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Power graphs and exchange property for resolving sets

Abstract: Classical applications of resolving sets and metric dimension can be observed in robot navigation, networking and pharmacy. In the present article, a formula for computing the metric dimension of a simple graph wihtout singleton twins is given. A sufficient condition for the graph to have the exchange property for resolving sets is found. Consequently, every minimal resolving set in the graph forms a basis for a matriod in the context of independence defined by Boutin [Determining sets, resolving set and the exchange property, Graphs Combin., 2009, 25, 789-806]. Also, a new way to define a matroid on finite ground is deduced. It is proved that the matroid is strongly base orderable and hence satisfies the conjecture of White [An unique exchange property for bases, Linear Algebra Appl., 1980, 31, 81-91]. As an application, it is shown that the power graphs of some finite groups can define a matroid. Moreover, we also compute the metric dimension of the power graphs of dihedral groups.

Keywords: basis, involution, metric dimension, matroid, power graph, resolving set

MSC: 05B35, 05C12

1 Introduction

Resolving sets and metric basis enjoys a lot of success due to its applications in computer science, medical sciences and chemistry. The concepts of metric dimension and resolving sets were initially drafted for the metric spaces in 1953 in [3] but did not receive much attention may be because of continuous nature of standard Euclidean spaces $\mathbf{R}^n$. These concepts were utilized for about twenty years later in 1975 [4]. Since then it has been artistically used in graphs, robotics, pharmacy, networking and in many other fields. Recently, the theory of metric dimension has been generalized for metric spaces and geometric spaces [5, 6].

Let Γ be a finite, simple, and connected graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. The *distance $d_\Gamma(u, v)$* between two vertices $u, v \in V(\Gamma)$ is the length of a shortest path between them. Let $W = \{w_1, w_2, \ldots, w_k\}$ be an ordered set of vertices of Γ, and let $v \in V(\Gamma)$. Then, the *representation* of v with respect to W is the k-tuple $(d_\Gamma(v, w_1), d_\Gamma(v, w_2), \ldots, d_\Gamma(v, w_k))$. Two vertices $u, v \in V(\Gamma)$ are said to be *resolved* by W if they have different representations. A subset W of vertices is a *resolving set* (or *locating set*) if every vertex of Γ

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is uniquely identified by its distances from the vertices of W. Thus, in a resolving set, every vertex of Γ has distinct representation. A resolving set of minimum cardinality is called a *basis* for Γ. The cardinality of such a resolving set is called the *metric dimension* of Γ and is denoted by $\beta(\Gamma)$ (see [7–14]). A resolving set is said to be *minimal* if it contains no resolving set as a proper subset. As an application, S. Khuller [15] considered the metric dimension and basis of a connected graph in robot navigation problems. In [16], authors computed metric dimension of flower graph and some families of convex polytopes. Chartrand et. al proved that a graph has constant metric dimension 1 if and only if it is a path. Murtaza et. al computed partial results of metric dimension of Mobius ladder in [17] whereas Munir et. al. computed exact and complete results for metric dimension of Mobius Ladders in [18]. C. Poisson et. al. computed metric dimension of uni-cyclic graphs in [19].

Whenever W_1 and W_2 are any two minimal resolving sets for Γ and for every $u \in W_1$, there is a vertex $v \in W_2$ such that $(W_1 \backslash \{u\}) \cup \{v\}$ is also a minimal resolving set. Then, resolving sets are said to have the *exchange property* in the graph Γ (for details, see [1]). All the graphs considered in this paper are finite, simple and connected. Also, all the groups considered are finite. Furthermore, the exchange property of a graph Γ always means the property for resolving sets.

The *open neighborhood* of a vertex $u \in V(\Gamma)$, denoted by $N(u)$, is the set

$$\{v \in V(\Gamma) : d_\Gamma(u, v) = 1\},$$

and the closed neighborhood of $u \in V(\Gamma)$, denoted by $N[u]$, is the set

$$\{v \in V(\Gamma) : d_\Gamma(u, v) = 1 \cup \{u\}\}.$$

The two vertices u and v in a graph Γ are called *twins*, denoted by $u \equiv v$, if either $N[u] = N[v]$ or $N(u) = N(v)$. The relation $\equiv$ is an equivalence relation (see [20]). Also, $d_\Gamma(u_1, w) = d_\Gamma(u_2, w)$ for $u_1 \equiv u_2$, and for all $w \in V(\Gamma) \backslash \{u_1, u_2\}$. Let $\overline{u}$ denote the twin-set of u with respect to the relation " $\equiv$ ", and let $\mathcal{U}(\Gamma) = \{\overline{u} | u \in V(\Gamma)\}$ be the set of all such twin-sets.

