Open Math. 2019; 17:1156–1167 **DE GRUYTER**

Open Mathematics

Research Article

Liuyang Shao*

Non-trivial solutions for Schrödinger-Poisson systems involving critical nonlocal term and potential vanishing at infinity

https://doi.org/10.1515/math-2019-0091
Received April 4, 2019; accepted August 16, 2019

Abstract: The present study is concerned with the following Schrödinger-Poisson system involving critical nonlocal term

$$\begin{cases} -\triangle u + V(x)u - l(x)\phi|u|^3u = \eta K(x)f(u), & \text{in } \mathbb{R}^3, \\ -\triangle\phi = l(x)|u|^5, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where the potential $V(x)$ and $K(x)$ are positive continuous functions that vanish at infinity, and $l(x)$ is bounded, nonnegative continuous function. Under some simple assumptions on V, K, l and f, we prove that the problem (1.1) has a non-trivial solution.

Keywords: Schrödinger-Poisson system, variational methods, critical nonlocal term, vanishing potential

MSC: 35B09, 35J20

1 Introduction and main results

The aim of this paper is to investigate the existence of non-trivial solutions for the following Schrödinger-Poisson system involving critical nonlocal term and potential vanishing at infinity

$$\begin{cases} -\triangle u + V(x)u - l(x)\phi|u|^3u = \eta K(x)f(u), & \text{in } \mathbb{R}^3, \\ -\triangle\phi = l(x)|u|^5, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $V, K \in C(\mathbb{R}^3, \mathbb{R})$, $f \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$, $l(x)$ is bounded, nonnegative continuous function, and V, K are nonnegative functions which can be vanishing at infinity. $\eta > 0$ is a parameter and $2^* := 6$ is the critical Sobolev exponent. Similar problems have been widely investigated, and it is well known that they have a strong physical meaning, because they appear in quantum mechanics models and in semiconductor theory. In particular, systems like (1.1) have been introduced in [1] as a model describing solitary waves, for nonlinear stationary equations of Schrödinger type interacting with an electrostatic field, and are usually known as Schrödinger-Poisson systems. Indeed, in (1.1) the first equation is a nonlinear stationary Schrödinger equation that is coupled with a Poisson equation, to be satisfied by ϕ, meaning that the potential is determined by the charge of the wave function. For more details, we refer the readers to [2–5] and the references therein.

In recent years, with the aid of variational methods, there has been increasing attention to problems like (1.1) on the existence and non-existence of positive solutions, positive ground states, multiple solutions, sign-changing solutions and so on. see for instance [6–21], and the references therein. In fact, the most of the

***Corresponding Author: Liuyang Shao:** School of Mathematics and Statistics, Guizhou University of Finance and Economics, Guiyang, 550025 Guizhou, P.R. China; E-mail: sliuyang316@163.com

above results focused on the subcritical nonlocal term. However, the system (1.1) with critical nonlocal term has only been studied in [22–25].

In [22], A. Azzollini and P. d'Avenia firstly studied the following Schrödinger-Poisson system with critical nonlocal term

$$\begin{cases} -\triangle u = \mu u + p\phi|u|^3 u = f(x,u), & x \in B_R, \\ -\triangle\phi = p|u|^5, & x \in B_R, \\ u = \phi = 0, & \text{on } \partial B_R. \end{cases} \tag{1.2}$$

They proved that the existence and nonexistence results for system (1.1) depend on the value of λ.

In [23], Liu studied the following Schrödinger-Poisson system with critical nonlocal term

$$\begin{cases} -\triangle u + V(x)u - K(x)\phi|u|^3 u = f(x,u), & \text{in } \mathbb{R}^3, \\ -\triangle\phi = K(x)|u|^5, & \text{in } \mathbb{R}^3. \end{cases} \tag{1.4}$$

Under the condition $V(x), K(x), f$ are asymptotically periodic, the author proved the system (1.4) has at least a positive solution by the mountain pass theorem and the concentration-compactness principle.

In [25], F. Li, Y. Li and J. Shi proved

$$\begin{cases} -\triangle u + bu - \phi|u|^3 u = f(u), & \text{in } \mathbb{R}^3, \\ -\triangle\phi = |u|^5, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.3}$$

possesses at least one positive radially symmetric solution when $b > 0$ is a constant. To the best of our knowledge, there seems to be little progress on the existence of nontrivial solution for Schrödinger-Poisson systems involving critical nonlocal term and potential vanishing at infinity.

By the motivation of above work, In our article, we establish the existence of non-trivial solution for problem (1.1) with critical nonlocal term and potential vanishing at infinity. Firstly the critical growth causes a lack of compactness, and it is much more difficult to obtain the existence of non-trivial solutions. Secondly since $V(x)$ is potential vanishing at infinity, which makes our studies more interesting. At last, we obtain a non-trivial solution by using the mountain pass theorem without (PS) condition.

Below, we assume that the pair (V, K) of continuous functions $V, K : \mathbb{R}^3 \to \mathbb{R}$ belongs to $\mathcal{K}$. Throughout the paper, $(V, K) \in \mathcal{K}$ means that

(VK_1): $V(x), K(x) > 0$ for all $x \in \mathbb{R}^3$ and $K \in L^\infty(\mathbb{R}^3)$.

(VK_2): If $\{A_n\}_n \subset \mathbb{R}^3$ is a sequence of Borel sets such that the Lebesgue measure meas$(A_n) \le R$, for all $n \in \mathbb{N}$ and some $R > 0$, then

$$\lim_{R\to+\infty} \int\limits_{A_n\cap B_R^c(0)} K(x)\mathrm{d}x = 0, \text{ uniformly in } n \in \mathbb{N}.$$

Furthermore, one of the below conditions occurs:

(VK_3): $K/V \in L^\infty(\mathbb{R}^3)$ or

(VK_4): there exists $p_0 \in (2, 6)$ such that

$$\frac{K(x)}{V(x)^{\frac{6-p_0}{4}}} \to 0 \text{ as } |x| \to +\infty.$$

The hypotheses (VK_1) – (VK_4) on functions V and K were first introduced in [26] and characterized problem (1.1) as zero mass. Problems of zero mass have been studied by many authors, see for example, [27–32] and references therein.

Finally, we assume the following growth conditions at the origin and at the infinity for the continuous function $f : \mathbb{R} \to \mathbb{R}$:

(f_1) $\lim\limits_{t\to 0} \frac{f(t)}{t} = 0$, if (VK_3) holds; or $\lim\limits_{t\to 0} \frac{f(t)}{|t|^{p_0-1}} < \infty$, if (VK_4) holds.

(f_2) f has a "quasicritical growth", namely, $\lim\limits_{t\to+\infty} \frac{f(t)}{t^5} = 0$.

(f_3) There exists a $\theta \in (2, 2^*)$ such that

$$0 \le \theta F(t) \le tf(t) \text{ for all } t \in \mathbb{R},$$

where $F(u) = \int_0^u f(s)\mathrm{d}s$.

