume that D is a smooth optimal distribution. Now, we assume that there are at most $2^{\alpha_1} + \beta_1$ pebbles can be put on x by a sequence of pebbling moves starting from D, and there are at most $2^{\alpha_2} + \beta_2$ pebbles can be put on y by a sequence of pebbling moves, where $\alpha_1 \geq 0, \alpha_2 \geq 0, 0 \leq \beta_i \leq 2^{\alpha_i} - 1$ for $i \in \{1, 2\}$. Thus, we assume $D(x) = 2^{\alpha_1} + \beta_1 - m_1$ and $D(y) = 2^{\alpha_2} + \beta_2 - m_2$, where $0 \leq m_i \leq min\{n, 2^{\alpha_i} + \beta_i\}$ for $i \in \{1, 2\}$. Note that $2^\ell \leq n < 2^{\ell+1}$ and $\ell \geq 1$. Clearly, $a \geq \min\{a, b\}$ for a and b are real numbers. We consider the following two cases.

Case 1. $\alpha_1 + \alpha_2 > k$.

Then $|D| = D(x) + D(y) + \sum_{i=1}^{n} D(A_i) \geq 2^{\alpha_1} + \beta_1 - m_1 + 2^{\alpha_2} + \beta_2 - m_2 + m_1 + m_2 \geq 2^{\alpha_1} + 2^{\alpha_2}$. Hence, $|D| \geq 2^{\alpha_1} + 2^{\alpha_2} \geq 2^{\lceil(\alpha_1+\alpha_2)/2\rceil} + 2^{\lfloor(\alpha_1+\alpha_2)/2\rfloor} > 2^{\lceil k/2\rceil} + 2^{\lfloor k/2\rfloor}$ since $2^a + 2^b \geq 2^{a-1} + 2^{b+1}$ for all integers satisfying $a > b$.

Assume that $k = 2\ell + 3p + s, s \in \{0, 1, 2\}$. By Lemma 2.5, we have that

$$\begin{aligned} |D| &> 2^{\lceil k/2\rceil} + 2^{\lfloor k/2\rfloor} \\ &= 2^{\lceil(2\ell+3p+s)/2\rceil} + 2^{\lfloor(2\ell+3p+s)/2\rfloor} \\ &= 2^\ell(2^{\lceil(3p+s)/2\rceil} + 2^{\lfloor(3p+s)/2\rfloor}) \\ &\geq 2^\ell(4p + s + 2). \end{aligned}$$

If $s = 0$, then $|D| > (4p + 2)2^\ell > 2^{\ell+1} + 2np$. If $s = 1$, then $|D| > 4p \times 2^\ell + 3 \times 2^\ell > 2^{\ell+1} + 2^\ell + 2np$. If $s = 2$, then $|D| > 4p \times 2^\ell + 4 \times 2^\ell > 2^{\ell+2} + 2np \geq \min\{2^{\ell+2} + 2np, 2^{\ell-1} + 2^\ell + 2n(p + 1)\}$. Thus, $|D| = \pi_{opt}(S(n, k)) > g(n, k)$. This is impossible as it would be greater than the known upper bound.

Case 2. $\alpha_1 + \alpha_2 \leq k$.

Denote $\omega = \alpha_1 + \alpha_2$. By Lemma 2.4, we get

$$|D| \geq 2^{\alpha_1} + 2^{\alpha_2} + n\lceil 2(k - \omega)/3\rceil \geq 2^{\lceil\omega/2\rceil} + 2^{\lfloor\omega/2\rfloor} + n\lceil 2(k - \omega)/3\rceil \tag{3.1}$$

as $2^a + 2^b \geq 2^{a-1} + 2^{b+1}$. Now we consider the following three subcases.

Subcase 2.1. $k < 2\ell$.

