Open Math. 2019; 17:1136–1146 DE GRUYTER

Open Mathematics

Research Article

Yuan Zhang, Ying Zhang, and Gang Wang*

Exclusion sets in the S-type eigenvalue localization sets for tensors

https://doi.org/10.1515/math-2019-0090
Received October 19, 2018; accepted August 26, 2019

Abstract: In this paper, we break the index set N into disjoint subsets S and its complement, and propose two S-type exclusion sets that all the eigenvalues do not belong to them. Furthermore, we establish new S-type eigenvalue inclusion sets, which can reduce computations and obtain more accurate numerical results. At the same time, we give two criteria for identifying nonsingular tensors. Finally, new S-type eigenvalue inclusion sets are shown to be sharper than existing results via two examples.

Keywords: S-type eigenvalue inclusion sets, S-type eigenvalue exclusion sets, Brauer-type eigenvalue inclusion sets

MSC: 15A18, 15A42

1 Introduction

Let $\mathbb{C}(\mathbb{R})$ be the set of all complex (real) numbers, $\mathbb{R}_+(\mathbb{R}_{++})$ be the set of all nonnegative (positive) numbers, $\mathbb{C}^n(\mathbb{R}^n)$ be the set of all dimension n complex (real) vectors, and $\mathbb{R}^n_+(\mathbb{R}^n_{++})$ be the set of all dimension n nonnegative (positive) vectors. An m order n dimensional tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ is a higher-order generalization of matrices, which consists of n^m entries:

$$a_{i_1 i_2 \cdots i_m} \in \mathbb{R}, \quad i_k \in N = \{1, 2, \ldots, n\}, \quad k = 1, 2, \ldots, m.$$

$\mathcal{A}$ is called nonnegative (positive) if $a_{i_1 i_2 \ldots i_m} \in \mathbb{R}_+ (a_{i_1 i_2 \ldots i_m} \in \mathbb{R}_{++})$.

Generally, tensor is a higher-order extension of matrix, and hence many concepts and the corresponding conclusions for matrices such as determinant, eigenvalue and singular value theory are extended to higher order tensors by studying their multilinear algebra properties [1, 2]. Based on matrix eigenvalues, tensor eigenvalue problems are developed [3, 4] which are important tools in tensor analysis and computing, and they are widely used in medical resonance imaging [5], data analysis [6], higher-order Markov chains [7, 8] and positive definiteness of even-order multivariate forms in automatical control [9–11]. At the same time, many effective algorithms for finding the eigenvalue of tensors have been presented; for more detailed discussions, see [12-22]. However, these algorithms cannot capture all eigenvalues, and their computational complexities are NP-hard. Hence, several inclusion sets in the complex plane for all the eigenvalues of a tensors have been considered: Gersgorin inclusion sets [4], Brauer inclusion sets [23-31], Brualdi inclusion sets [32] and the minimal Gersgorin sets [33]. For example, Gersgorin inclusion sets can be described as follows:

Yuan Zhang: School of Management Science, Qufu Normal University, Rizhao 276800, P.R. China, E-mail: zymath1115@163.com
Ying Zhang: School of Basic Teaching, Shandong Water Conservancy Vocational College, Rizhao 276800, P.R. China, E-mail: 20371202@qq.com
***Corresponding Author: Gang Wang:** School of Management Science, Qufu Normal University, Rizhao 276800, P.R. China, E-mail: wgglj1977@163.com

Lemma 1.1. *(Theorem 6 of [4]) Let $\mathcal{A}$ be a complex tensor of order m and dimension n. Then, all eigenvalues of $\mathcal{A}$ are located in the union of the following sets:*

$$\sigma(\mathcal{A}) \subseteq \Gamma(\mathcal{A}) = \bigcup_{i \in N} \Gamma_i(\mathcal{A}).$$

where $\sigma(\mathcal{A})$ denotes the set of all the set of all eigenvalues of $\mathcal{A}$, $\Gamma_i(\mathcal{A}) = \{z \in \mathbb{C} : |z - a_{i\dots i}| \le r_i(\mathcal{A})\}$, and $r_i(\mathcal{A}) = \sum_{\delta_{ii_2\dots i_m}=0} |a_{ii_2\dots i_m}|$.

On the other hand, by the discreteness of the eigenvalues [4], we know that all the eigenvalues are excluded some open sets. For this, Li *et al.* [34] proposed some exclusion sets, which exclude open sets respectively from the Gersgorin eigenvalue inclusion set in [4] and the Brauer-type eigenvalue inclusion set in [23] as follows:

Lemma 1.2. *(Theorem 2 of [34]) Let $\mathcal{A}$ be a complex tensor of order m and dimension $n \ge 2$. Then, all eigenvalues of $\mathcal{A}$ are located in the union of the following sets:*

$$\sigma(\mathcal{A}) \subseteq \Omega(\mathcal{A}) = \bigcup_{i \in N} \Omega_i(\mathcal{A}).$$

where $\Omega_i(\mathcal{A}) = \Gamma_i(\mathcal{A}) \backslash \Delta_i(\mathcal{A})$, $\Delta_i(\mathcal{A}) = \bigcup_{j \ne i} \Delta_{ij}(\mathcal{A})$ and

$$\Delta_{ij}(\mathcal{A}) = \{z \in \mathbb{C} : |z - a_{j\dots j}| < 2|a_{ji\dots i}| - r_j(\mathcal{A})\}.$$

Lemma 1.3. *(Theorem 4 of [24]) Let $\mathcal{A}$ be a complex tensor of order m and dimension $n \ge 2$. Then, all eigenvalues of $\mathcal{A}$ are located in the union of the following sets:*

$$\sigma(\mathcal{A}) \subseteq \Theta(\mathcal{A}) = \bigcup_{i,j \in N, j \ne i} \Theta_{ij}(\mathcal{A}),$$

where $\Theta_{ij}(\mathcal{A}) = \mathcal{K}_{i,j}(\mathcal{A}) \backslash \Lambda_i(\mathcal{A})$, $\Lambda_i(\mathcal{A}) = \bigcup_{j \ne i} \Lambda_{ij}(\mathcal{A})$ and

$$\mathcal{K}_{i,j}(\mathcal{A}) = \{z \in \mathbb{C} : (|z - a_{i\dots i}| - (r_i(\mathcal{A}) - |a_{ij\dots j}|))|z - a_{j\dots j}| \le |a_{ij\dots j}| r_j(\mathcal{A})\},$$

$$\Lambda_{ij}(\mathcal{A}) = \{z \in \mathbb{C} : (|z - a_{i\dots i}| + (r_i(\mathcal{A}) - |a_{ij\dots j}|))|z - a_{j\dots j}| < |a_{ij\dots j}|(2|a_{ji\dots i}| - r_j(\mathcal{A}))\}.$$

From Lemmas 1.1-1.3, we obtain inclusion sets $\Gamma(\mathcal{A})$ and $\mathcal{K}(\mathcal{A})$ by computing n sets $\Gamma_i(\mathcal{A})$ and $n(n-1)$ sets $\mathcal{K}_{i,j}(\mathcal{A})$, respectively. It is noted that $\mathcal{K}(\mathcal{A})$ is much sharper than $\Gamma(\mathcal{A})$ [23]. Similarly, the exclusion sets $\Delta(\mathcal{A})$ and $\Lambda(\mathcal{A})$ are composed of $n(n-1)$ sets $\Delta_{i,j}(\mathcal{A})$ and $n(n-1)$ sets $\Lambda_{ij}(\mathcal{A})$, respectively. To reduce computations, Li *et al.* [23] gave an S-type eigenvalue inclusion set by selecting subset S of N, which $\mathcal{K}^S(\mathcal{A})$ is made up of $2|S|(n - |S|)$ sets $\mathcal{K}_{i,j}(\mathcal{A})$.