Furthermore, we make the following hypotheses on the function $l(x)$.
(l_1) There exists x_0, such that

$$l(x_0) = \sup_{x\in\mathbb{R}^3} l(x).$$

(l_2) For x close to x_0 we have

$$l(x) = l(x_0) + O(|x - x_0|), \text{ as } x \to x_0.$$

Now we state our main results as follows.

Theorem 1.1. *Suppose that* $(V, K) \in \mathcal{K}$, *and* f *satisfies* (f_1), (f_2), (f_3), $l(x)$ *satisfies* $(l_1) - (l_2)$.
(i) If $\theta \in (1, 3]$, *for sufficiently lange* $\eta > 0$, *then system* (1.1) *has at least one non-trivial solution.*
(ii) If $\theta \in (3, 5)$, *for any* $\eta > 0$, *then system* (1.1) *has at least one non-trivial solution.*

Notation. In this paper we make use of the following notations: C will denote various positive constants; the strong (respectively weak) convergence is denoted by $\to$ (respectively $\rightharpoonup$); $o(1)$ denotes $o(1) \to 0$ as $n \to \infty$, $B_\rho(0)$ denotes a ball centered at the origin with radius $\rho > 0$.

The remainder of this paper is organized as follows. In Section 2, some preliminary results are presented. In Section 3, we give the proof of our main results.

2 Variational setting and preliminaries

Let us consider the space

$$E = \left\{ u \in D^{1,2}(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)|u|^2 \mathrm{d}x < +\infty \right\}$$

endowed with the norm

$$\|u\|_E^2 = \int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x + \int_{\mathbb{R}^3} V(x)|u|^2 \mathrm{d}x.$$

Recall that a weak solution of problem (1.1) is a function $u \in E$ such that

$$\int_{\mathbb{R}^3} \nabla u \nabla \varphi \mathrm{d}x + \int_{\mathbb{R}^3} V(x)u\varphi \mathrm{d}x - \int_{\mathbb{R}^3} l(x)\phi|u|^4\varphi \mathrm{d}x - \eta \int_{\mathbb{R}^3} K(x)f(u)\varphi \mathrm{d}x = 0, \tag{2.1}$$

for all $\varphi \in E$.
Then, the weak solutions of (1.1) are the critical points of the energy functional defined on E by

$$J(u) := \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)|u|^2)\mathrm{d}x - \frac{1}{10}\int_{\mathbb{R}^3} l(x)\phi_u|u|^5 \mathrm{d}x - \eta \int_{\mathbb{R}^3} K(x)f(u)\mathrm{d}x, \tag{2.2}$$

where $F(u) = \int_0^u f(s)\mathrm{d}s$. More precisely, $J \in C^1(E, \mathbb{R})$ and its differential $J' : E \to E'$ is defined as

$$\langle J'(u), v\rangle = \int_{\mathbb{R}^3} (|\nabla u||\nabla v| + V(x)uv)\mathrm{d}x - \int_{\mathbb{R}^3} l(x)\phi_u|u|^4 v\mathrm{d}x - \eta \int_{\mathbb{R}^3} K(x)f(u)v\mathrm{d}x, \tag{2.3}$$

for all $v \in E$, where E' is the dual space of E.

We define the Lebesgue space $L^p_K(\mathbb{R}^N)$ composed by all measurable functions $u : \mathbb{R}^N \to \mathbb{R}$ such that

$$L^p_K(\mathbb{R}^3) = \left\{ u : \mathbb{R}^3 \to \mathbb{R} | u \text{ is measurable and } \int_{\mathbb{R}^3} K(x)|u|^p \mathrm{d}x < +\infty, \right\}$$

endowed with norm

$$\|u\|_{L^p_K(\mathbb{R}^3)} := \left(\int\limits_{\mathbb{R}^3} K(x)|u|^p \mathrm{d}x\right)^{\frac{1}{p}},$$

and we will state, without proof, two important results of Alves and Souto (see [[26], Lemmas 2.1 and 2.2]).

Proposition 2.1. *Assume $(V, K) \in \mathcal{K}$ holds, E is compactly embedded in $L^p_K(\mathbb{R}^N)$ for every $p \in (2, 6)$.If (VK_4) holds, E is compactly embedded in $L^p_K(\mathbb{R}^N)$.*

Proposition 2.2. *Suppose that f satisfies (f_1) and (f_2) and $(V, K) \in \mathcal{K}$, Let $\{v_n\}$ be such that $v_n \rightharpoonup v$ in E. Then,*

$$\int\limits_{\mathbb{R}^3} K(x)F(v_n)\mathrm{d}x \to \int\limits_{\mathbb{R}^3} K(x)F(v)\mathrm{d}x \tag{2.4}$$

and

$$\int\limits_{\mathbb{R}^3} K(x)f(v_n)v_n\mathrm{d}x \to \int\limits_{\mathbb{R}^3} K(x)f(v)v\mathrm{d}x. \tag{2.5}$$

Lemma 2.3 [25, Lemma 2.1] *For every $u \in L^6(\mathbb{R}^3)$, there exists a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ which is the solution of*

$$-\triangle\phi = |u|^5, \quad in\ \mathbb{R}^3,$$

here ϕ_u can be expressed by the from

$$\phi_u(x) = \int\limits_{\mathbb{R}^3} \frac{|u(y)|^5}{|x-y|}\mathrm{d}y$$

Moreover,
(i) $\|\phi_u\|^2_{D^{1,2}(\mathbb{R}^3)} = \int_{\mathbb{R}^3} \phi_u|u|^5$;
(ii) $\phi_u(x) > 0$ for $x \in \mathbb{R}^3$;
(iii) for any $\theta > 0$, $\phi_{u_\theta} = \theta^2(\phi_u)_\theta$, where $u_\theta(\cdot) = u(\cdot/\theta)$;
(iv) for any $t > 0$, $\phi_{tu} = t^5\phi_u$;
(v) for any $u \in L^6(\mathbb{R}^3)$,

$$\|\phi_u\|_{D^{1,2}(\mathbb{R}^3)} \le S^{-\frac{1}{2}}|u|_6^5, \int\limits_{\mathbb{R}^3} \phi_u|u|^5 \le S^{-1}|u|_6^{10},$$

where S is defined in (1.6);
(vi) if $u_n \rightharpoonup u$ in $L^6(\mathbb{R}^3)$ and $u_n \to u$ a.e in $\mathbb{R}^3$ as $n \to \infty$, then $\phi_{u_n} \to \phi_u$ in $D^{1,2}(\mathbb{R}^3)$.

