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Continuous linear operators on Orlicz-Bochner spaces

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Abstract: Let (Ω, Σ, μ) be a complete σ-finite measure space, φ a Young function and X and Y be Banach spaces. Let $L^\varphi(X)$ denote the corresponding Orlicz-Bochner space and $\mathcal{T}_\varphi^\wedge$ denote the finest Lebesgue topology on $L^\varphi(X)$. We examine different classes of $(\mathcal{T}_\varphi^\wedge, \|\cdot\|_Y)$-continuous linear operators $T : L^\varphi(X) \to Y$: weakly compact operators, order-weakly compact operators, weakly completely continuous operators, completely continuous operators and compact operators. The relationships among these classes of operators are established.

Keywords: Orlicz-Bochner spaces, Lebesgue topologies, weakly compact operators, compact operators, weakly completely continuous operators, completely continuous operators

MSC: 47B38, 46E40, 28A25

1 Introduction and preliminaries

Throughout the paper, $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ denote real Banach spaces and X^* and Y^* denote their Banach duals, respectively. By B_X we denote the closed unit ball in X. Let $\mathcal{L}(X, Y)$ stand for the Banach space of all bounded linear operators from X to Y, equipped with the uniform operator norm $\|\cdot\|$.

Continuous linear operators on Banach spaces of vector-valued function spaces (in particular, Orlicz-Bochner spaces $L^\varphi(X)$ and Lebesgue-Bochner spaces $L^p(X)$ $(1 \le p \le \infty)$) has been the object of much study (see [1–13]). Andrews ([5, Theorems 2 and 5], [6, Theorem 3]) proved the Dunford-Pettis-Phillips type theorems for compact and weakly compact operators from $L^1(X)$ to a Banach space Y.

Now we recall the basic concepts and properties of Orlicz-Bochner spaces (see [11, 12, 14 – 16] for more details).

By a *Young function* we mean here a continuous convex mapping $\varphi : [0, \infty) \to [0, \infty)$ that vanishes only at 0 and $\varphi(t)/t \to 0$ as $t \to 0$ and $\varphi(t)/t \to \infty$ as $t \to \infty$. Let φ^* stand for the complementary Young function of φ in the sense of Young.

We assume that (Ω, Σ, μ) is a complete σ-finite measure space. Denote by $\Sigma_f(\mu)$ the δ-ring of sets $A \in \Sigma$ with $\mu(A) < \infty$. By $L^0(X)$ we denote the linear space of μ-equivalence classes of all strongly Σ-measurable functions $f : \Omega \to X$.

Let $L^\varphi(X)$ (resp., L^φ) denote the *Orlicz-Bochner space* (resp., *Orlicz space*) defined by a Young function φ, i.e.,

$$\begin{aligned} L^\varphi(X) &:= \left\{ f \in L^0(X) : \int_\Omega \varphi(\lambda \|f(\omega)\|_X) d\mu < \infty \text{ for some } \lambda > 0 \right\} \\ &= \left\{ f \in L^0(X) : \|f(\cdot)\|_X \in L^\varphi \right\}. \end{aligned}$$

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Then $L^\varphi(X)$, equipped with the topology $\mathcal{T}_\varphi$ of the norm

$$\|f\|_\varphi := \inf\left\{\lambda > 0 : \int_\Omega \varphi\left(\frac{\|f(\omega)\|_X}{\lambda}\right) d\mu \le 1\right\}$$

is a Banach space.

For a bounded linear operator $T : L^\varphi(X) \to Y$ let

$$m(A)(x) := T(\mathbb{1}_A \otimes x) \ \ for \ \ A \in \Sigma_f(\mu), \ x \in X.$$

One can easily show that $m(A) \in \mathcal{L}(X, Y)$ for $A \in \Sigma_f(\mu)$. Then the mapping $m : \Sigma_f(\mu) \to \mathcal{L}(X, Y)$ will be called the *representing measure* of T.

Recall that a subset H of $L^\varphi(X)$ is said to be *solid* whenever $\|f_1(\omega)\|_X \le \|f_2(\omega)\|_X$ μ-a.e. and $f_1 \in L^\varphi(X)$, $f_2 \in H$ imply $f_1 \in H$. A linear topology ξ on $L^\varphi(X)$ is said to be *locally solid* if it has a local basis at 0 consisting of solid sets (see [16]).