In this paper, based on S-type eigenvalue inclusion sets [23, 24], we propose S-type eigenvalue exclusion sets, which can reduce computations and achieve more accurate numerical results. Furthermore, we establish new S-type eigenvalue inclusion sets and propose criteria for identifying nonsingular tensors. To end this section, we introduce some fundamental notion and properties related to eigenvalue of a tensor [2–4] and propose structure of the article.

Let $\mathcal{A}$ be an m-order n-dimensional tensor. Assume that $\mathcal{A}x^{m-1}$ is not identical to 0. We say that $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \backslash \{0\})$ is an eigenvalue-eigenvector of $\mathcal{A}$ if

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]},$$

where $(\mathcal{A}x^{m-1})_i = \sum_{i_2,\dots,i_m=1}^{n} a_{ii_2\dots i_m} x_{i_2} \dots x_{i_m}$, $x^{[m-1]} = [x_1^{m-1}, x_2^{m-1}, \dots, x_n^{m-1}]^T$, and (λ, x) is called an H-eigenpair if they are both real.

We define the following m-order Kronecker delta

$$\delta_{i_1 i_2 \dots i_m} = \begin{cases} 1, & \text{if } i_1 = i_2 = \dots = i_m, \\ 0, & \text{otherwise.} \end{cases}$$

This paper is organized as follows. In Section 1, we introduce important notation and recall fundamental results. In Section 2, we propose two S-type exclusion sets and establish new S-type eigenvalue inclusion sets. Meanwhile, we give two sufficient conditions to verify whether the determinant of a tensor is zero. In Section 3, we show that two new S-type eigenvalue inclusion sets are sharper than existing results via two examples.

2 S-type eigenvalue exclusion sets for tensors

In this section, we present two new S-type eigenvalue exclusion sets and show that these exclusion sets are included S-type inclusion set in [23, 24].

Lemma 2.1. *(Theorem 2.2 of [23]) Let $\mathcal{A}$ be a complex tensor of order m and dimension $n \geq 2$ and S be a nonempty proper subset of N. Then, all eigenvalues of $\mathcal{A}$ are located in the union of the following sets:*

$$\sigma(\mathcal{A}) \subseteq \mathcal{K}^S(\mathcal{A}) = (\bigcup_{i\in S, j\in \bar{S}} \mathcal{K}_{i,j}(\mathcal{A}))\bigcup(\bigcup_{i\in \bar{S}, j\in S} \mathcal{K}_{i,j}(\mathcal{A})),$$

where $\mathcal{K}_{i,j}(\mathcal{A})$ is defined in Lemma 1.3, $\bar{S}$ is the complement of S in N.

Next we try to find some proper subsets of $\mathcal{K}^S(\mathcal{A})$ which do not include any eigenvalue of a tensor $\mathcal{A}$.

Theorem 2.2. *Let $\mathcal{A}$ be a complex tensor of order m and dimension $n \geq 2$ and S be a nonempty proper subset of N. Then, all eigenvalues of $\mathcal{A}$ are located in the union of the following sets:*

$$\sigma(\mathcal{A}) \subseteq \Psi^S(\mathcal{A}) = (\bigcup_{i\in S, j\in \bar{S}} \Psi_{i,j}(\mathcal{A}))\bigcup(\bigcup_{i\in \bar{S}, j\in S} \Psi_{i,j}(\mathcal{A})),$$

where $\Psi_{i,j}(\mathcal{A}) = \mathcal{K}_{i,j}(\mathcal{A}) \setminus \mathcal{U}_{i,j}(\mathcal{A})$ and

$$\mathcal{U}_{i,j}(\mathcal{A}) = \{z \in \mathbb{C} : (|z - a_{i\ldots i}| + r_i(\mathcal{A}) - |a_{ij\ldots j}|)|z - a_{j\ldots j}| < |a_{ij\ldots j}|(2|a_{ji\ldots i}| - r_j(\mathcal{A}))\}.$$

Furthermore, $\Psi^S(\mathcal{A}) \subseteq \mathcal{K}^S(\mathcal{A})$.

Proof. Let λ be an eigenvalue of $\mathcal{A}$ with corresponding eigenvector x, i.e.,

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]}. \tag{1}$$

Let $|x_t| = \max\{|x_i| : i \in S\}$ and $|x_p| = \max\{|x_i| : i \in \bar{S}\}$. Then at least one of $|x_t|$ and $|x_p|$ is nonzero. We break the proof into three cases.

Case 1. If $|x_t||x_p| \neq 0$ and $|x_t| \geq |x_p|$, then $|x_t| = \max\{|x_i| : i \in N\} \neq 0$. By (1), it holds that

$$(\lambda - a_{p\ldots p})x_p^{m-1} = \sum_{\substack{\delta_{ti_2\ldots i_m}=0\\ \delta_{pi_2\ldots i_m}=0}} a_{pi_2\ldots i_m} x_{i_2}\ldots x_{i_m} + a_{pt\ldots t}x_t^{m-1}$$

and

$$a_{pt\ldots t}x_t^{m-1} = (\lambda - a_{p\ldots p})x_p^{m-1} - \sum_{\substack{\delta_{ti_2\ldots i_m}=0\\ \delta_{pi_2\ldots i_m}=0}} a_{pi_2\ldots i_m} x_{i_2}\ldots x_{i_m}.$$

Taking modulus in the above equation and using the triangle inequality give

$$\begin{aligned}|a_{pt\ldots t}||x_t|^{m-1} &\leq |\lambda - a_{p\ldots p}||x_p|^{m-1} + \sum_{\substack{\delta_{ti_2\ldots i_m}=0\\ \delta_{pi_2\ldots i_m}=0}} |a_{pi_2\ldots i_m}||x_t|^{m-1}\\ &= |\lambda - a_{p\ldots p}||x_p|^{m-1} + (r_p(\mathcal{A}) - |a_{pt\ldots t}|)|x_t|^{m-1},\end{aligned}$$

equivalently,

$$(2|a_{pt\ldots t}| - r_p(\mathcal{A}))|x_t|^{m-1} \leq |\lambda - a_{p\ldots p}||x_p|^{m-1}. \tag{2}$$

Similarly, by the t-th equation of (1), we obtain

$$(\lambda - a_{t\dots t})x_t^{m-1} = \sum_{\substack{\delta_{ti_2\dots i_m}=0\\ \delta_{pi_2\dots i_m}=0}} a_{ti_2\dots i_m}x_{i_2}\dots x_{i_m} + a_{tp\dots p}x_p^{m-1}$$

and

$$\begin{aligned}|a_{tp\dots p}||x_p|^{m-1} &\le |\lambda - a_{t\dots t}||x_t|^{m-1} + \sum_{\substack{\delta_{ti_2\dots i_m}=0\\ \delta_{pi_2\dots i_m}=0}} |a_{ti_2\dots i_m}||x_t|^{m-1}\\ &= (|\lambda - a_{t\dots t}| + r_t(\mathcal{A}) - |a_{tp\dots p}|)|x_t|^{m-1}.\end{aligned} \tag{3}$$

Multiplying (2) with (3), we yield

$$(2|a_{pt\dots t}| - r_p(\mathcal{A}))|a_{tp\dots p}||x_t|^{m-1}|x_p|^{m-1} \le |\lambda - a_{p\dots p}|(|\lambda - a_{t\dots t}| + r_t(\mathcal{A}) - |a_{tp\dots p}|)|x_t|^{m-1}|x_p|^{m-1}$$

and

$$(2|a_{pt\dots t}| - r_p(\mathcal{A}))|a_{tp\dots p}| \le |\lambda - a_{p\dots p}|(|\lambda - a_{t\dots t}| + r_t(\mathcal{A}) - |a_{tp\dots p}|). \tag{4}$$

Hence, $\lambda \notin \mathcal{U}_{t,p}(\mathcal{A})$ and $\lambda \in \bigcup_{i\in S, j\in\bar{S}} \Psi_{i,j}(\mathcal{A}))$.