Lemma 2.4. [25, Lemma 2.3] *If $u_n \rightharpoonup u$ in $L^6(\mathbb{R}^3)$, a,e in $\mathbb{R}^3$, then as $n \to \infty$,*

$$|u_n|^5 - |u_n - u|^5 - |u|^5 \to 0 \quad in\ L^{\frac{6}{5}}(\mathbb{R}^3),$$

$$\phi_{u_n} - \phi_{u_n - u} - \phi_u \to 0, \quad in\ D^{1,2}(\mathbb{R}^3),$$

$$\int\limits_{\mathbb{R}^3} \phi_{u_n}|u_n|^5\mathrm{d}x - \int\limits_{\mathbb{R}^3} \phi_{u_n-u}|u_n - u|^5\mathrm{d}x - \int\limits_{\mathbb{R}^3} \phi_u|u|^5\mathrm{d}x \to 0,$$

and

$$|u_n|^3u_n - |u_n - u|(u_n - u) - |u|^3u \to 0, \ in\ D^{1,2}(\mathbb{R}^3).$$

3 Proof of Theorem 1.1

To prove Lemma 3.2, we need the following results.

Lemma 3.1. *Suppose that $(V, K) \in \mathcal{K}$ hold. Then for $p \in [2, 6]$, there is $C > 0$ such that*

$$\|u\|_{L^p_k(\mathbb{R}^3)} \le C\|u\|_E, \quad \forall\, u \in E.$$

Proof. First we suppose that (VK_2) holds. The proof is trivial if $p = 2$ or 6. Now we prove that the embedding is true for $p \in (2,6)$ under the assumption (VK_3). For fixed $p \in (2,6)$, define $m = \frac{(6-p)}{4}$, and hence $p = 2m + (1-m)6$. so we have that

$$
\begin{aligned}
\int_{\mathbb{R}^3} K(x)|u|^p \mathrm{d}x &= \int_{\mathbb{R}^3} |u|^{2m}|u|^{(1-m)6}\mathrm{d}x \\
&\leq \left(\int_{\mathbb{R}^3} |K(x)|^{\frac{1}{m}}|u|^2\mathrm{d}x\right)^m \left(\int_{\mathbb{R}^3} |u|^6\mathrm{d}x\right)^{1-m} \\
&\leq \left(\sup_{x\in\mathbb{R}^3} \frac{|K(x)|}{|V(x)|^m}\right)\left(\int_{\mathbb{R}^3} V(x)u^2\mathrm{d}x\right)^m \left(\int_{\mathbb{R}^3} |u|^6\mathrm{d}x\right)^{1-m} \\
&\leq C\left(\sup_{x\in\mathbb{R}^3} \frac{|K(x)|}{|V(x)|^m}\right)\left(\int_{\mathbb{R}^3} V(x)u^2\mathrm{d}x\right)^m \left(\int_{\mathbb{R}^3} |\nabla u|^2\mathrm{d}x\right)^{3(1-m)} \\
&\leq C\left(\sup_{x\in\mathbb{R}^3} \frac{|K(x)|}{|V(x)|^m}\right)\left(\int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2)\mathrm{d}x\right)^{m+3(1-m)} \\
&= C\left(\sup_{x\in\mathbb{R}^3} \frac{|K(x)|}{|V(x)|^m}\right)\sup_{x\in\mathbb{R}^3}\left(\int_{\mathbb{R}^3} |\nabla u|^2 + V(x)|u|^2\mathrm{d}x\right)^{\frac{p}{2}}.
\end{aligned}
$$

Since $K(x) \in L^\infty(\mathbb{R}^3)$ and $K/V \in L^\infty(\mathbb{R}^3)$, we have that

$$\|u\|_{L^p_K(\mathbb{R}^3)} \leq C\|u\|_E, \quad \text{for } p \in (2,6).$$

Next, we suppose that (VK_4) holds. Using the same argument as above, we define $m_0 = \frac{(6-p_0)}{4}$, and hence $p_0 = 2m_0 + (1-m_0)6$ so that we have

$$
\begin{aligned}
\int_{\mathbb{R}^3} K(x)|u|^{p_0} \mathrm{d}x &= \int_{\mathbb{R}^3} K(x)|u|^{2m_0}|u|^{(1-m_0)6}\mathrm{d}x \\
&\leq \left(\int_{\mathbb{R}^3} |K(x)|^{\frac{1}{m_0}}|u|^2\mathrm{d}x\right)^{m_0} \left(\int_{\mathbb{R}^3} |u|^6\mathrm{d}x\right)^{1-m_0} \\
&\leq \left(\sup_{x\in\mathbb{R}^3} \frac{|K(x)|}{|V(x)|^{m_0}}\right)\left(\int_{\mathbb{R}^3} V(x)|u|^2\mathrm{d}x\right)^{m_0} \left(\int_{\mathbb{R}^3} |u|^6\mathrm{d}x\right)^{1-m_0} \\
&\leq C\left(\sup_{x\in\mathbb{R}^3} \frac{|K(x)|}{|V(x)|^{m_0}}\right)\left(\int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)|u|^2)\mathrm{d}x\right)^{\frac{p_0}{2}}.
\end{aligned}
$$

From (VK_3) we deduce that $\frac{|K(x)|}{|V(x)|^{m_0}} \in L^\infty(\mathbb{R}^3)$. It follows from the above inequality that

$$\|u\|_{L^{p_0}_K(\mathbb{R}^3)} \leq C\|u\|_E.$$

we complete the proof. □

The functional J satisfies the mountain pass geometry.

Lemma 3.2. *The functional J satisfies the following conditions:*
(i) There exist ρ and $\alpha > 0$ such that $J(u) \geq \alpha$ with $\|u\|_E = \rho$.

(ii) *There exists* $e \in B_\rho(0)$ *with* $J(e) < 0$.

Proof. (i) Now, we distinguish two case.

Case 1. We suppose that (VK_3) is true. For any $\varepsilon > 0$, it follows from (f_1) and (f_2) that there exists $C_\varepsilon > 0$ such that

$$F(u) \leq \frac{\varepsilon}{2}|u|^2 + C_\varepsilon|u|^6, \quad \text{for all } u \in E. \tag{3.1}$$

Thus, by (3.1) and Lemma 3.1, we get that

$$\begin{aligned}\int_{\mathbb{R}^3} K(x)F(u)\mathrm{d}x &\leq \frac{\varepsilon}{2}\int_{\mathbb{R}^3} K(x)|u|^2\mathrm{d}x + C_\varepsilon\int_{\mathbb{R}^3} K(x)|u|^6\mathrm{d}x \\ &\leq \frac{\varepsilon}{2}\|u\|_E^2 + C_\varepsilon\|u\|_E^6.\end{aligned}$$

Hence, in view of Lemma 2.3, we obtain

$$\begin{aligned}J(u) &= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + V(x)|u|^2\mathrm{d}x - \frac{1}{10}\int_{\mathbb{R}^3} l(x)\phi_u|u|^5\mathrm{d}x - \eta\int_{\mathbb{R}^3} K(x)F(u)\mathrm{d}x \\ &\geq \frac{1}{2}\|u\|_E^2 - C\|u\|_E^{10} - \frac{\varepsilon}{2}\|u\|_E^2 - C_\varepsilon\|u\|_E^6 \\ &= (\frac{1-\varepsilon}{2})\|u\|_E^2 - C\|u\|_E^{10} - C_\varepsilon\|u\|_E^6.\end{aligned}$$

So, taking $\varepsilon = \frac{1}{2}$, there exists enough small $\|u\|_E = \rho$, such that $J(u) \geq \alpha$.