Following [17, Definition 2.2], [13] we have

Definition 1.1. A locally solid topology ξ on $L^\varphi(X)$ is said to be a *Lebesgue topology* if for a net (f_α) in $L^\varphi(X)$, $\|f_\alpha(\cdot)\|_X \xrightarrow{(o)} 0$ in L^φ implies $f_\alpha \to 0$ in ξ.

In view of the super Dedekind completeness of L^φ one can restrict in the above definition to usual sequences (f_n) in $L^\varphi(X)$ (see [17, Definition 2.2, p. 173]).

Note that for a sequence (f_n) in $L^\varphi(X)$, $\|f_n(\cdot)\|_X \xrightarrow{(o)} 0$ in L^φ if and only if $\|f_n(\omega)\|_X \to 0$ μ-a.e. and $\|f_n(\omega)\|_X \le u(\omega)$ μ-a.e. for some $0 \le u \in L^\varphi$.

For $\varepsilon > 0$ let $U_\varphi(\varepsilon) = \{f \in L^\varphi(X) : \int_\Omega \varphi(\|f(\omega)\|_X)\, d\mu \le \varepsilon\}$. Then the family of all sets of the form:

$$(\star) \qquad \bigcup_{n=1}^{\infty}\left(\sum_{i=1}^{n} U_\varphi(\varepsilon_i)\right),$$

where (ε_n) is a sequence of positive numbers, is a local basis at 0 for a linear topology $\mathcal{T}_\varphi^\wedge$ on $L^\varphi(X)$ (see [13, 16] for more details). Using [16, Lemma 1.1] one can show that the sets of the form ($\star$) are convex and solid, so $\mathcal{T}_\varphi^\wedge$ is a locally convex-solid topology.

We now recall terminology and basic facts concerning the spaces of weak*-measurable functions $g : \Omega \to X^*$ (see [18, 19]). Given a function $g : \Omega \to X^*$ and $x \in X$, let $g_x(\omega) = g(\omega)(x)$ for $\omega \in \Omega$. By $L^0(X^*, X)$ we denote the linear space of the weak*-equivalence classes of all weak*-measurable functions $g : \Omega \to X^*$. In view of the super Dedekind completeness of L^0 the set $\{|g_x| : x \in B_X\}$ is order bounded in L^0 for each $g \in L^0(X^*, X)$. Thus one can define the so called *abstract norm* $\vartheta : L^0(X^*, X) \to L^0$ by

$$\vartheta(g) := \sup\left\{|g_x| : x \in B_X\right\} \ \text{in } L^0.$$

It is known that for $f \in L^0(X)$, $g \in L^0(X^*, X)$, the function $\langle f, g\rangle : \Omega \to \mathbb{R}$ defined by $\langle f, g\rangle(\omega) = \langle f(\omega), g(\omega)\rangle$ is measurable and

$$|\langle f(\omega), g(\omega)\rangle| \le \|f(\omega)\|_X\, \vartheta(g)(\omega) \ \ \mu\text{-a.e.}$$

Moreover, $\vartheta(g) = \|g(\cdot)\|_{X^*}$ for $g \in L^0(X^*)$. Let

$$L^{\varphi^*}(X^*, X) := \left\{g \in L^0(X^*, X) : \vartheta(g) \in L^{\varphi^*}\right\}.$$

Clearly $L^{\varphi^*}(X^*) \subset L^{\varphi^*}(X^*, X)$. If, in particular, X^* has the Radon-Nikodym property (i.e., X is an *Asplund space* see [20, p. 213]), then $L^{\varphi^*}(X^*, X) = L^{\varphi^*}(X^*)$. Note that every reflexive Banach space X is an Asplund space.

Let $(L^\varphi(X), \mathcal{T}_\varphi^\wedge)^*$ denote the topological dual of $(L^\varphi(X), \mathcal{T}_\varphi^\wedge)$.

Now we present basic properties of the topology $\mathcal{T}_\varphi^\wedge$ on $L^\varphi(X)$.