Case 2. If $|x_t||x_p| \ne 0$ and $|x_p| \ge |x_t|$, then $|x_p| = \max\{|x_i| : i \in N\} \ne 0$. Similar to the proof of Case 1, by (2.1) it holds that

$$(2|a_{tp\dots p}| - r_t(\mathcal{A}))|x_p|^{m-1} \le |\lambda - a_{t\dots t}||x_t|^{m-1}. \tag{5}$$

Similarly, by the p-th equation of (2.1), we obtain

$$\begin{aligned}|a_{pt\dots t}||x_t|^{m-1} &\le |\lambda - a_{p\dots p}||x_p|^{m-1} + \sum_{\substack{\delta_{pi_2\dots i_m}=0\\ \delta_{ti_2\dots i_m}=0}} |a_{pi_2\dots i_m}||x_p|^{m-1}\\ &= (|\lambda - a_{p\dots p}| + r_p(\mathcal{A}) - |a_{pt\dots t}|)|x_p|^{m-1}.\end{aligned} \tag{6}$$

Multiplying (5) with (6), we yield

$$(2|a_{tp\dots p}| - r_t(\mathcal{A}))|a_{pt\dots t}| \le |\lambda - a_{t\dots t}|(|\lambda - a_{p\dots p}| + r_p(\mathcal{A}) - |a_{pt\dots t}|). \tag{7}$$

Hence, $\lambda \notin \mathcal{U}_{p,t}(\mathcal{A})$ and $\lambda \in \bigcup_{i\in\bar{S}, j\in S} \Psi_{i,j}(\mathcal{A})$.

Case 3. If $|x_t||x_p| = 0$, $|x_t| \ne 0$ and $|x_p| = 0$, then $|x_t| \ge |x_p|$. From (2), we have $2|a_{pt\dots t}| - r_p(\mathcal{A}) \le 0$. Since $|\lambda - a_{p\dots p}|(|\lambda - a_{t\dots t}| + r_t(\mathcal{A}) - |a_{tp\dots p}|) \ge 0$, it holds that

$$(2|a_{pt\dots t}| - r_p(\mathcal{A}))|a_{tp\dots p}| \le |\lambda - a_{p\dots p}|(|\lambda - a_{t\dots t}| + r_t(\mathcal{A}) - |a_{tp\dots p}|),$$

which implies $\lambda \notin \mathcal{U}_{t,p}(\mathcal{A})$ and $\lambda \in \bigcup_{i\in S, j\in\bar{S}} \Psi_{i,j}(\mathcal{A}))$.

If $|x_t||x_p| = 0$, $|x_p| \ne 0$ and $|x_t| = 0$, then $|x_p| \ge |x_t|$. By (5), it holds that $2|a_{tp\dots p}| - r_t(\mathcal{A}) \le 0$. Since $|\lambda - a_{t\dots t}|(|\lambda - a_{p\dots p}| + r_p(\mathcal{A}) - |a_{pt\dots t}|) \ge 0$, we obtain

$$(2|a_{tp\dots p}| - r_t(\mathcal{A}))|a_{pt\dots t}| \le |\lambda - a_{t\dots t}|(|\lambda - a_{p\dots p}| + r_p(\mathcal{A}) - |a_{pt\dots t}|),$$

which implies $\lambda \notin \mathcal{U}_{p,t}(\mathcal{A})$ and $\lambda \in \bigcup_{i\in\bar{S}, j\in S} \Psi_{i,j}(\mathcal{A})$.

Combining the discussions for the above three cases, we obtain $\lambda \in \bigcup_{i\in S, j\in\bar{S}} \Psi_{i,j}(\mathcal{A}))\bigcup \bigcup_{i\in\bar{S}, j\in S} \Psi_{i,j}(\mathcal{A})$.

Next, we show the exclusion set $\mathcal{U}_{i,j}(\mathcal{A}) \subseteq \mathcal{K}_{i,j}(\mathcal{A})$. For any $\tilde{\lambda} \in \mathcal{U}_{t,p}(\mathcal{A})$, it holds that

$$(|\tilde{\lambda} - a_{t\dots t}| + r_t(\mathcal{A}) - |a_{tp\dots p}|)|\tilde{\lambda} - a_{p\dots p}| < (2|a_{pt\dots t}| - r_p(\mathcal{A}))|a_{tp\dots p}|. \tag{8}$$

The following argument is divided into two cases.

Case 1. If $(2|a_{pt\dots t}| - r_p(\mathcal{A}))|a_{tp\dots p}| \le 0$, then $\mathcal{U}_{t,p}(\mathcal{A}) = \emptyset$. Obviously, $\mathcal{U}_{t,p}(\mathcal{A}) \subseteq \mathcal{K}_{t,p}(\mathcal{A})$.

Case 2. If $(2|a_{pt\ldots t}| - r_p(\mathcal{A}))|a_{tp\ldots p}| > 0$, from (8), we have

$$\frac{|\tilde{\lambda} - a_{t\ldots t}| + r_t(\mathcal{A}) - |a_{tp\ldots p}|}{|a_{tp\ldots p}|} \cdot \frac{|\tilde{\lambda} - a_{p\ldots p}|}{2|a_{pt\ldots t}| - r_p(\mathcal{A})} < 1. \tag{9}$$

Noting that $|a_{pt\ldots t}| \le r_p(\mathcal{A})$, i.e., $2|a_{pt\ldots t}| - r_p(\mathcal{A}) \le r_p(\mathcal{A})$, we obtain

$$\frac{|\tilde{\lambda} - a_{p\ldots p}|}{r_p(\mathcal{A})} \le \frac{|\tilde{\lambda} - a_{p\ldots p}|}{2|a_{pt\ldots t}| - r_p(\mathcal{A})}. \tag{10}$$

Meanwhile,

$$\frac{|\tilde{\lambda} - a_{t\ldots t}| - (r_t(\mathcal{A}) - |a_{tp\ldots p}|)}{|a_{tp\ldots p}|} \le \frac{|\tilde{\lambda} - a_{t\ldots t}| + r_t(\mathcal{A}) - |a_{tp\ldots p}|}{|a_{tp\ldots p}|}. \tag{11}$$

It follows from (9), (10) and (11) that

$$\frac{|\tilde{\lambda} - a_{t\ldots t}| - (r_t(\mathcal{A}) - |a_{tp\ldots p}|)}{|a_{tp\ldots p}|} \cdot \frac{|\tilde{\lambda} - a_{p\ldots p}|}{r_p(\mathcal{A})} \le 1,$$

which implies

$$(|\tilde{\lambda} - a_{t\ldots t}| - (r_t(\mathcal{A}) - |a_{tp\ldots p}|))|\tilde{\lambda} - a_{p\ldots p}| \le |a_{tp\ldots p}| r_p(\mathcal{A}).$$

So, $\tilde{\lambda} \in \mathcal{K}_{t,p}(\mathcal{A})$ and $\mathcal{U}_{t,p}(\mathcal{A}) \subseteq \mathcal{K}_{t,p}(\mathcal{A})$. For the arbitrariness of t, p, we have

$$\lambda \in \bigcup_{i \in S, j \in \bar{S}} \left(\mathcal{K}_{i,j}(\mathcal{A}) \setminus \mathcal{U}_{i,j}(\mathcal{A})\right) = \bigcup_{i \in S, j \in \bar{S}} \Psi_{i,j}(\mathcal{A}).$$