Case 2. We suppose that (VK_4) holds. By (f_1) and (f_2), there exist $C'_\varepsilon > 0$ and $C''_\varepsilon > 0$ such that

$$F(u) \leq C'_\varepsilon|u|^{p_0} + C''_\varepsilon|u|^6, \quad \text{for all } u \in E.$$

Therefore

$$\begin{aligned}J(u) &= \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|_E^2 + V(x)u^2\mathrm{d}x - \frac{1}{10}\int_{\mathbb{R}^3} l(x)\phi_u|u|^5\mathrm{d}x - \eta\int_{\mathbb{R}^3} K(x)F(u)\mathrm{d}x \\ &\geq \frac{1}{2}\|u\|_E^2 - C\|u\|_E^6 - C\|u\|_E^{10} - C\|u\|_E^{p_0}.\end{aligned}$$

The same as Case 1, we can take $\|u\|_E = \rho$ such that $J(u) \geq \alpha$.
(ii) For every $t > 0$ we obtain

$$J(tu) = \frac{t^2}{2}\|u\|_E^2 - \frac{t^{10}}{10}\int_{\mathbb{R}^3} K(x)\phi_u|u|^5\mathrm{d}x - \eta\int_{\mathbb{R}^3} K(x)F(tu)\mathrm{d}x.$$

From (f_3), we obtain $J(tu) \to -\infty$ as $t \to +\infty$, so it is satisfies (ii). We complete the proof. □

As a consequence of Lemma 3.2, we can find a (PS) sequence of the functional $J(u)$ at the level

$$c := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J(\gamma(t)) > 0, \tag{3.2}$$

where the set of paths is defined as

$$J := \{\gamma \in C([0,1], H^1(\mathbb{R}^3)) : \gamma(0) = 0, J(\gamma(1)) < 0\}.$$

Lemma 3.3. *Let* $\{u_n\}$ *be a* $(PS)_c$ *sequence for* J. *Then* $\{u_n\}$ *is bounded in* E.
Proof. Let $\{u_n\} \subset E$ be a $(PS)_c$ sequence for J, that is

$$J(u_n) \to c \text{ and } J'(u_n) \to 0 \text{ as } n \to +\infty.$$

Therefore, From (f_3), we have

$$c+1+\|u_n\|_E \geq J(u_n)-\frac{1}{\theta}\langle J'(u_n),u_n\rangle$$
$$=\left(\frac{1}{2}-\frac{1}{\theta}\right)\|u_n\|_E^2+\frac{\eta}{\theta}\int_{\mathbb{R}^3}K(x)(f(u_n)u_n-\theta F(u_n))\mathrm{d}x+\left(\frac{1}{\theta}-\frac{1}{10}\right)\int_{\mathbb{R}^3}l(x)\phi_u|u|^5\mathrm{d}x$$
$$\geq\left(\frac{1}{2}-\frac{1}{\theta}\right)\|u_n\|_E^2,$$

which implies that $\{u_n\}$ is bounded in E. □

Because of the appearance of the critical nonlocal term, we have to estimate the Mountain-pass value given by (3.2) carefully, To do it, we choose the extremal function $U_\varepsilon(x)=\frac{(3\varepsilon^2)^{\frac{1}{4}}}{(\varepsilon^2+|x-x_0|^2)^{\frac{1}{2}}}$ to solve $-\triangle u=u^5$ in $\mathbb{R}^3$. Let $\varphi\in C_0^\infty(\mathbb{R}^3)$ be a cut-off function verifying that $0\leq\varphi(x)\leq1$ for all $x\in\mathbb{R}^3$, $supp\varphi\subset B_2(x_0)$, and $\varphi(x)\equiv1$ on $B_1(x_0)$. Set $V_\varepsilon=\varphi U_\varepsilon$, then thanks to the asymptotic estimates from [8], we have

$$|\nabla v_\varepsilon|_2^2=S^{\frac{3}{2}}+O(\varepsilon),\quad |v_\varepsilon|_6^2=S^{\frac{1}{2}}+O(\varepsilon)$$

and for $s\in[2,6)$

$$|v_\varepsilon|_s^s=\begin{cases}O(\varepsilon^{\frac{s}{2}}), & \text{if } s\in[2,3),\\ O(\varepsilon^{\frac{3}{2}}|log\varepsilon|), & \text{if } s=3,\\ O(\varepsilon^{\frac{6-s}{2}}), & \text{if } s\in(3,6),\end{cases}$$

where S denotes the best constant for the embedding $D^{1,2}(\mathbb{R}^3)\hookrightarrow L^6(\mathbb{R}^3)$, namely,

$$S:=\inf_{u\in D^{1,2}(\mathbb{R}^3)}\left\{\int_{\mathbb{R}^3}|\nabla u|^2dx,\int_{\mathbb{R}^3}|u|^6dx=1\right\}.$$

We define

$$V_{max}:=\max_{x\in B_{2R}(x_0)}V(x)$$

and

$$K_{min}:=\min_{x\in B_{2R}(x_0)}K(x).$$

By the assumption (l_2) we also have

$$l(x_0)\int_{B_{2R}(x_0)}\phi|u|^5\mathrm{d}x\leq\int_{B_{2R}(x_0)}l(x)\phi|u|^5\mathrm{d}x.$$

Lemma 3.4. *Suppose that $(V,K)\in\mathcal{K}$, $(f_1)-(f_3)$ hold, and $l(x)$ satisfies $(l_1)-((l_2)$. Then, there exists a $u_0\in E\backslash\{0\}$ such that*

$$0<\sup_{t\geq0}J(tu_0)<\frac{2}{5}S^{\frac{3}{2}}|l(x)|_{L^\infty(\mathbb{R}^3)}^{-\frac{1}{2}}.$$

Proof. We now consider

$$I(tv_\varepsilon)=\frac{t^2}{2}\int_{\mathbb{R}^3}(|\nabla v_\varepsilon|^2+V(x)|v_\varepsilon|^2)\mathrm{d}x-\frac{t^{10}}{10}\int_{\mathbb{R}^3}l(x)\phi_{v_\varepsilon}|v_\varepsilon|^5\mathrm{d}x-\eta\int_{\mathbb{R}^3}K(x)F(tv_\varepsilon)\mathrm{d}x.$$

By Lemma 3.1, we know that, there exists $t_\varepsilon>0$ such that $\sup_{t\geq0}J(tv_\varepsilon)>0$ is attained and $\lim_{t\to\infty}J(tv_\varepsilon)=-\infty$ for any $\varepsilon>0$.