Theorem 1.1. *Let φ be a Young function. Then the following statements hold:*

(i) $\mathcal{T}_\varphi^\wedge \subset \mathcal{T}_\varphi$ *and* $\mathcal{T}_\varphi^\wedge = \mathcal{T}_\varphi$ *if* φ *satisfies the* Δ_2*-condition, i.e.,* $\varphi(2t) \le d\varphi(t)$ *for some* $d > 1$ *and all* $t \ge 0$.

(ii) $\mathcal{T}_\varphi^\wedge$ *is the finest Lebesgue topology on* $L^\varphi(X)$.

(iii) $(L^\varphi(X), \mathcal{T}_\varphi^\wedge)^* = \{F_g : g \in L^{\varphi^*}(X^*, X)\}$,

where $$F_g(f) = \int_\Omega \langle f(\omega), g(\omega)\rangle \, d\mu \text{ for } f \in L^\varphi(X).$$

(iv) *If* X *is an Asplund space, then the space* $(L^\varphi(X), \mathcal{T}_\varphi^\wedge)$ *is strongly Mackey; hence* $\mathcal{T}_\varphi^\wedge$ *coincides with the Mackey topology* $\tau(L^\varphi(X), L^{\varphi^*}(X^*))$.

(v) *If a subset* H *of* $L^\varphi(X)$ *is* $\mathcal{T}_\varphi^\wedge$*-bounded, then* $\sup_{f\in H} \|f\|_\varphi < \infty$.

Proof. (i)–(ii) This follows from [16, Theorem 6.1 and Theorem 6.3].

(iii) In view of [13, Corollary 4.4 and Theorem 1.2], we get $(L^\varphi(X), \mathcal{T}_\varphi^\wedge)^* = L^\varphi(X)_n^\sim$, where $L^\varphi(X)_n^\sim$ stands for the order continuous dual of $L^\varphi(X)$ (see [13, 18] for more details). According to [18, Theorem 4.1] $L^\varphi(X)_n^\sim = \{F_g : g \in L^{\varphi^*}(X^*, X)\}$. Thus the proof is complete.

(iv) See [13, Theorem 4.5].

(v) Assume that a subset H of $L^\varphi(X)$ is $\mathcal{T}_\varphi^\wedge$-bounded. Then by (iv) H is $\sigma(L^\varphi(X), L^{\varphi^*}(X^*, X))$-bounded. Hence in view of [21, Proposition 1.3], the set $\{\|f(\cdot)\|_X : f \in H\}$ in L^φ is $\sigma(L^\varphi, L^{\varphi^*})$-bounded. Since L^{φ^*} is a norming subset of $(L^\varphi)^*$ (see [22, p. 12]), by [22, Lemma 1, p. 20], we get $\sup_{f\in H} \|f\|_\varphi = \sup_{f\in H} \| \, \|f(\cdot)\|_X \, \|_\varphi < \infty$. □

The following result establishes relationships between different classes of linear operators on $L^\varphi(X)$.

Proposition 1.2. *For a linear operator* $T : L^\varphi(X) \to Y$ *consider the following statements:*

(i) T *is* $(\mathcal{T}_\varphi^\wedge, \|\cdot\|_Y)$*-continuous.*

(ii) T *is* $(\mathcal{T}_\varphi^\wedge, \|\cdot\|_Y)$*-sequentially continuous.*

(iii) $\|T(f_n)\|_Y \to 0$ *if* $\|f_n(\omega)\|_X \to 0$ μ*-a.e. and* $\|f_n(\omega)\|_X \le u(\omega)$ μ*-a.e. for some* $0 \le u \in L^\varphi$ *and all* $n \in \mathbb{N}$.

(iv) *For every* $y^* \in Y^*$, $y^* \circ T \in (L^\varphi(X), \mathcal{T}_\varphi^\wedge)^*$.