In a similar way, we also get $\lambda \in \bigcup_{i \in \bar{S}, j \in S} \left(\mathcal{K}_{i,j}(\mathcal{A}) \setminus \mathcal{U}_{i,j}(\mathcal{A})\right) = \bigcup_{i \in \bar{S}, j \in S} \Psi_{i,j}(\mathcal{A})$. So,

$$\lambda \in \Big(\bigcup_{i \in S, j \in \bar{S}} \Psi_{i,j}(\mathcal{A})\Big) \bigcup \Big(\bigcup_{i \in \bar{S}, j \in S} \Psi_{i,j}(\mathcal{A})\Big) = \Psi^S(\mathcal{A}) \subseteq \mathcal{K}^S(\mathcal{A}).$$

□

The determinant of a tensor $\mathcal{A}$, denoted by $det(\mathcal{A})$, is the resultant of the ordered system of homogeneous equations $\mathcal{A}x^{m-1} = 0$ of [4], and is closely related to the eigenvalue inclusion sets of a tensor $\mathcal{A}$. From Theorem 2.2, by verifying $2|S|(n - |S|)$ sets $\mathcal{K}_{i,j}(\mathcal{A})$ or $2|S|(n - |S|)$ sets $\mathcal{U}_{i,j}(\mathcal{A})$, we obtain the following condition such that $det(\mathcal{A}) \neq 0$. Compared with Corollary 3 of [24], Corollary 2.3 reduces computations and cuts down the validation conditions with $det(\mathcal{A}) \neq 0$.

Corollary 2.3. *Let $\mathcal{A}$ be a complex tensor of order m and dimension $n \ge 2$ and S be a nonempty proper subset of N. For $i \in S, j \in \bar{S}$ and $i \in \bar{S}, j \in S$, if either*

$$(|a_{i\ldots i}| - (r_i(\mathcal{A}) - |a_{ij\ldots j}|))|a_{j\ldots j}| > |a_{ij\ldots j}| r_j(\mathcal{A})$$

or

$$(|a_{i\ldots i}| + r_i(\mathcal{A}) - |a_{ij\ldots j}|)|a_{j\ldots j}| < |a_{ij\ldots j}|(2|a_{ji\ldots i}| - r_j(\mathcal{A})),$$

then $det(\mathcal{A}) \neq 0$.

To obtain tighter inclusion sets than $\mathcal{K}^S(\mathcal{A})$, Li *et al.* [33] established new S-type inclusion set for tensors. According to Section 2 of [33], for simplicity, we denote

$$\Delta^N = \{(i_2, i_3, \ldots, i_m) : each\ i_j \in N\ for\ j = 2, \ldots, m\},$$

$$\Delta^S = \{(i_2, i_3, \ldots, i_m) : each\ i_j \in S\ for\ j = 2, \ldots, m\},$$

where S is nonempty proper subset of N, and then $\overline{\Delta^S} = \Delta^N \setminus \Delta^S$.

Given a tensor $\mathcal{A} = (a_{i_1\ldots i_m})$, let

$$r_i^{\Delta^S}(\mathcal{A}) = \sum_{\substack{(i_2,\ldots,i_m) \in \Delta^S \\ \delta_{ii_2\ldots i_m} = 0}} |a_{ii_2\ldots i_m}|, \quad r_i^{\overline{\Delta^S}}(\mathcal{A}) = \sum_{(i_2,\ldots,i_m) \in \overline{\Delta^S}} |a_{ii_2\ldots i_m}|.$$

Obviously, $r_i(\mathcal{A}) = r_i^{\Delta^S}(\mathcal{A}) + r_i^{\overline{\Delta^S}}(\mathcal{A})$.

Lemma 2.4. *(Theorem 4 of [24]) Let $\mathcal{A}$ be a complex tensor of order m and dimension $n \geq 2$ and S be a nonempty proper subset of N. Then, all eigenvalues of $\mathcal{A}$ are located in the union of the following sets:*

$$\sigma(\mathcal{A}) \subseteq \Omega^S(\mathcal{A}) = \Big(\bigcup_{i\in S, j\in \bar{S}} \Omega^S_{i,j}(\mathcal{A})\Big)\bigcup\Big(\bigcup_{i\in \bar{S}, j\in S} \Omega^{\bar{S}}_{i,j}(\mathcal{A})\Big),$$

where

$$\Omega^S_{i,j}(\mathcal{A}) = \{z \in \mathbb{C} : |z - a_{i\ldots i}|(|z - a_{j\ldots j}| - r_j^{\overline{\Delta^S}}(\mathcal{A})) \leq r_i(\mathcal{A}) r_j^{\Delta^S}(\mathcal{A})\}$$

and

$$\Omega^{\bar{S}}_{i,j}(\mathcal{A}) = \{z \in \mathbb{C} : |z - a_{i\ldots i}|(|z - a_{j\ldots j}| - r_j^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})) \leq r_i(\mathcal{A}) r_j^{\Delta^{\bar{S}}}(\mathcal{A})\}.$$

Based on sharp S-type eigenvalue inclusion sets of [24], we are at the position to establish the following theorem.

Theorem 2.5. *Let $\mathcal{A}$ be a complex tensor of order m and dimension $n \geq 2$ and S be a nonempty proper subset of N. Then, all eigenvalues of $\mathcal{A}$ are located in the union of the following sets:*

$$\sigma(\mathcal{A}) \subseteq \Phi^S(\mathcal{A}) = \Big(\bigcup_{i\in S, j\in \bar{S}} \Phi^S_{i,j}(\mathcal{A})\Big)\bigcup\Big(\bigcup_{i\in \bar{S}, j\in S} \Phi^{\bar{S}}_{i,j}(\mathcal{A})\Big),$$

where $\Phi^S_{i,j}(\mathcal{A}) = \Omega^S_{i,j}(\mathcal{A}) \setminus \mathcal{V}^S_{i,j}(\mathcal{A})$, $\Phi^{\bar{S}}_{i,j}(\mathcal{A}) = \Omega^{\bar{S}}_{i,j}(\mathcal{A}) \setminus \mathcal{V}^{\bar{S}}_{i,j}(\mathcal{A})$ and

$$\mathcal{V}^S_{i,j}(\mathcal{A}) = \{z \in \mathbb{C} : (|z - a_{i\ldots i}| + r_i(\mathcal{A}) - |a_{ij\ldots j}|)(|z - a_{j\ldots j}| + r_j^{\overline{\Delta^S}}(\mathcal{A})) < |a_{ij\ldots j}|(2|a_{ji\ldots i}| - r_j^{\Delta^S}(\mathcal{A}))\},$$

$$\mathcal{V}^{\bar{S}}_{i,j}(\mathcal{A}) = \{z \in \mathbb{C} : (|z - a_{i\ldots i}| + r_i(\mathcal{A}) - |a_{ij\ldots j}|)(|z - a_{j\ldots j}| + r_j^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})) < |a_{ij\ldots j}|(2|a_{ji\ldots i}| - r_j^{\Delta^{\bar{S}}}(\mathcal{A}))\}.$$

Furthermore, $\Phi^S(\mathcal{A}) \subseteq \Omega^S(\mathcal{A})$.

Proof. Let (λ, x) be an eigenvalue-eigenvector of $\mathcal{A}$. Setting $|x_t| = \max\{|x_i| : i \in S\}$ and $|x_p| = \max\{|x_i| : i \in \bar{S}\}$, we obtain at least one of $|x_t|$ and $|x_p|$ is nonzero. The following argument is divided into three cases.