We suppose that there exists ϱ_1,ϱ_2 such that $\varrho_1<t_\varepsilon<\varrho_2$ for small enough $\varepsilon>0$. In fact, $J(t_\varepsilon v_\varepsilon)=\sup_{t\geq0}J(tv_\varepsilon)$, and hence $dJ(tv_\varepsilon)/dt|_{t=t_\varepsilon}=0$, we obtain that

$$t_\varepsilon\int_{B_{2R}(x_0)}(|\nabla v_\varepsilon|^2+V(x)|v_\varepsilon|^2)\mathrm{d}x-\eta\int_{B_{2R}(x_0)}K(x)f(t_\varepsilon v_\varepsilon)v_\varepsilon\mathrm{d}x-t_\varepsilon^{10}\int_{B_{2R}(x_0)}l(x)\phi_{v_\varepsilon}|v_\varepsilon|^5\mathrm{d}x=0. \quad (3.3)$$

Now, we prove that $t_\varepsilon \to +\infty$ as $\varepsilon_n \to 0^+$ does not hold. By (3.3)

$$t_{\varepsilon_n} \int_{B_{2R}(x_0)} (|\nabla v_{\varepsilon_n}|^2 + V(x)|v_{\varepsilon_n}|^2)\mathrm{d}x \geq t_{\varepsilon_n}^{10} \int_{B_{2R}(x_0)} l(x)\phi_{v_\varepsilon}|v_\varepsilon|^5 \mathrm{d}x,$$

which is a contradiction when $t_\varepsilon \to +\infty$.

Similarly, we suppose that there is a sequence $\widetilde{t}_{\varepsilon_n} \to 0$ as $\varepsilon_n \to 0^+$. Firstly, if (VK_3) holds, from (f_1) and (f_2), for all $\delta > 0$ there exists $C_\delta > 0$ such that

$$\begin{aligned}\int_{\mathbb{R}^3} K(x)f(\widetilde{t}_{\varepsilon_n} v_{\varepsilon_n})v_{\varepsilon_n}\mathrm{d}x &\leq \delta \widetilde{t}_{\varepsilon_n} \int_{\mathbb{R}^3} K(x)|v_{\varepsilon_n}|^2\mathrm{d}x + C_\delta(\widetilde{t}_{\varepsilon_n})^5 \int_{\mathbb{R}^3} K(x)|v_{\varepsilon_n}|^6\mathrm{d}x \\ &\leq \delta C\widetilde{t}_{\varepsilon_n} \int_{\mathbb{R}^3} (|\nabla v_{\varepsilon_n}|^2 + V(x)|v_{\varepsilon_n}|^2)\mathrm{d}x + C_\delta(\widetilde{t}_{\varepsilon_n})^5 \int_{\mathbb{R}^3} K(x)|v_{\varepsilon_n}|^6\mathrm{d}x.\end{aligned}$$

Choosing $\delta = \frac{1}{2C}$, it follows from (3.3) that

$$\frac{\widetilde{t}_{\varepsilon_n}}{2} \int_{\mathbb{R}^3} (|\nabla v_{\varepsilon_n}|^2 + V(x)|v_{\varepsilon_n}|^2)\mathrm{d}x \leq C_\delta(\widetilde{t}_{\varepsilon_n})^5 \int_{\mathbb{R}^3} K(x)|v_{\varepsilon_n}|^6\mathrm{d}x + (\widetilde{t}_{\varepsilon_n})^9 \int_{\mathbb{R}^3} l(x)\phi_{v_n}|v_{\varepsilon_n}|^5\mathrm{d}x.$$

Next, we suppose that (VK_4) holds. By (f_1), (f_2), there is a constant $\overline{C} > 0$, such that

$$\int_{\mathbb{R}^3} K(x)f(\widetilde{t}_{\varepsilon_n} v_{\varepsilon_n})\mathrm{d}x \leq (\widetilde{t}_{\varepsilon_n})^{p_0-1} \int_{\mathbb{R}^3} K(x)|v_{\varepsilon_n}|^{p_0}\mathrm{d}x + \overline{C}(\widetilde{t}_{\varepsilon_n})^5 \int_{\mathbb{R}^3} K(x)|v_{\varepsilon_n}|^6\mathrm{d}x.$$

It again follows from (3.3) that

$$\begin{aligned}\widetilde{t}_{\varepsilon_n} \int_{\mathbb{R}^3} (|\nabla v_{\varepsilon_n}|^2 + V(x)|v_{\varepsilon_n}|^2)\mathrm{d}x \leq& (\widetilde{t}_{\varepsilon_n})^{p_0-1} \int_{\mathbb{R}^3} K(x)|v_{\varepsilon_n}|^{p_0}\mathrm{d}x + \overline{C}(\widetilde{t}_{\varepsilon_n})^5 \int_{\mathbb{R}^3} K(x)|v_{\varepsilon_n}|^6\mathrm{d}x \\ &+ (\widetilde{t}_{\varepsilon_n})^9 \int_{\mathbb{R}^3} l(x)\phi_{v_{\varepsilon_n}}|v_{\varepsilon_n}|^5\mathrm{d}x,\end{aligned}$$

We arrive at a contradiction because $p_0 > 2$. So we complete the proof.
Since $0 < \varrho_1 < t_\varepsilon < \varrho_2 < \infty$, together with the definitions of V_{max} and K_{min}, we have

$$\begin{aligned}J(tv_\varepsilon) =& \frac{t^2}{2} \int_{\mathbb{R}^3} (|\nabla v_\varepsilon|^2 + V(x)|v_\varepsilon|^2)\mathrm{d}x - \frac{t^{10}}{10} \int_{\mathbb{R}^3} l(x)\phi_{v_\varepsilon}|v_\varepsilon|^5\mathrm{d}x - \eta \int_{\mathbb{R}^3} K(x)F(tv_\varepsilon)\mathrm{d}x \\ \leq& \frac{t_\varepsilon^2}{2} \int_{\mathbb{R}^3} (|\nabla v_\varepsilon|^2 + V(x)|v_\varepsilon|^2)\mathrm{d}x - \frac{t_\varepsilon^{10}}{10} \int_{\mathbb{R}^3} l(x)\phi_{v_\varepsilon}|v_\varepsilon|^5\mathrm{d}x - \eta \int_{\mathbb{R}^3} K(x)F(t_\varepsilon v_\varepsilon)\mathrm{d}x \\ \leq& \frac{t_\varepsilon^2}{2} \int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2\mathrm{d}x + \frac{t_\varepsilon^2}{2} V_{max}(x) \int_{\mathbb{R}^3} |v_\varepsilon|^2\mathrm{d}x - \frac{t_\varepsilon^{10}}{10} \int_{\mathbb{R}^3} l(x)\phi_{v_\varepsilon}|v_\varepsilon|^5\mathrm{d}x - \eta K_{min}(x) \int_{\mathbb{R}^3} F(t_\varepsilon v_\varepsilon)\mathrm{d}x.\end{aligned}$$

We define

$$h(t) := \frac{t^2}{2} \int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2\mathrm{d}x - \frac{t^{10}}{10} \int_{\mathbb{R}^3} l(x)\phi_{v_\varepsilon}|v_\varepsilon|^5\mathrm{d}x.$$