The following definition is helpful in proving the main results of this paper.

Definition 1.1. *A vertex u is called a singleton twin if $\overline{u} = \{u\}$.*

2 Main results

In this section, we formulate our main results. Proofs of these results are given in the next sections. Our first result gives a formula to compute the metric dimension of a graph without singleton twins.

Theorem 2.1. *Let Γ be a graph without singleton twins. If there are n non-singleton twin-sets, each of size m_i. Then,*

$$\beta(\Gamma) = \sum_{i=1}^{n} m_i - n.$$

Moreover, every minimal resolving set is a basis of the graph Γ.

Our second result provides a sufficient condition for a graph to have the exchange property.

Theorem 2.2. *A graph Γ without singleton twins has the exchange property.*

The concept of a matroid is a generalization of the notion of linear independence. More precisely, a finite matroid $M = (H, I)$ is a pair (H, I), where H is a finite set called the *ground set* and I is a family of subsets of H called *independent sets* with the following properties:
a) the empty set is independent;
b) every subset of an independent set is independent (this property is called *hereditary property*); and

c) if A and B are two independent sets of I with $|A| > |B|$, then there exist $x \in A \setminus B$, such that $B \cup \{x\}$ is also independent (this is called *augmentation property*).

A maximal independent set is called a *basis* of the matroid M.

We say, as defined in [1], a set W of vertices in a graph Γ is resolving independent, denoted by *res-independent*, if for every $v \in W$, $W-\{v\}$ is not a resolving set. With this definition, a maximal res-independent set is a minimal resolving set. This definition of independence defines a hereditary system

$$M_\Gamma = \{W \subset V(\Gamma) : W \setminus \{u\} \text{ is not resolving for all } u \in W\}$$

in the graph Γ. The question of whether the exchange property holds in Γ is equivalent to the question whether the hereditary system M_Γ is a matroid (see [21] for further details).

As an application of Theorem 2.2 to matroid theory, the following corollary can be used to define a matroid on a finite ground set.

Corollary 2.3. *The hereditary system M_Γ is a matroid for a graph Γ without singleton twin and every minimal resolving set is a basis for the matriod.*

A matroid M is called *strongly base orderable* if for any two bases B_1 and B_2 there is a bijection $\pi : B_1 \to B_2$ such that $B_1 \cup \pi(A) \setminus A$ is a basis for any subset $A \subset B_1$.

Theorem 2.4. *The matriod M_Γ is strongly base orderable.*

Conjecture 2.5. *[2] For every matroid M, its toric ideal is generated by quadratic binomials corresponding to symmetric exchanges.*

It is proved in [22] that the White conjecture is true for every strongly orderable matroid. Therefore, the conjecture is true for M_Γ.

Let G be a finite group. An *undirected power graph* $\mathcal{P}_G$ associated to G, is a graph whose vertices are the elements of G, and there is an edge between two vertices x and y if either $x^m = y$ or $y^m = x$, for some positive integer m. The *power digraph* of G is a digraph $\mathcal{P}_G$ with the vertex set G, and there is an arc from vertex x to y if $x^m = y$, for some positive integer m. The directed power graph of a group was introduced by Kelarev and Quinn [23]. The definition was formulated so that it applied to semigroups as well. The power graphs of semigroups were first considered in [24–26]. All of these papers used only the brief term 'power graph', even though they covered both directed and undirected power graphs. The investigation of graphs of this sort is very important, because they have serious applications and are related to automata theory (see [27, 28] and the books [29, 30], where applications are presented). It is also explained in the survey [31] that the definition given in [23] covers all undirected graphs as well. Chakrabarty, Ghosh, and Sen [32] also studied undirected power graphs of semigroups. Recently, many interesting results on the power graphs of finite groups have been obtained (see [16, 33–36]). It is obvious that the power graph of a finite group is always connected. For other results and open questions on power graphs, we refer to the survey [31].