Case 1. If $|x_t||x_p| \neq 0$ and $|x_t| \geq |x_p|$, then $|x_t| = \max\{|x_i| : i \in N\} \neq 0$. By the p-th equation of (1), we have

$$\begin{aligned}(\lambda - a_{p\ldots p})x_p^{m-1} &= \sum_{\substack{(i_2,\ldots,i_m)\in\overline{\Delta^S}\\ \delta_{pi_2\ldots i_m}=0}} a_{pi_2\ldots i_m}x_{i_2}\cdots x_{i_m} + \sum_{(i_2,\ldots,i_m)\in\Delta^S} a_{pi_2\ldots i_m}x_{i_2}\cdots x_{i_m}\\ &= \sum_{\substack{(i_2,\ldots,i_m)\in\overline{\Delta^S}\\ \delta_{pi_2\ldots i_m}=0}} a_{pi_2\ldots i_m}x_{i_2}\cdots x_{i_m} + a_{pt\ldots t}x_t^{m-1} + \sum_{\substack{(i_2,\ldots,i_m)\in\Delta^S\\ \delta_{ti_2\ldots i_m}=0}} a_{pi_2\ldots i_m}x_{i_2}\cdots x_{i_m}\end{aligned}$$

and

$$a_{pt\ldots t}x_t^{m-1} = (\lambda - a_{p\ldots p})x_p^{m-1} - \sum_{\substack{(i_2,\ldots,i_m)\in\overline{\Delta^S}\\ \delta_{pi_2\ldots i_m}=0}} a_{pi_2\ldots i_m}x_{i_2}\cdots x_{i_m} - \sum_{\substack{(i_2,\ldots,i_m)\in\Delta^S\\ \delta_{ti_2\ldots i_m}=0}} a_{pi_2\ldots i_m}x_{i_2}\cdots x_{i_m}.$$

Taking modulus in the above equation and using the triangle inequality gives

$$\begin{aligned}|a_{pt\ldots t}||x_t|^{m-1} &\leq |\lambda - a_{p\ldots p}||x_p|^{m-1} + \sum_{\substack{(i_2,\ldots,i_m)\in\overline{\Delta^S}\\ \delta_{pi_2\ldots i_m}=0}} |a_{pi_2\ldots i_m}||x_p|^{m-1}\\ &\quad + \sum_{\substack{(i_2,\ldots,i_m)\in\Delta^S\\ \delta_{ti_2\ldots i_m}=0}} |a_{pi_2\ldots i_m}||x_t|^{m-1}\\ &= |\lambda - a_{p\ldots p}||x_p|^{m-1} + r_p^{\overline{\Delta^S}}(\mathcal{A})|x_p|^{m-1} + (r_p^{\Delta^S}(\mathcal{A}) - |a_{pt\ldots t}|)|x_t|^{m-1},\end{aligned}$$

equivalently,

$$(2|a_{pt\ldots t}| - r_p^{\Delta^S}(\mathcal{A}))|x_t|^{m-1} \leq (|\lambda - a_{p\ldots p}| + r_p^{\overline{\Delta^S}}(\mathcal{A}))|x_p|^{m-1}. \tag{12}$$

Noting the t-th equation of (2.1), we obtain

$$(\lambda - a_{t\ldots t})x_t^{m-1} = \sum_{\substack{\delta_{ti_2\ldots i_m}=0\\ \delta_{pi_2\ldots i_m}=0}} a_{ti_2\ldots i_m}x_{i_2}\cdots x_{i_m} + a_{tp\ldots p}x_p^{m-1}$$

and

$$|a_{tp\dots p}||x_p|^{m-1} \le |\lambda - a_{t\dots t}||x_t|^{m-1} + \sum_{\substack{\delta_{ti_2\dots i_m}=0 \\ \delta_{pi_2\dots i_m}=0}} |a_{ti_2\dots i_m}||x_t|^{m-1},$$

equivalently,

$$|a_{tp\dots p}||x_p|^{m-1} \le (|\lambda - a_{t\dots t}| + r_t(\mathcal{A}) - |a_{tp\dots p}|)|x_t|^{m-1}. \tag{13}$$

Multiplying (12) with (13), one has

$$|a_{tp\dots p}|(2|a_{pt\dots t}| - r_p^{\Delta^S}(\mathcal{A})) \le (|\lambda - a_{t\dots t}| + r_t(\mathcal{A}) - |a_{tp\dots p}|)(|\lambda - a_{p\dots p}| + r_p^{\overline{\Delta^S}}(\mathcal{A})), \tag{14}$$

which implies $\lambda \notin \mathcal{V}_{t,p}^S(\mathcal{A})$ and $\lambda \in \bigcup_{i\in S, j\in \bar{S}} \Omega_{i,j}^S(\mathcal{A})$.

Case 2. If $|x_t||x_p| \ne 0$ and $|x_p| \ge |x_t|$, then $|x_p| = \max\{|x_i| : i \in N\} \ne 0$. Similar to the proof of Case 1, by the t-th equation of (2.1), we have

$$a_{tp\dots p}x_p^{m-1} = (\lambda - a_{t\dots t})x_t^{m-1} - \sum_{\substack{(i_2,\dots,i_m)\in \overline{\Delta^{\bar{S}}} \\ \delta_{ti_2\dots i_m}=0}} a_{ti_2\dots i_m}x_{i_2}\dots x_{i_m} - \sum_{\substack{(i_2,\dots,i_m)\in \Delta^{\bar{S}} \\ \delta_{pi_2\dots i_m}=0}} a_{ti_2\dots i_m}x_{i_2}\dots x_{i_m}.$$

and

$$(2|a_{tp\dots p}| - r_t^{\Delta^{\bar{S}}}(\mathcal{A}))|x_p|^{m-1} \le (|\lambda - a_{t\dots t}| + r_t^{\overline{\Delta^{\bar{S}}}}(\mathcal{A}))|x_t|^{m-1}. \tag{15}$$

Recalling the p-th equation of (2.1), we obtain

$$\begin{aligned} |a_{pt\dots t}||x_t|^{m-1} &\le |\lambda - a_{p\dots p}||x_p|^{m-1} + \sum_{\substack{\delta_{pi_2\dots i_m}=0 \\ \delta_{ti_2\dots i_m}=0}} |a_{pi_2\dots i_m}||x_p|^{m-1} \\ &\le (|\lambda - a_{p\dots p}| + r_p(\mathcal{A}) - |a_{pt\dots t}|)|x_p|^{m-1}. \end{aligned} \tag{16}$$

Multiplying (15) with (16) yields

$$|a_{pt\dots t}|(2|a_{tp\dots p}| - r_t^{\Delta^{\bar{S}}}(\mathcal{A})) \le (|\lambda - a_{p\dots p}| + r_p(\mathcal{A}) - |a_{pt\dots t}|)(|\lambda - a_{t\dots t}| + r_t^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})), \tag{17}$$

which shows $\lambda \notin \mathcal{V}_{p,t}^{\bar{S}}(\mathcal{A})$ and $\lambda \in \bigcup_{i\in \bar{S}, j\in S} \Phi_{i,j}^{\bar{S}}(\mathcal{A})$.

Case 3. If $|x_t||x_p| = 0$, $|x_t| \ne 0$ and $|x_p| = 0$, then $|x_t| \ge |x_p|$. From (12), we obtain $2|a_{pt\dots t}| - r_p^{\Delta^S}(\mathcal{A}) \le 0$. Noting that $(|\lambda - a_{t\dots t}| + r_t(\mathcal{A}) - |a_{tp\dots p}|)(|\lambda - a_{p\dots p}| + r_p^{\overline{\Delta^S}}(\mathcal{A})) \ge 0$, we have

$$|a_{tp\dots p}|(2|a_{pt\dots t}| - r_p^{\Delta^S}(\mathcal{A})) \le (|\lambda - a_{t\dots t}| + r_t(\mathcal{A}) - |a_{tp\dots p}|)(|\lambda - a_{p\dots p}| + r_p^{\overline{\Delta^S}}(\mathcal{A})),$$

which shows $\lambda \notin \mathcal{V}_{t,p}^S(\mathcal{A})$ and $\lambda \in \bigcup_{i\in S, j\in \bar{S}} \Omega_{i,j}^S(\mathcal{A})$.