(v) T *is* $(\sigma(L^\varphi(X), L^{\varphi^*}(X^*, X)), \sigma(Y, Y^*))$*-continuous.*

(vi) T *is* $(\tau(L^\varphi(X), L^{\varphi^*}(X^*, X)), \|\cdot\|_Y)$*-continuous.*

Then the following implications hold:
(i)⇒(ii)⇒(iii)⇒(iv)⇒(v) ⇒(vi).
If, in particular, X is an Asplund space, then (vi)⇒(i), *that is, all the statements* (i)–(vi) *are equaivalent.*

Proof. (i)⇒(ii)⇒(iii) Obvious because $\mathcal{T}_\varphi^\wedge$ is a Lebesgue topology.

(iii)⇒(iv) Assume that (iii) holds. Then for every $y^* \in Y^*$, $y^* \circ T \in L^\varphi(X)_c^\sim$, where $L^\varphi(X)_c^\sim$ stands for the σ-order continuous dual of $L^\varphi(X)$ (see [17] for more details). In view of the super Dedekind completeness of L^0 we have $L^\varphi(X)_c^\sim = L^\varphi(X)_n^\sim$ (see [17]). Since $L^\varphi(X)_n^\sim = (L^\varphi(X), \mathcal{T}_\varphi^\wedge)^*$, the proof is complete.

(iv)⇒(v) See [23, Theorem 9.26].

(v))⇒(vi) See [24, Theorem 8.6.1].

Assume that X is an Asplund space. Then (vi)⇒(i) holds because $\mathcal{T}_\varphi^\wedge = \tau(L^\varphi(X), L^{\varphi^*}(X^*, X))$ (see Theorem 1.1). □

In this paper, using the results of [21], concerning conditional $\sigma(L^\varphi(X), L^{\varphi^*}(X^*, X))$-compactness and relative $\sigma(L^\varphi(X), L^{\varphi^*}(X^*, X))$-compactness in $L^\varphi(X)$, we examine different classes of $(\mathcal{T}_\varphi^\wedge, \|\cdot\|_Y)$-continuous linear operators $T : L^\varphi(X) \to Y$: weakly compact operators, order-weakly compact operators, weakly completely continuous operators, completely continuous operators and compact operators. We establish relationships among these classes of operators.

2 Order-weakly compact and order-almost weakly compact operators on $L^\varphi(X)$

Dodds [25] studied the class of order-weakly compact operators on Banach lattices (see also [23, Section 18]). Following [25] one can define order-weakly compact and order-almost weakly compact operators on Orlicz-Bochner spaces $L^\varphi(X)$ (see [12]).

For $0 \le u \in L^\varphi$, let $I_u = \{f \in L^\varphi(X) : \|f(\omega)\|_X \le u(\omega)\ \mu\text{-a.e.}\}$.

Definition 2.1. A bounded linear operator $T : L^\varphi(X) \to Y$ is said to be *order-weakly compact* (resp. *order-almost weakly compact*) if for every $0 \le u \in L^\varphi$, the set $T(I_u)$ is a relatively weakly compact (resp., conditionally weakly compact) set in Y.

Recall that a Banach space X is called *almost reflexive* if every bounded set in X is conditionally $\sigma(X, X^*)$-compact. The fundamental ℓ^1-Rosenthal theorem says that a Banach space X is almost reflexive if and only if it contains no isomorphic copy of ℓ^1. Moreover, X contains no isomorphic copy of ℓ^1 if and only if X^* has the weak Radon-Nikodym property (see [26]).

Proposition 2.1. *Assume that a Banach space X is almost reflexive (resp., X is reflexive). Then for every $0 \le u \in L^\varphi$, the set I_u is conditionally $\sigma(L^\varphi(X), L^{\varphi^*}(X^*, X))$-compact (resp., relatively $\sigma(L^\varphi(X), L^{\varphi^*}(X^*))$-compact).*

Proof. Let $0 \le u \in L^\varphi$. Then I_u is a norm bounded subset of $L^\varphi(X)$ and for every $v \in L^{\varphi^*}$, we have $uv \in L^1$ and

$$p_{I_u}(v) := \sup_{f \in I_u} \int_\Omega \|f(\omega)\|_X\, |v(\omega)|\, d\mu \le \int_\Omega |u(\omega)v(\omega)|\, d\mu.$$