In our next theorem, the metric dimension of the power graph of the dihedral group D_{2n} of order $2n$ is computed.

Theorem 2.6. $\beta(\mathcal{P}_{D_{2n}}) = \beta(\mathcal{P}_{Z_n}) + n - 2$, *where Z_n is a cyclic group of order n.*

In the following theorem, we identify some finite groups whose corresponding power graph define a matroid on the group.

Theorem 2.7. *Let G be a finite group and $\mathcal{P}_G$ be the power graph associated to G. Then, $M_{\mathcal{P}_G}$ is a matroid if G is cyclic and $|G| = 2k + 1$ for positive integers k.*

3 Proofs

3.1 Exchange property

Every vector in a finite dimensional vector space is uniquely determined (written as a linear combination) by the elements of a basis of the vector space. A basis of a vector space has the exchange property. Similarly, each vertex of a finite graph can be uniquely identified by the vertices of a minimal resolving set. Therefore, resolving sets of a finite graph behave like bases in a finite dimensional vector space. Unlike a linear basis of a vector space, the minimal resolving sets do not always have the exchange property. Results about the exchange property for different graphs can be found in the literature. For example, the exchange property holds for resolving sets in trees; for $n \geq 8$, the exchange property does not hold in wheels W_n [1].

Lemma 3.1. *[20] Suppose that u, v are twins in a connected graph Γ and W resolves Γ. Then, u or v is in W. Moreover, if $u \in W$ and $v \notin W$, then $(W \setminus \{u\}) \cup \{v\}$ also resolves Γ.*

Proof of Theorem 2.1: Since the graph Γ contains n non-singleton twin-sets, a basis W, by Lemma 3.1, contains $m_i - 1$ vertices of each twin-set of size m_i. Let u and v be two vertices which are not twins. Then, there must be some $w \in W$ such that $d_\Gamma(u, w) \neq d_\Gamma(v, w)$; otherwise $d_\Gamma(u, x) = d_\Gamma(v, x)$ for all $x \in V(\Gamma)$ which means that u and v are twins, a contradiction. Consequently, exactly one representative from each twin-set stays outside W. Therefore,

$$\beta(\Gamma) = \sum_{i=1}^{n} m_i - n.$$

The cardinality of a minimal resolving set W_1 is $\geq \beta(\Gamma)$. Now, W_1 must have exactly $\beta(\Gamma) = \sum_{i=1}^{n} m_i - n$ vertices. Otherwise, W_1 contains an entire twin-set $\overline{u}$ of a vertex u and $W_1 \setminus \{u\}$ is again resolving set, a contradiction. Therefore, every minimal resolving set is a basis.

Proof of Theorem 2.2: Let W_1 and W_2 be two different minimal resolving sets in a graph Γ, and let $u_1 \in W_1$. If $u_1 \in W_2$, then obviously $\{W_1 \setminus \{u_1\}\} \cup \{u_1\}$ is a minimal resolving set. For $u_1 \notin W_2$ there exists a vertex $u_2 \notin W_1$ such that $u_1 \equiv u_2$. Otherwise, W_1 contains an entire twin-set and W_1 is not minimal by Theorem 2.1, a contradiction. By Lemma 3.1, $u_2 \in W_2$ and every vertex in $V(\Gamma) \setminus \{u_1, u_2\}$ is at same distance from the vertices u_1 and u_2. Therefore, the vertices which are resolved by u_1 are also resolved by u_2 and vice versa. Consequently, $(\{W_1 \setminus \{u_1\}) \cup \{u_2\}$ is again a minimal resolving set.

Proof of Theorem 2.4: Let W_1 and W_2 are two bases. Define a bijection $\pi : W_1 \to W_2$ as follows

$$\pi(w) = \begin{cases} w, & \text{if } w \in W_2; \\ u \in \overline{w} \text{ and } u \neq w & \text{if } w \notin W_2. \end{cases}$$

The graph Γ is without singleton twins. Therefore, $W_1 \cup \pi(U) \setminus U$ is a minimal resolving set (basis for the matroid M_Γ) for all $U \subset W_1$.