If $|x_t||x_p| = 0$, $|x_t| = 0$ and $|x_p| \ne 0$, then $|x_p| \ge |x_t|$. Using (15), it holds that $2|a_{tp\dots p}| - r_t^{\Delta^{\bar{S}}}(\mathcal{A}) \le 0$. From $(|\lambda - a_{p\dots p}| + r_p(\mathcal{A}) - |a_{pt\dots t}|)(|\lambda - a_{t\dots t}| + r_t^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})) \ge 0$, we deduce

$$|a_{pt\dots t}|(2|a_{tp\dots p}| - r_t^{\Delta^{\bar{S}}}(\mathcal{A})) \le (|\lambda - a_{p\dots p}| + r_p(\mathcal{A}) - |a_{pt\dots t}|)(|\lambda - a_{t\dots t}| + r_t^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})),$$

which implies $\lambda \notin \mathcal{V}_{p,t}^{\bar{S}}(\mathcal{A})$ and $\lambda \in \bigcup_{i\in \bar{S}, j\in S} \Phi_{i,j}^{\bar{S}}(\mathcal{A})$.

To sum up above three discussions, we yield that $\lambda \notin \mathcal{V}_{t,p}^S(\mathcal{A}) \bigcup \mathcal{V}_{p,t}^{\bar{S}}(\mathcal{A})$.

Next, we establish $\mathcal{V}_{t,p}^S(\mathcal{A}) \subseteq \Omega_{t,p}^S(\mathcal{A})$. For any $\tilde{\lambda} \in \mathcal{V}_{t,p}^S(\mathcal{A})$, we have

$$(|\tilde{\lambda} - a_{t\dots t}| + r_t(\mathcal{A}) - |a_{tp\dots p}|)(|\tilde{\lambda} - a_{p\dots p}| + r_p^{\overline{\Delta^S}}(\mathcal{A})) < |a_{tp\dots p}|(2|a_{pt\dots t}| - r_p^{\Delta^S}(\mathcal{A})). \tag{18}$$

The following argument is divided into two cases.

Case 1. If $|a_{tp\dots p}|(2|a_{pt\dots t}| - r_p^{\Delta^S}(\mathcal{A})) \le 0$, then $\mathcal{V}_{t,p}^S(\mathcal{A}) = \emptyset$. Obviously, $\mathcal{V}_{t,p}^S(\mathcal{A}) \subseteq \Omega_{t,p}^S(\mathcal{A})$.

Case 2. If $|a_{tp\dots p}|(2|a_{pt\dots t}| - r_p^{\Delta^S}(\mathcal{A})) > 0$, dividing (18) through by $|a_{tp\dots p}|(2|a_{pt\dots t}| - r_p^{\Delta^S}(\mathcal{A}))$, we have

$$\frac{|\tilde{\lambda} - a_{t\dots t}| + r_t(\mathcal{A}) - |a_{tp\dots p}|}{|a_{tp\dots p}|} \cdot \frac{|\tilde{\lambda} - a_{p\dots p}| + r_p^{\overline{\Delta^S}}(\mathcal{A})}{2|a_{pt\dots t}| - r_p^{\Delta^S}(\mathcal{A})} < 1. \tag{19}$$

From $|a_{pt\dots t}| \leq r_p^{\Delta^S}(\mathcal{A})$, we know $2|a_{pt\dots t}| - r_p^{\Delta^S}(\mathcal{A}) \leq r_p^{\Delta^S}(\mathcal{A})$. Hence,

$$\frac{|\tilde{\lambda} - a_{p\dots p}| - r_p^{\overline{\Delta^S}}(\mathcal{A})}{r_p^{\Delta^S}(\mathcal{A})} \leq \frac{|\tilde{\lambda} - a_{p\dots p}| + r_p^{\overline{\Delta^S}}(\mathcal{A})}{2|a_{pt\dots t}| - r_p^{\Delta^S}(\mathcal{A})}. \tag{20}$$

By $|a_{tp\dots p}| \leq r_t(\mathcal{A})$, it holds that

$$\frac{|\tilde{\lambda} - a_{t\dots t}|}{r_t(\mathcal{A})} \leq \frac{|\tilde{\lambda} - a_{t\dots t}| + r_t(\mathcal{A}) - |a_{tp\dots p}|}{|a_{tp\dots p}|}. \tag{21}$$

It follows from (19), (20) and (21) that

$$\frac{|\tilde{\lambda} - a_{t\dots t}|}{r_t(\mathcal{A})} \cdot \frac{|\tilde{\lambda} - a_{p\dots p}| - r_p^{\overline{\Delta^S}}(\mathcal{A})}{r_p^{\Delta^S}(\mathcal{A})} \leq 1,$$

which implies

$$|\tilde{\lambda} - a_{t\dots t}|(|\tilde{\lambda} - a_{p\dots p}| - r_p^{\overline{\Delta^S}}) \leq r_t(\mathcal{A}) r_p^{\Delta^S}(\mathcal{A})$$

and $\mathcal{V}_{t,p}^S(\mathcal{A}) \subseteq \Omega_{t,p}^S(\mathcal{A})$. For the arbitrariness of t, p, we have

$$\lambda \in \bigcup_{i \in S, j \in \bar{S}} \left(\Omega_{i,j}^S(\mathcal{A}) \setminus \mathcal{V}_{i,j}^S(\mathcal{A})\right) = \bigcup_{i \in S, j \in \bar{S}} \Phi_{i,j}^S(\mathcal{A}).$$

Similarly, we can obtain

$$\lambda \in \bigcup_{i \in \bar{S}, j \in S} \left(\Omega_{i,j}^{\bar{S}}(\mathcal{A}) \setminus \mathcal{V}_{i,j}^{\bar{S}}(\mathcal{A})\right) = \bigcup_{i \in \bar{S}, j \in S} \Phi_{i,j}^{\bar{S}}(\mathcal{A}).$$

Furthermore,

$$\lambda \in (\bigcup_{i \in S, j \in \bar{S}} \Phi_{i,j}^S(\mathcal{A})) \bigcup (\bigcup_{i \in \bar{S}, j \in S} \Phi_{i,j}^{\bar{S}}(\mathcal{A})) = \Phi^S(\mathcal{A}) \subseteq \Omega^S(\mathcal{A}).$$

□

Corollary 2.6. *Let $\mathcal{A}$ be a complex tensor of order m and dimension $n \geq 2$ and S be a nonempty proper subset of N. For $i \in S, j \in \bar{S}$ and $i \in \bar{S}, j \in S$, if either*

$$|a_{i\dots i}|(|a_{j\dots j}| - r_j^{\overline{\Delta^S}}(\mathcal{A})) > r_i(\mathcal{A}) r_j^{\Delta^S}(\mathcal{A})$$

or

$$(|a_{i\dots i}| + r_i(\mathcal{A}) - |a_{ij\dots j}|)(|a_{j\dots j}| + r_j^{\overline{\Delta^S}}(\mathcal{A})) < |a_{ij\dots j}|(2|a_{ji\dots i}| - r_j^{\Delta^S}(\mathcal{A})),$$

then $det(\mathcal{A}) \neq 0$.

3 Numerical examples

The following example exhibits the superiority of the results given in Theorem 2.2.

Example 3.1. *Consider 3 order 3 dimensional tensor $\mathcal{A} = (a_{ijk})$ defined by*

$$a_{ijk} = \begin{cases} a_{222} = a_{133} = 2; a_{333} = \frac{3}{2}; \\ a_{122} = a_{211} = a_{311} = 1; a_{233} = \frac{1}{2}; \\ a_{ijk} = 0, \quad otherwise. \end{cases}$$

For simplicity, we take λ as a real number, where λ is an eigenvalue of $\mathcal{A}$.