By some elementary calculations, we obtain

$$\max_{t\geq 0} h(t) = \frac{4(\frac{1}{2}\int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2\mathrm{d}x)^{\frac{5}{4}}}{5(\frac{1}{2}\int_{\mathbb{R}^3} l(x)\phi_{v_\varepsilon}|v_\varepsilon|^5\mathrm{d}x)^{\frac{1}{4}}} = \frac{2}{5}\frac{(\int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2\mathrm{d}x)^{\frac{5}{4}}}{(\int_{\mathbb{R}^3} l(x)\phi_{v_\varepsilon}|v_\varepsilon|^5\mathrm{d}x)^{\frac{1}{4}}}.$$

The Poisson equation $-\triangle\phi_{v_\varepsilon} = |v_\varepsilon|^5$ and Cauchy's inequality give

$$\int_{\mathbb{R}^3} |v_\varepsilon|^6\mathrm{d}x = \int_{\mathbb{R}^3} \nabla\phi_{v_\varepsilon}|\nabla v_\varepsilon|\mathrm{d}x$$

$$\leq \frac{1}{2|l(x)|_\infty}\int_{\mathbb{R}^3}|\nabla\phi_{v_\varepsilon}|^2\mathrm{d}x+\frac{|l(x)|_\infty}{2}\int_{\mathbb{R}^3}|\nabla v_\varepsilon|^2\mathrm{d}x$$
$$=\frac{1}{2|l(x)|_\infty}\int_{\mathbb{R}^3}\phi_{v_\varepsilon}|v_\varepsilon|^5\mathrm{d}x+\frac{|l(x)|_\infty}{2}\int_{\mathbb{R}^3}|\nabla v_\varepsilon|^2\mathrm{d}x.$$

This implies that

$$\int_{\mathbb{R}^3} l(x)\phi_{v_\varepsilon}|v_\varepsilon|^5\mathrm{d}x \geq 2|l(x)|_\infty\int_{\mathbb{R}^3} l(x)|v_\varepsilon|^6\mathrm{d}x-|l(x)|_\infty^2\int_{\mathbb{R}^3}|\nabla v_\varepsilon|^2\mathrm{d}x$$
$$=|l(x)|_\infty^2 S^{\frac{3}{2}}+O(\varepsilon).$$

As a consequence of the above fact, one has

$$\max_{t\geq 0} h(t)\leq \frac{2}{5}\frac{(S^{\frac{2}{5}}+O(\varepsilon))^{\frac{5}{4}}}{(|l(x)|_\infty^2 S^{\frac{3}{2}}+O(\varepsilon))^{\frac{1}{4}}}=\frac{2}{5}|l(x)|_\infty S^{\frac{3}{2}}+O(\varepsilon).$$

On the other hand, from (f_3), we obtain that $F(s)\geq Cs^\theta$, for $s>0$. Hence,

$$\int_{B_{2R}(x_0)} F(t_\varepsilon v_\varepsilon)\mathrm{d}x \geq C\int_{B_{2R}(x_0)}(t_\varepsilon v_\varepsilon)^\theta\mathrm{d}x\geq C\varrho_1^\theta\int_{B_R(x_0)}(v_\varepsilon)^\theta\mathrm{d}x=\begin{cases} C\varrho_1^\theta O(\varepsilon^{\frac{\theta+1}{2}}), \text{ if } \theta\in[1,2),\\ C\varrho_1^\theta O(\varepsilon^{\frac{3}{2}}|log\varepsilon|), \text{ if } \theta=2,\\ C\varrho_1^\theta O(\varepsilon^{\frac{5-\theta}{2}}), \text{ if } \theta\in(2,5).\end{cases} \tag{3.4}$$

We have $\max_{t\geq 0} J(tv_\varepsilon)=J(t_\varepsilon v_\varepsilon)$ at the beginning, that is,

$$\max_{t\geq 0} J(tv_\varepsilon)=\frac{t_\varepsilon^2}{2}\int_{\mathbb{R}^3}|\nabla v_\varepsilon|^2+V(x)|v_\varepsilon|^2\mathrm{d}x-\frac{t_\varepsilon^{10}}{10}\int_{\mathbb{R}^3} l(x)\phi_{v_\varepsilon}|v_\varepsilon|^5\mathrm{d}x-\eta\int_{\mathbb{R}^3}K(x)F(tv_\varepsilon)\mathrm{d}x$$
$$\leq h(t_\varepsilon)+\frac{V_{max}}{2}\int_{\mathbb{R}^3}|v_\varepsilon|^2\mathrm{d}x-\eta\int_{\mathbb{R}^3}K(x)F(tv_\varepsilon)\mathrm{d}x$$
$$\leq \frac{2}{5}|l(x)|_\infty S^{\frac{3}{2}}+CO(\varepsilon)-\eta\int_{\mathbb{R}^3}K(x)F(tv_\varepsilon)\mathrm{d}x.$$

Using (3.4), we have

$$CO(\varepsilon)-\eta\int_{\mathbb{R}^3}K(x)F(tv_\varepsilon)\mathrm{d}x\leq CO(\varepsilon)-\begin{cases} C\eta\varrho_1^\theta O(\varepsilon^{\frac{\theta+1}{2}}), & \text{if } \theta\in[1,2),\\ C\eta\varrho_1^\theta O(\varepsilon^{\frac{3}{2}}|log\varepsilon|), & \text{if } \theta=2,\\ C\eta\varrho_1^\theta O(\varepsilon^{\frac{5-\theta}{2}}), & \text{if } \theta\in(2,5).\end{cases}$$

If $(1,2)$ and $\eta=\varepsilon^{-\frac{1}{2}}$, then $\frac{1}{2}<\frac{\theta+1}{2}-\frac{1}{2}<1$ and hence for small enough $\varepsilon>0$

$$CO(\varepsilon)-\eta\int_{\mathbb{R}^3}K(x)F(tv_\varepsilon)\mathrm{d}x\leq CO(\varepsilon)-C\varrho_1^\theta\varepsilon^{-\frac{1}{2}}O(\varepsilon^{\frac{\theta+1}{2}})<0,$$

when $\eta>0$ is enough large.
If $\theta=2$ and $\eta=\varepsilon^{-\frac{1}{2}}$, then for small enough $\varepsilon>0$

$$CO(\varepsilon)-\eta\int_{\mathbb{R}^3}K(x)F(tv_\varepsilon)\mathrm{d}x\leq CO(\varepsilon)-C\varrho_1^\theta\varepsilon^{-\frac{1}{2}}O(\varepsilon^{\frac{3}{2}})|log\varepsilon|<0,$$

when $\eta>0$ is enough large.
If $\theta\in(2,3]$ and $\eta=\varepsilon^{-\frac{1}{2}}$, then $\frac{1}{2}\leq\frac{5-\theta}{2}-\frac{1}{2}<1$ and hence for small enough $\varepsilon>0$