3.2 Power graph of finite groups

Proposition 3.2. *[16] Suppose x and y are two elements of an abelian group G, then x and y have the same closed neighborhoods in the power graph $\mathcal{P}_G$ if and only if one of the followings holds:*
(i) $\langle x \rangle = \langle y \rangle$;
(ii) G is cyclic, and one of x and y is a generator of G and the other is the identity e; and
(iii) G is cyclic of prime order (x and y are arbitrary).

Definition 3.3. *[37] For elements x and y in a group G, write $R\{x, y\} = \{z : z \in V(\mathcal{P}_G), d_{\mathcal{P}_G}(x, z) \neq d_{\mathcal{P}_G}(y, z)\}$.*

An *involution* is a non-identity element of order 2 in a group G. A *resolving involution*, in a power graph $\mathcal{P}_G$ of a group G, is an involution w which satisfies that there exist two vertices $x, y \in V(\mathcal{P}_G) \backslash \overline{w}$ with $R\{x, y\} = \{x, y, w\}$. Let $W(\mathcal{P}_G)$ denotes the set of all resolving involutions of $\mathcal{P}_G$.

Example 3.4. *Let $G = \{e, x, x^2, x^3, x^4, x^5\}$ be the cyclic group of order 6. Note that $R\{x, y\} = \{u, v, x^3\}$ for $u \in \{x, x^5\}$ and $v \in \{x^2, x^4\}$. Therefore, x^3 is a resolving involution of $\mathcal{P}_G$.*

Let Ψ denote the set of noncyclic groups G such that there exists an odd prime p such that the following conditions hold (see [37]):
(C_1) the prime divisors of $|G|$ are 2 and p;
(C_2) the subgroup of order p is unique;
(C_3) there is no element of order 4 in G; and
(C_4) each involution of G is contained in a cyclic subgroup of order $2p$.
   In the original paper [37], for a finite group G, the notations $|G|$; $|\mathcal{U}(G)|$; and $|W(G)|$ were used for $|V(\mathcal{P}_G)|$; $|\mathcal{U}(\mathcal{P}_G)|$; and $|W(\mathcal{P}_G)|$ respectively. We give the following results in our notations.

Theorem 3.5. *[37]*
(i) If $G \in \Psi$, then

$$\beta(\mathcal{P}_G) = |V(\mathcal{P}_G)| - |\mathcal{U}(\mathcal{P}_G)| + 1.$$

(ii) If $G \notin \Psi$, then

$$\beta(\mathcal{P}_G) = |V(\mathcal{P}_G)| - |\mathcal{U}(\mathcal{P}_G)| + |W(\mathcal{P}_G)|.$$

Corollary 3.6. *[37] Suppose that $n = p_1^{r_1} \cdots p_t^{r_t}$, where $p_1, \ldots, p_t$ are primes with $p_1 < \cdots < p_t$, and $r_1, \ldots, r_t$ are positive integers. Let Z_n denotes the cyclic group of order n. Then*

$$\beta(\mathcal{P}_{Z_n}) = \begin{cases} n - 1, & \text{if } t = 1; \\ n - 2r_2, & \text{if } (t, p_1, t_1) = (2, 2, 1); \\ n - 2r_1, & \text{if } (t, p_1, t_2) = (2, 2, 1); \\ n + 1 - \prod_{t=1}^{t}(r_i + 1), & \text{otherwise}. \end{cases}$$

A *dihedral group* is presented as:

$$D_{2n} = < a, b | a^n = b^2 = e, (ab)^2 = e >.$$

D_{2n} is the disjoint union of the cyclic subgroup $Z_n \cong \langle a \rangle = \{e, a, a^2, \ldots, a^{n-1}\}$, and the set of involution $B = \{b, ab, a^2 b, \ldots, a^{n-1} b\}$.

Lemma 3.7. *Let $w \in B$ then, in the power graph $\mathcal{P}_{D_{2n}}$, the following are true:*
(i) $\overline{w} = B$;
(ii) w is not a resolving involution.