According to Theorem 2 of [24], we have

$$\Omega(\mathcal{A}) = \bigcup_{i\in N} \Omega_i(\mathcal{A}) = \{\lambda \in \mathbb{C} : -3 \le \lambda \le 3.5\}.$$

From Theorem 4 of [24], we obtain

$$\Theta(\mathcal{A}) = \bigcup_{i,j\in N, i\neq j} \Theta_{i,j}(\mathcal{A}) = \{\lambda \in \mathbb{C} : -2.345 \le \lambda \le 3.386\},$$

where

$\Theta_{1,2} \cup \Theta_{1,3} = \{\lambda \in \mathbb{C} : -2.345 \le \lambda \le -0.186\} \cup \{\lambda \in \mathbb{C} : 2.137 \le \lambda \le 3.225\}$
$\Theta_{2,1} \cup \Theta_{2,3} = \{\lambda \in \mathbb{C} : -1.137 \le \lambda \le 3.386\}$
$\Theta_{3,1} \cup \Theta_{3,2} = \{\lambda \in \mathbb{C} : -1.137 \le \lambda \le -0.5\} \cup \{\lambda \in \mathbb{C} : 2 \le \lambda \le 2.5\}$
$\mathcal{K}_{1,2} = \{\lambda \in \mathbb{C} : -2.345 \le \lambda \le 3.225\}, \mathcal{K}_{1,3} = \{\lambda \in \mathbb{C} : -1.637 \le \lambda \le 2.686\}$
$\mathcal{K}_{2,1} = \{\lambda \in \mathbb{C} : -1.137 \le \lambda \le 3.386\}, \mathcal{K}_{2,3} = \{\lambda \in \mathbb{C} : 0.5 \le \lambda \le 3.281\}$
$\mathcal{K}_{3,1} = \{\lambda \in \mathbb{C} : -1.137 \le \lambda \le 2.637\}, \mathcal{K}_{3,2} = \{\lambda \in \mathbb{C} : 0.5 \le \lambda \le 2.5\}$
$\mathcal{U}_1 = \{\lambda \in \mathbb{C} : -0.186 < \lambda < 2.137\}, \mathcal{U}_3 = \{\lambda \in \mathbb{C} : -0.5 < \lambda < 2\}, \mathcal{U}_2 = \emptyset.$

Without loss of generality, we select $S = \{3\}$, $\bar{S} = \{1, 2\}$. According to Lemma 2.1 and Theorem 2.2, we get

$$\begin{aligned}\mathcal{K}^S(\mathcal{A}) &= (\mathcal{K}_{3,1}(\mathcal{A}) \cup \mathcal{K}_{3,2}(\mathcal{A})) \bigcup (\mathcal{K}_{1,3}(\mathcal{A}) \cup \mathcal{K}_{2,3}(\mathcal{A})) \\ &= \{\lambda \in \mathbb{C} : -1.637 \le \lambda \le 3.281\}\end{aligned}$$

and

$$\begin{aligned}\Psi^S(\mathcal{A}) &= (\Psi_{3,1}(\mathcal{A}) \cup \Psi_{3,2}(\mathcal{A})) \bigcup (\Psi_{1,3}(\mathcal{A}) \cup \Psi_{2,3}(\mathcal{A})) \\ &= \{\lambda \in \mathbb{C} : -1.637 \le \lambda \le -0.186\} \cup \{\lambda \in \mathbb{C} : 0.5 \le \lambda \le 3.281\}.\end{aligned}$$

From the fact that

$$\Omega(\mathcal{A}) = \{\lambda \in \mathbb{C} : -3 \le \lambda \le 3.5\}, \Theta(\mathcal{A}) = \{\lambda \in \mathbb{C} : -2.345 \le \lambda \le 3.386\}$$

and

$$\Psi^S(\mathcal{A}) = \{\lambda \in \mathbb{C} : -1.637 \le \lambda \le -0.186\} \cup \{\lambda \in \mathbb{C} : 0.5 \le \lambda \le 3.281\},$$

we conclude that the result given in Theorem 2.2 is more accurate than Theorem 2 and Theorem 4 of [24].

The following example exhibits the efficiency of the new inclusion sets given in Theorem 2.5.

Example 3.2. *Consider* 3 *order* 3 *dimensional tensor* $\mathcal{A} = (a_{ijk})$ *defined by*

$$a_{ijk} = \begin{cases} a_{111} = \frac{1}{2}; a_{222} = \frac{1}{4}; a_{333} = -3; a_{122} = 2; \\ a_{112} = a_{131} = \frac{1}{10}; a_{211} = a_{232} = \frac{1}{8}; a_{311} = a_{332} = \frac{1}{16}; \\ a_{ijk} = 0, \quad otherwise. \end{cases}$$

For simplicity, we take λ as a real number, where λ is an eigenvalue of $\mathcal{A}$.

According to Theorem 2 of [24], we have

$$\Omega(\mathcal{A}) = \bigcup_{i\in N} \Omega_i(\mathcal{A}) = \{\lambda \in \mathbb{C} : -3.125 \le \lambda \le -2.875\} \bigcup \{\lambda \in \mathbb{C} : -1.7 \le \lambda \le 2.7\}.$$

According to Theorem 4 of [24], we have

$$\Theta(\mathcal{A}) = \bigcup_{i,j\in N, i\neq j} \Theta_{i,j}(\mathcal{A}) = \{\lambda \in \mathbb{C} : -3.125 \le \lambda \le -2.875\} \bigcup \{\lambda \in \mathbb{C} : -1.7 \le \lambda \le 2.7\}.$$

Without loss of generality, we choose $S = \{1\}$, $\bar{S} = \{2, 3\}$. According to Lemma 2.4 and Theorem 2.5, we obtian

$$\begin{aligned}\Omega^S(\mathcal{A}) &= (\Omega^S_{1,2}(\mathcal{A}) \cup \Omega^S_{1,3}(\mathcal{A})) \bigcup (\Omega^{\bar{S}}_{2,1}(\mathcal{A}) \cup \Omega^{\bar{S}}_{3,1}(\mathcal{A}))\\ &= \{\lambda \in \mathbb{C} : -3.101 \le \lambda \le -2.897\} \bigcup \{\lambda \in \mathbb{C} : -0.433 \le \lambda \le 1.217\}\end{aligned}$$

and

$$\begin{aligned}\Phi^S(\mathcal{A}) &= (\Phi^S_{1,2}(\mathcal{A}) \cup \Phi^S_{1,3}(\mathcal{A})) \bigcup (\Phi^{\bar{S}}_{2,1}(\mathcal{A}) \cup \Phi^{\bar{S}}_{3,1}(\mathcal{A}))\\ &= \{\lambda \in \mathbb{C} : -3.101 \le \lambda \le -2.897\} \bigcup \{\lambda \in \mathbb{C} : -0.433 \le \lambda \le 0.012\} \bigcup \{\lambda \in \mathbb{C} : 0.222 \le \lambda \le 1.217\}.\end{aligned}$$

So, we conclude that the result given in Theorem 2.5 is much sharper than Theorem 2 and Theorem 4 of [24] in some cases.