If $\omega = k$, by (3.1), then $|D| \geq 2^{\lceil k/2\rceil} + 2^{\lfloor k/2\rfloor}$. If $\omega = k - 1$, by (3.1), then

$$\begin{aligned} |D| &\geq 2^{\lceil\omega/2\rceil} + 2^{\lfloor\omega/2\rfloor} + n\lceil 2(k - \omega)/3\rceil \\ &\geq 2^{\lceil(k-1)/2\rceil} + 2^{\lfloor(k-1)/2\rfloor} + 2^\ell \\ &\geq 2^{\lceil(k-1)/2\rceil} + 2^{\lfloor(k-1)/2\rfloor} + 2^{\lceil k/2\rceil} \\ &\geq 2^{\lceil k/2\rceil} + 2^{\lfloor k/2\rfloor}. \end{aligned}$$

If $\omega \leq k - 2$, then

$$|D| \geq 2^{\lceil\omega/2\rceil} + 2^{\lfloor\omega/2\rfloor} + n\lceil 2(k - \omega)/3\rceil \geq 2^{\lceil\omega/2\rceil} + 2^{\lfloor\omega/2\rfloor} + 2 \times 2^\ell \geq 2^{\lceil k/2\rceil} + 2^{\lfloor k/2\rfloor}.$$

Subcase 2.2. $k = 2\ell + 3p + s$ and $0 \leq \omega \leq 2\ell + s$, where $s \in \{0, 1, 2\}$.

Let $\omega = 2\ell + s - j, 0 \leq j \leq 2\ell + s$. By (3.1), we get

$$|D| \geq 2^{\lceil\omega/2\rceil} + 2^{\lfloor\omega/2\rfloor} + n\lceil(6p + 2j)/3\rceil \geq 2^{\lceil(2\ell+s-j)/2\rceil} + 2^{\lfloor(2\ell+s-j)/2\rfloor} + 2np + n\lceil 2j/3\rceil. \tag{3.2}$$

If $s = 0$ and $0 \le j \le 1$, by (3.2), then $|D| \ge 2^{\ell+1} + 2np$. If $s = 0$ and $j \ge 2$, then

$$|D| \ge 2^{\lceil (2\ell-j)/2\rceil} + 2^{\lfloor (2\ell-j)/2\rfloor} + 2np + 2^{\ell}\lceil 2j/3\rceil > 2^{\ell+1} + 2np.$$

If $s = 1$ and $j \le 3$, then

$$|D| \ge 2^{\lceil (2\ell+1-j)/2\rceil} + 2^{\lfloor (2\ell+1-j)/2\rfloor} + 2np + 2^{\ell}\lceil 2j/3\rceil \ge 2^{\ell+1} + 2^{\ell} + 2np.$$

If $s = 1$ and $j \ge 4$, then

$$|D| \ge 2^{\lceil (2\ell+1-j)/2\rceil} + 2^{\lfloor (2\ell+1-j)/2\rfloor} + 2np + 2^{\ell}\lceil 2j/3\rceil > 2^{\ell+1} + 2^{\ell} + 2np.$$

Assume that $s = 2$. If $j = 0$, then $|D| \ge 2^{\ell+2} + 2np \ge \min\{2^{\ell+2} + 2np, 2^{\ell-1} + 2^{\ell} + 2n(p+1)\}$. If $j = 1$, then $|D| \ge 2^{\ell+1} + 2^{\ell} + 2np + n \ge 2^{\ell+2} + 2np \ge \min\{2^{\ell+2} + 2np, 2^{\ell-1} + 2^{\ell} + 2n(p+1)\}$. If $2 \le j \le 3$, then $|D| \ge 2^{\ell} + 2^{\ell-1} + 2np + 2n \ge \min\{2^{\ell+2} + 2np, 2^{\ell-1} + 2^{\ell} + 2n(p+1)\}$. If $j = 4$, then $|D| \ge 2^{\ell-1} + 2^{\ell-1} + 2np + 3n \ge 2^{\ell+2} + 2np \ge \min\{2^{\ell+2} + 2np, 2^{\ell-1} + 2^{\ell} + 2n(p+1)\}$. If $j \ge 5$, then $|D| \ge 2np + 4n \ge 2^{\ell+2} + 2np \ge \min\{2^{\ell+2} + 2np, 2^{\ell-1} + 2^{\ell} + 2n(p+1)\}$.

Subcase 2.3. $k = 2\ell + 3p + s$ and $2\ell + s < \omega \le 2\ell + 3p + s$, where $s \in \{0, 1, 2\}$.

Note that $p \ge 1$. Let $\omega = 2\ell + 3p + s - j$, $0 \le j \le 3p - 1$. By (3.1), we have

$$|D| \ge 2^{\ell}(2^{\lceil (3p+s-j)/2\rceil} + 2^{\lfloor (3p+s-j)/2\rfloor}) + n\lceil 2j/3\rceil. \tag{3.3}$$

Assume that $3p + s - j = 1$. Then $|D| \ge 3 \times 2^{\ell} + n\lceil 2j/3\rceil = 2^{\ell+1} + 2^{\ell} + 2np + n\lceil (2s-2)/3\rceil$. If $s = 0$, then $|D| > 2^{\ell+1} + 2np$. If $s = 1$, then $|D| \ge 2^{\ell+1} + 2^{\ell} + 2np$. If $s = 2$, then $|D| \ge 2^{\ell+1} + 2^{\ell} + 2np + n > 2^{\ell+2} + 2np$.