$$CO(\varepsilon)-\eta\int_{\mathbb{R}^3}K(x)F(tv_\varepsilon)\mathrm{d}x\leq CO(\varepsilon)-C\varrho_1^\theta\varepsilon^{-\frac{1}{2}}O(\varepsilon^{\frac{5-\theta}{2}})<0,$$

when $\eta > 0$ is enough large.
If $\theta \in (3,5)$ then $0 < \frac{5-\theta}{2} < 1$ and for small enough $\varepsilon > 0$

$$CO(\varepsilon) - \eta \int_{\mathbb{R}^3} K(x)F(tv_\varepsilon)\mathrm{d}x \le CO(\varepsilon) - C\eta\varrho_1^\theta O(\varepsilon^{\frac{5-\theta}{2}}) < 0,$$

for any $\eta > 0$.
Consequently, we show that for $\theta \in (3,5)$ with any $\eta > 0$, or $\theta \in (1,3]$ with enough large $\eta > 0$

$$CO(\varepsilon) - \eta \int_{\mathbb{R}^3} K(x)F(tv_\varepsilon)\mathrm{d}x < 0.$$

So, we can obtain that $\sup_{t\ge 0} J(tv_\varepsilon) < \frac{2}{5}S^{\frac{3}{2}}|l(x)|_{L^\infty(\mathbb{R}^3)}^{-\frac{1}{2}}$. □

Proof of Theorem 1.1

Proof. From Lemma 3.3, we know that $\{u_n\}_n$ is bounded in E. Up to a subsequence, we have

$$\begin{aligned} &u_n \rightharpoonup u \quad \text{in } E,\\ &u_n \to u \quad \text{in } L^r_{loc}(\mathbb{R}^3), \quad \text{for } r \in [2,6),\\ &u_n \to u \quad a.e \text{ in } \mathbb{R}^3. \end{aligned} \tag{3.5}$$

By (2.1) – (2.3), we get

$$\begin{aligned} J(u_n) =& \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u_n|^2 + V(x)|u_n|^2)\mathrm{d}x - \eta\int_{\mathbb{R}^3} K(x)F(u_n)\mathrm{d}x - \frac{1}{10}\int_{\mathbb{R}^3} l(x)\phi_{u_n}|u_n|^5\mathrm{d}x\\ =& c + o(1) \end{aligned}$$

and

$$\begin{aligned} \langle J'(u_n), u_n\rangle =& \int_{\mathbb{R}^3}(|\nabla u_n|^2 + V(x)|u_n|^2)\mathrm{d}x - \eta\int_{\mathbb{R}^3} K(x)F(u_n)\mathrm{d}x - \int_{\mathbb{R}^3} l(x)\phi_{u_n}|u_n|^5\mathrm{d}x\\ =& o(1). \end{aligned}$$

Assuming $v_n = u_n - u$, in view of (3.5), Proposition 2.2 , Lemma 2.4 and the Brézis-Lieb Lemma [33, 34],

$$J(u_n) = J(u) + \frac{1}{2}\int_{\mathbb{R}^3}(|\nabla v_n|^2 + V(x)|v_n|^2)\mathrm{d}x - \frac{1}{10}\int_{\mathbb{R}^3} l(x)\phi_{v_n}|v_n|^5\mathrm{d}x \tag{3.6}$$

and

$$\begin{aligned} \langle J'(u_n), u_n\rangle =& \int_{\mathbb{R}^3}(|\nabla u_n|^2 + V(x)|u_n|^2)\mathrm{d}x - \eta\int_{\mathbb{R}^3} K(x)f(u_n)u_n\mathrm{d}x - \int_{\mathbb{R}^3} l(x)\phi_{u_n}|u_n|^5\mathrm{d}x\\ =& \int_{\mathbb{R}^3} |\nabla u|^2 + V(x)|u|^2)\mathrm{d}x - \int_{\mathbb{R}^3} l(x)\phi_u|u|^5\mathrm{d}x - \eta\int_{\mathbb{R}^3} K(x)f(u)u\mathrm{d}x\\ &+ \int_{\mathbb{R}^3}(|\nabla v_n|^2 + V(x)|v_n|^2)\mathrm{d}x - \int_{\mathbb{R}^3} l(x)\phi_{v_n}|v_n|^5\mathrm{d}x + o_n(1). \end{aligned} \tag{3.7}$$

Since $J'(u_n) \to 0$ as $n \to +\infty$ and by (3.5) again, we get

$$\langle J'(u_n), u\rangle = \int_{\mathbb{R}^3}(|\nabla u|^2 + V(x)u^2)\mathrm{d}x - \int_{\mathbb{R}^3} l(x)\phi_u|u|^5\mathrm{d}x - \eta\int_{\mathbb{R}^3} K(x)f(u)u\mathrm{d}x. \tag{3.8}$$

From (3.7) and (3.8), we obtain

$$\int_{\mathbb{R}^3} |\nabla v_n|^2 dx + \int_{\mathbb{R}^3} V(x)|v_n|^2 dx - \int_{\mathbb{R}^3} l(x)\phi_{v_n}|v_n|^5 dx \to 0 \text{ as } n \to +\infty \tag{3.9}$$

and

$$\begin{aligned} J(u) &= \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)|u|^2)dx - \frac{1}{10}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} l(x)\phi_u|u|^5 dx - \eta\int_{\mathbb{R}^3} K(x)F(u)dx \\ &= \frac{1}{2}\eta\int_{\mathbb{R}^3} K(x)f(u)u dx - \eta\int_{\mathbb{R}^3} K(x)F(u)dx + \frac{2}{5}\int_{\mathbb{R}^3} l(x)\phi_u|u|^5 dx \\ &\geq 0. \end{aligned} \tag{3.10}$$

Without loss of generality we can suppose that

$$\int_{\mathbb{R}^3} (|\nabla v_n|^2 + V(x)|v_n|^2)dx \to l \text{ as } n \to \infty \tag{3.11}$$

and from (3.9)

$$\int_{\mathbb{R}^3} l(x)\phi_{v_n}|v_n|^5 dx \to l \text{ as } n \to \infty. \tag{3.12}$$

By estimate

$$\int_{\mathbb{R}^3} l(x)\phi_{v_n}|v_n|^5 dx \leq \frac{|l(x)|_\infty^2 |v_n|_6^{10}}{S} \leq \frac{|l(x)|_\infty^2 \|v_n\|^{10}}{S^6}. \tag{3.13}$$

Combining (3.10) – (3.13), which implies $l \leq \frac{|l(x)|_\infty^2 l^5}{S^6}$. Therefore, either $l = 0$ or $l \geq |l(x)|_\infty^{-\frac{1}{2}} S^{\frac{3}{2}}$.
If $l > 0$, we have $l \geq |l(x)|_\infty^{-\frac{1}{2}} S^{\frac{3}{2}}$. Taking the limit in (3.6) as $n \to +\infty$, and using (3.10), we obtain $c_0 \geq \frac{2}{5}|l(x)|_\infty^{-\frac{1}{2}} S^{\frac{3}{2}}$. On the other hand, from (3.2) and Lemma 3.4, we obtain $c_0 \leq \frac{2}{5}|l(x)|_\infty^{-\frac{1}{2}} S^{\frac{3}{2}}$. We get a contradiction. This shows that $l = 0$. Thus

$$J(u) = c > 0 \text{ and } J'(u) = 0,$$

i.e. u is a non-trivial solution of (1.1). We complete the proof. □

Acknowledgements: The authors thank the referees for valuable comments and suggestions which improved the presentation of this manuscript. This work is partially supported by the Fundamental Research Funds for the National Natural Science Foundation of China 11671403 and Guizhou University of Finance and Economics of 2018YJ20.