Proof. The neighborhood in the graph $\mathcal{P}_{D_{2n}}$, of every involution $w \in B$ is $\{e\}$. Therefore, $\overline{w} = B$. If $x, y \in V(\mathcal{P}_{D_{2n}}) \backslash B$, then there are two possibilities:
1) $x = a^s, y = e, \quad 1 \le s \le n - 1$;
2) $x = a^{s_1}, y = a^{s_2}, 1 \le s_1, s_2 \le n - 1$.
In the above two cases, one can see that $R\{x, y\} \ne \{x, y, w\}$. Therefore, $w \in B$ is not a resolving involution. □

Proof of Theorem 2.6: By part (ii) of Lemma 3.7, every resolving involution in $\mathcal{P}_{D_{2n}}$ belongs to the subgraph $P_{\langle a \rangle}$, corresponding to the cyclic subgroup $\langle a \rangle$, of D_{2n}. Therefore, $W(\mathcal{P}_{D_{2n}}) = W(\mathcal{P}_{\langle a \rangle})$. In the subgraph $\mathcal{P}_{\langle a \rangle}$,

the identity e and the generator a are twins. However, e is the unique singleton twin in $\mathcal{P}_{D_{2n}}$. By part (i) of Lemma 3.7, all $w \in B$ are in the same twin-set. Therefore, the set $\mathcal{U}(\mathcal{P}_{D_{2n}})$ is the disjoint union of $\mathcal{U}(\mathcal{P}_{\langle a \rangle})$; the twin-set of e, and the twin-set of w, for $w \in B$. Consequently, $|\mathcal{U}(\mathcal{P}_{D_{2n}})| = |\mathcal{U}(\mathcal{P}_{\langle a \rangle})| + 2$. A dihedral group D_{2n} does not satisfy the condition (C_4); therefore, $D_{2n} \notin \Psi$. Now, put $|V(\mathcal{P}_{D_{2n}})| = |V(\mathcal{P}_{\langle a \rangle})| + n$; $|\mathcal{U}(\mathcal{P}_{D_{2n}})| = |\mathcal{U}(\mathcal{P}_{\langle a \rangle})| + 2$; and $|W(\mathcal{P}_{D_{2n}})| = |W(\mathcal{P}_{\langle a \rangle})|$ in the equation of part (ii) of Theorem 3.5 to complete the proof.

To compute the exact value of $\beta(\mathcal{P}_{D_{2n}})$, one can use Theorem 2.6 and corollary 3.6.

Lemma 3.8. *A singleton twin x, in the power graph $\mathcal{P}_G$, is either an involution or the identity e in the group G.*
Proof. If x in G is not an involution or e, then the order of x is ≥ 3 and $N[x] = N[x^{-1}]$, a contradiction.

Proof of Theorem 2.7: Let G be a cyclic group of odd order and y is a generator of G. Then, there is no involution in the group G. Also, part (ii) of Proposition 3.2 implies that $\overline{y} = \overline{e}$, and e is not a singleton twin. Therefore, by Lemma 3.7, the graph $\mathcal{P}_G$ has no singleton twin. Hence, the exchange property holds in $\mathcal{P}_G$ by Theorem 2.2.

$\square$

The following example shows that the converse of Theorem 2.2 and Theorem 2.7 is not true.

Example 3.9. *Let $\mathcal{P}_{z_6}$ be the power graph, where $Z_6 \cong \langle x \rangle = \{e, x, x^2, x^3, x^4, x^5\}$. Then, the order of the group is even and not a power of a prime. Furthermore, the power graph contains the singleton twin x^3. Still, the graph has the exchange property for resolving sets.*

4 Conclusions

We give a new formula for computing the metric dimension of a simple graph without singleton twins. We also give sufficient conditions for a graph to have the exchange property for resolving sets. Moreover, we deduce a new way to define a matroid on finite group. It is proved that the new matroid is strongly base orderable and hence satisfies the conjecture of White [2]. We also compute the metric dimension of the power graphs of dihedral groups. We did not encounter a power graph of a finite group which does not have the exchange property. Therefore, the following question makes sense to be posed.

Question 4.1. *Does there exist a finite group whose power graph does not hold the exchange property?*

It is worth mentioning that the authors of [38] have cited the pre-published version of the present article and have answered Question 4.1. They, in fact, give a necessary and sufficient condition for resolving sets to have the exchange property in the power graphs of finite groups.

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