Assume that $3p + s - j = 2$. Then $|D| \ge 2 \times 2^{\ell+1} + n\lceil 2j/3\rceil = 2^{\ell+2} + 2np + n\lceil (2s-4)/3\rceil$. If $s = 0$, then $|D| > 2^{\ell+1} + 2np$. If $s = 1$, then $|D| > 2^{\ell+1} + 2^{\ell} + 2np$. If $s = 2$, then $|D| \ge 2^{\ell+2} + 2np$.

Assume that $3p + s - j = 3$. Then $|D| \ge 6 \times 2^{\ell} + n\lceil 2j/3\rceil = 2^{\ell+2} + 2^{\ell+1} + 2np + n\lceil (2s-6)/3\rceil$. If $s = 0$, then $|D| > 2^{\ell+1} + 2np$. If $s = 1$, then $|D| \ge 2^{\ell+1} + 2^{\ell} + 2np$. If $s = 2$, then $|D| > 2^{\ell+2} + 2np$.

Assume that $3p + s - j = 4$. Then $|D| \ge 2^{\ell+3} + 2np + n\lceil (2s-8)/3\rceil$. If $s = 0$, then $|D| > 2^{\ell+1} + 2np$. If $s = 1$, then $|D| > 2^{\ell+1} + 2^{\ell} + 2np$. If $s = 2$, then $|D| > 2^{\ell+2} + 2np$.

Assume that $3p + s - j = 2c$ and $c \ge 3$. By (3.3), and

$$p + s + \lceil 2j/3\rceil - j \ge (j+1-s)/3 + s + \lceil 2j/3\rceil - j = \lceil 2j/3\rceil - 2j/3 + (1+2s)/3 > 0,$$

we have that

$$\begin{aligned}
|D| &\ge 2^{\ell}(2^{\lceil (3p+s-j)/2\rceil} + 2^{\lfloor (3p+s-j)/2\rfloor}) + n\lceil 2j/3\rceil\\
&= 2^{\ell+1}2^{c} + n\lceil 2j/3\rceil\\
&\ge 2^{\ell+1}(2+2c) + n\lceil 2j/3\rceil\\
&= 2^{\ell+2} + (3p+s-j)2^{\ell+1} + n\lceil 2j/3\rceil\\
&> 2^{\ell+2} + (3p+s-j)n + n\lceil 2j/3\rceil\\
&\ge 2^{\ell+2} + 2np + n(p + s + \lceil 2j/3\rceil - j)\\
&> 2^{\ell+2} + 2np.
\end{aligned}$$

Now, we assume that $3p + s - j = 2c + 1$ and $c \ge 2$. By (3.3), and

$$\frac{p+3s+1-3j}{4} + \lceil 2j/3\rceil \ge \frac{j+1-s}{12} + \frac{3s+1-3j}{4} + \lceil 2j/3\rceil = \lceil 2j/3\rceil - 2j/3 + \frac{1+2s}{3} > 0,$$

we have that

$$\begin{aligned}
|D| &\ge 2^{\ell}(2^{\lceil (3p+s-j)/2\rceil} + 2^{\lfloor (3p+s-j)/2\rfloor}) + n\lceil 2j/3\rceil\\
&= 3 \times 2^{\ell} \times 2^{c} + n\lceil 2j/3\rceil\\
&\ge 3 \times 2^{\ell}(2+c) + n\lceil 2j/3\rceil\\
&= 2^{\ell+2} + 2^{\ell+1} + 3c \times 2^{\ell} + n\lceil 2j/3\rceil\\
&> 2^{\ell+2} + n(1 + 3c/2 + \lceil 2j/3\rceil)\\
&> 2^{\ell+2} + n(1 + \tfrac{9p+3s-3j-3}{4} + \lceil 2j/3\rceil)\\
&> 2^{\ell+2} + 2np.
\end{aligned}$$

Therefore, $\pi_{opt}(S(n,k)) \ge g(n,k)$. The proof of Theorem 1.1 is completed. □

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