References

[1] Benci V., Fortunato D., An eigenvalue problem for the Schrödinger-Maxwell equations, Topol. Methods Nonlinear Anal., 1998, 11, 283–293.
[2] Ritchie B., Relativistic self-focusing and channel formation in laser-plasma interactions, Phys. Rev. E., 1994, 50, 687–689.
[3] Brandi H.S., Manus C., Mainfray G., Lehner T., Bonnaud G., Relativistic and ponderomotive self-focusing of a laser beam in a radially inhomogeneous plasma, Phys. Fluids B., 1993, 5, 3539–3550.
[4] Bass F.G., Nasanov N.N., Nonlinear electromagnetic spin waves, Physics Reports, 1990, 189, 165–223.
[5] Ruiz D., The Schrödinger-Poisson equation under the effect of a nonlinear local term, J. Funct. Anal., 2006, 237, 655–674.
[6] Coclite G. M., A multiplicity result for the nonlinear Schrödinger-Maxwell equations, Commun. Appl. Anal., 2003, 7, 417–423.
[7] Carriäo P.C., Cunha P.L., Miyagaki O.H., Existence results for the Klein-Gordon-Maxell equations in higher dimensions with critical exponents, Commun. Pure. Appl. Anal., 2011, 10, 709–718.
[8] Lions P.L., The concentration compactness principle in the calculus of variations: The locally compact case, Parts 1,2, Ann. Inst. H. H. Poincar. Anal. Non Linaire., 1984b, 2, 223–283.

[9] Alves C.O., Souto A.S., Soared H.M., Schrödinger-Poisson equations without Ambrosetti-Rabinowitz condition, J. Math. Anal. Appl., 2011, 377, 584–592.

[10] Rabinowitz P.H., On a class of nonlinear Schrödinger equations, Z. Angew. Math. Phys., 1992, 43, 270–291.

[11] Shao L.Y., Chen H.B., Multiple solutions for Schrödinger-Poisson systems with sign-changing potential and critical nonlinearity, Electron. J. Differential Equations, 2016, 276, 1–8.

[12] Kurihura S., Large-amplitude quasi-solitons in superfluid films, J. Phys. Soc. Japan., 1981, 50, 3262–3267 .

[13] Brezis H., Nirenberg L., Positive solutions of nonlinear elliptic problems involving critical Sobolev exponent, Comm. Pure Appl. Math., 1983, 36, 437–477.

[14] Azzollini A., d'Avenia P., Pomponio A., On the Schrödinger-Maxwell equations under the effect of a general nonlinear term, Ann. Inst. H. Poincaré Anal. Non Linéaire., 2010, 27, 779–791.

[15] Cerami G., Vaira G., Positive solutions for some non-autonomous Schrödinger-Poisson systems, J. Differential Equations, 2010, 248, 521–543.

[16] Alves C.O., do Ó J.M., Souto A.S., Local mountain-pass for a class of elliptic problem in $\mathbb{R}^N$ involving critical exponents, Nonlinear Anal. TMA., 2001, 46, 495–510.

[17] D'Aprile T., Mugnai D., Solitary waves for nonlinear Klein-Gordon-Maxwell and Schrodinger-Maxwell equations, Proc. Roy. Soc. Edinburgh Sect. A., 2004, 134, 893–906.

[18] Liu Z.S., Guo S.J., On ground state solutions for the Schrödinger-Poisson equations with critical growth, J. Math. Anal. Appl., 2014, 412, 435–448.

[19] Wang Z., Zhou H., Positive solution for a nonlinear stationary Schrödinger-Poisson system in $\mathbb{R}^3$, Discrete Contin. Dyn. Syst., 2007, 18, 809–816.

[20] Nakamura A., On nonlinear Schrödinger-Poisson equations with general potentials, J. Math. Anal. Appl., 2013, 401, 672–681.

[21] Sun J.T., Chen H.B., Nieto J.J., On ground state solutions for some non-autonomous Schrödinger-Poisson systems, J. Differential Equations, 2012, 252, 3365–3380.

[22] Azzollini A., d'Avenia P., On a system involving a critically growing nonlinearity, J. Math. Anal. Appl., 2012, 387, 433–435.

[23] Li Y.H., Li F.Y., Existence and multiplicity of positive solutions to Schrödinger-Poisson type systems with critical nonlocal, Calc. Var. Partial Differential Equations., 2017, 56(134), 1–17.

[24] Liu H.D., Positive solutions of an asymptotically periodic Schrödinger-Poisson system with critical exponent, Nonlinear Anal. Real World Appl., 2016, 32, 198–212.

[25] Li F.Y., Li Y.H., Shi J.P., Existence of positive solutions to Schrödinger-Poisson type systems with critical exponent, Commun. Contemp. Math., 2014, 16, 517–520.

[26] Alves C.O., Souto A.S., Existence of solutions for a class of nonlinear Schrödinger equations with potential vanishing at infinity, J. Differential Equations, 2013, 254, 1977–1991.

[27] Su J. B., Wang Z.Q., Willem M., Weighted Sobolev embedding with unbounded and decaying radial potentials, J. Differential Equations, 2007, 238, 201–219.

[28] Ambrosetti A., Felli V., Malchiodi A., Ground states of nonlinear Schrödinger equations with potentials vanishing at infinity, J. Eur. Math. Soc. (JEMS), 2005, 7, 117–144.

[29] Ambrosetti A., Wang Z.Q., Nonlinear Schrödinger equations with vanishing and decaying potentials, Differential Integral Equations, 2005, 18(12), 1321–1332.

[30] Bonheure D., Scaftingen J.Van., Ground states for nonlinear Schrödinger equation with potential vanishing at infinity, Ann. Mat. Pura. Appl., 2010, 189, 273–301.

[31] Sun J., Chen H., Yang L., Positive solutions of asymptotically linear Schrödinger-Poisson systems with a radial potential vanishing at infinity, Nonlinear Anal., 2011, 74, 413–423.

[32] Mercuri C., Positive solutions of nonlinear Schrödinger-Poisson systems with radial potentials vanishing at infinity, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl., 2008, 19, 211–227.

[33] Millem M., Minimax Theorems., Birkh$\ddot{a}$user, Berlin, 1996.

[34] Struwe M., Variational Methods., Springer-Verlag Berlin Heidelberg, 2008.