Remark 3.3. *It is worth noting that $\Omega^S(\mathcal{A}) \subseteq \mathcal{K}^S(\mathcal{A}) \subseteq \mathcal{K}(\mathcal{A}) \subseteq \Gamma(\mathcal{A})$ by Theorem 2.3 of [23] and Theorem 6 of [24]. However, $\Phi^S(\mathcal{A}) \subseteq \Psi^S(\mathcal{A})$ may not hold in general because the exclusion sets $\mathcal{V}^S_{i,j}(\mathcal{A}) \subseteq \mathcal{U}_{i,j}(\mathcal{A})$.*

In order to compare two new S-type inclusion sets, we further compute the above two examples. We apply Theorem 2.2 to example 3.2 and obtain that

$$\Psi^S(\mathcal{A}) = \{\lambda \in \mathbb{C} : -3.125 \le \lambda \le -2.875\} \cup \{\lambda \in \mathbb{C} : -1.7 \le \lambda \le 2.7\},$$

which shows that the result of Theorem 2.5 is sharper than that of Theorem 2.2 in this case.

On the other hand, we apply Theorem 2.5 to example 3.1 and obtain that

$$\Phi^S(\mathcal{A}) = \{\lambda \in \mathbb{C} : -1.345 \le \lambda \le 3.386\},$$

which shows that the result of Theorem 2.2 is tighter than that of Theorem 2.5 in this case. Thus, Theorem 2.2 and Theorem 2.5 have their own advantages.

4 Conclusions

In the paper, we focused on S-type eigenvalue exclusion sets for tensors. S-type eigenvalue exclusion sets have two advantages: (i) all eigenvalues of tensor are not contained; (ii) they may reduce computations and achieve more accurate numerical results. Based on characterizations of S-type eigenvalue exclusion sets, we proposed new S-type inclusion sets $\Psi^S(\mathcal{A})$ and $\Phi^S(\mathcal{A})$. Furthermore, we showed that new S-type eigenvalue inclusion sets are sharper than those of results in [23, 24, 34] via two examples. It is worth noting that the proper selection of S has a great influence on the numerical effect of S-type inclusion sets. Therefore, how to select the proper S is our further research.

Acknowledgement: This work is supported by the Natural Science Foundation of China (11671228) and the Natural Science Foundation of Shandong Province (ZR2016AM10).

References

[1] Cichocki A., Zdunek R., Phan A., Amari S., Nonnegative Matrix and Tensor Factorizations, John Wiley, 2009.
[2] Qi L., Luo Z., Tensor Analysis: Spectral Theory and Special Tensors, SIAM, 2017.
[3] Lim L.H., Singular values and eigenvalues of tensors: A variational approach, In: Proceedings of the IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing (CAMSAP'05), 2005, 129–132.
[4] Qi L., Eigenvalues of a real supersymmetric tensor, J. Symb. Comput., 2005, 40(6), 1302–1324.
[5] Bloy L., Verma R., On computing the underlying fiber directions from the diffusion orientation distribution function in Medical Image Computing and Computer-Assisted Intervention, Springer, 2008.
[6] Kofidis E., Regalia R., On the best rank-1 approximation of higher-order supersymmetric tensors, SIAM. J. Matrix. Anal. Appl., 2002, 23(3), 863–884.
[7] Li W., Ng M., On the limiting probability distribution of a transition probability tensor, Linear Multilinear Algebra, 2014, 62(3), 362–385.

[8] Ng M., Qi L., Zhou G., Finding the largest eigenvalue of a nonnegative tensor, SIAM. J. Matrix. Anal. Appl., 2009, 31(4), 1090—1099.
[9] Gao L., Wang D., Wang G., Further results on exponential stability for impulsive switched nonlinear time-delay systems with delayed impulse effects, Appl. Math. Comput., 2015, 268, 186–200.
[10] Gao L., Wang D., Input-to-state stability and integral input-to-state stability for impulsive switched systems with time-delay under asynchronous switching, Nonlinear Anal. Hybrid Syst., 2016, 20, 55–71.
[11] Ni Q., Qi L., Wang F., An eigenvalue method for testing the positive definiteness of a multivariate form, IEEE. Trans. Automat. Contr., 2008, 53(5), 1096–1107.
[12] Chen H., Wang Y., On computing the minimal H-eigenvalue of sign-structured tensors, Front. Math. China., 2017, 12(6), 1289–1202.
[13] Chen H., Huang Z., Qi L., Copositive tensor detection and its applications in physics and hypergraphs, Comput. Optim. Appl., 2018, 69, 133–158.
[14] Friedland S., Gaubert S., Han L., Perron-Frobenius theorem for nonnegative multilinear forms and extensions. Linear Algebra Appl., 2013, 438(2), 738–749.
[15] Wang Y., Zhou G., Caccetta L., Convergence analysis of a block improvement method for polynomial optimization over unit spheres, Numer. Linear Algebra Appl., 2015, 22(6), 1059–1076.
[16] Wang Y., Zhang K., Sun, H., Criteria for strong H-tensors, Front. Math. China., 2016, 11(3), 577–592.
[17] Wang Y., Sun X., Meng F., On the conditional and partial trade credit policy with capital constraints, Appl. Math. Model., 2016, 40, 1–18.
[18] Wang Y., Zhou G., Caccetta L., Nonsingular H-tensor and its Criteria, J. Ind. Manage. Optim., 2016, 12, 1173–1186.
[19] Wang X., Chen H., Wang Y., Solution structures of tensor complementarity problem, Front. Math. China., 2018, 13(6), 935–945.
[20] Wang X., Chen H., Wang Y., On the solution existence of Cauchy tensor variational inequality problems, Pac. J. Optim., 2018, 14(3), 479–487.
[21] Yang Y., Yang Q., Further results for Perron-Frobenius theorem for nonnegative tensors I, SIAM. J. Matrix. Anal. Appl., 2010, 31(5), 2517–2530.
[22] Zhang K., Wang Y., An H-tensor based iterative scheme for identifying the positive definiteness of multivariate homogeneous forms, J. Comput. Appl. Math., 2016, 305, 1–10.
[23] Li C., Li Y., Kong, X., New eigenvalue inclusion sets for tensors, Numer. Linear Algebra Appl., 2014, 21(1), 39–50.
[24] Li C., Jiao A., Li Y., An S-type eigenvalue localization set for tensors, Linear Algebra Appl., 2016, 493, 469–483.
[25] Liu Q., Li Y., Bounds for the Z-eigenpair of general nonnegative tensors, Open Math., 2016, 14, 181–194.
[26] Wang G., Zhou G., Caccetta L., Z-eigenvalue inclusion theorems for tensors, Discrete Cont. Dyn-B., 2017, 22(1), 187–198.
[27] Wang G., Zhou G., Caccetta L., Sharp Brauer-type eigenvalue inclusion theorems for tensors, Pac. J. Optim., 2018, 14(2), 227–244.
[28] Wang G., Wang Y., Liu L., Bound estimations on the eigenvalues for Fan Product of M-tensors, Taiwan. J. Math. 2019, 23, 751–766.
[29] Wang G., Wang Y.N., Wang Y., Some Ostrowski-type bound estimations of spectral radius for weakly irreducible nonnegative tensors, Linear Multilinear Algebra, 2018, https://doi.org/10.1080/03081087.2018.1561823.
[30] Wang Y., Wang G., Two S-type Z-eigenvalue inclusion sets for tensors, J. Inequal. Appl., 2017, 2017:152.
[31] Zhao J., Sang C., Two new eigenvalue localization sets for tensors and theirs applications, Open Math., 2017, 15, 1267–1276.
[32] Bu C., Wei Y., Sun L., Zhou J., Brualdi-type eigenvalue inclusion sets of tensors, Linear Algebra Appl., 2015, 480, 168–175.
[33] Li C., Zhang C., Li Y., Minimal Gerschgorin tensor eigenvalue inclusion set and its approximation, J. Comput. Appl. Math., 2016, 302, 200–210.
[34] Li C., Li S., Liu Q., Li Y., Exclusion sets in eigenvalue inclusion sets for tensors, arXiv:1706.00944.