To show that p_{I_u} is an order continuous seminorm on L^{φ^*}, assume that (v_n) is a sequence in L^{φ^*} such that $v_n \xrightarrow{(o)} 0$ in L^{φ^*}, i.e., $v_n(\omega) \to 0$ μ-a.e. and $|v_n(\omega)| \le v(\omega)$ μ-a.e. for some $0 \le v \in L^{\varphi^*}$ and all $n \in \mathbb{N}$. Since $u\,v \in L^1$, by the Lebesgue dominated convergence theorem $p_{I_u}(v_n) \to 0$. In view of [21, Proposition 1.1] (resp. [21, Proposition 1.1] and [22, Lemma 11(a), p. 31]) the set I_u is conditionally $\sigma(L^\varphi, L^{\varphi^*})$-compact (resp., relatively $\sigma(L^\varphi, L^{\varphi^*})$-compact). □

Theorem 2.2. *Assume that a Banach space X is reflexive. Then every $(\mathcal{T}_\varphi^\wedge \| \cdot \|_Y)$-continuous linear operator $T : L^\varphi(X) \to Y$ is order-weakly compact.*

Proof. Let $T : L^\varphi(X) \to Y$ be a $(\mathcal{T}_\varphi^\wedge, \|\cdot\|_Y)$-continuous linear operator and $0 \le u \in L^\varphi$. Then by Proposition 2.1 I_u is a relatively $\sigma(L^\varphi(X), L^{\varphi^*}(X^*))$-compact set in $L^\varphi(X)$. Since T is $(\sigma(L^\varphi(X), L^{\varphi^*}(X^*)), \sigma(Y, Y^*))$-continuous, $T(I_u)$ is relatively $\sigma(Y, Y^*)$-compact in Y. □

Using Proposition 2.1 and arguing as in the proof of Theorem 2.2, we get:

Theorem 2.3. *Assume that a Banach space X is almost reflexive. Then every $(\mathcal{T}_\varphi^\wedge, \| \cdot \|_Y)$-continuous linear operator $T : L^\varphi(X) \to Y$ is order-almost weakly compact.*

3 Weakly compact and almost weakly compact operators on $L^\varphi(X)$

We say that a Young function ψ *increases more rapidly* than another φ (in symbols, $\varphi \prec \psi$) if for arbitrary $c > 0$ there exists $d > 0$ such that $c\varphi(t) \le \frac{1}{d}\psi(dt)$ for all $t \ge 0$. Recall that a Young function φ satisfies the $\bigtriangledown_2$-*condition*, if $\varphi(t) \le \frac{1}{2d}\varphi(dt)$ for some $d > 1$ and all $t \ge 0$. It is known that φ satisfies the $\bigtriangledown_2$-condition if and only if φ^* satisfies the Δ_2-condition (see [27, Theorem 2.2.3, pp. 22-23]).

The following results will be useful (see [27, Theorem 5.3.3, p. 171]).

Proposition 3.1. *Let φ and ψ be Young functions such that $\varphi \prec \psi$. Then $L^\psi \subset L^\varphi$ and every norm bounded set in L^ψ is relatively $\sigma(L^\varphi, L^{\varphi^*})$-compact in L^φ.*

Theorem 3.2. *Let φ be a Young function. Then for a subset H of L^φ the following statements are equivalent:*

(i) *H is conditionally $\sigma(L^\varphi, L^{\varphi^*})$-compact.*

(ii) *H is relatively $\sigma(L^\varphi, L^{\varphi^*})$-sequentially compact.*

(iii) *H is relatively $\sigma(L^\varphi, L^{\varphi^*})$-compact.*

(iv) *There exists a Young function ψ with $\varphi \prec \psi$ such that $H \subset L^\psi$ and $\sup\{\|u\|_\psi : u \in H\} \leq 1$.*

Proof. (i)⇔(ii) See [21, Proposition 1.1].
(ii)⇔(iii) This follows from [22, Lemma 11(a), p. 31].
(iii)⇔(iv) This follows from [28, Theorem 1.2]. □

Remark. For a finite measure space (Ω, Σ, μ), the equivalence (iii)⇔(iv) in Theorem 3.2 was established by Ando (see [29, Theorem 2]).

If $\varphi \prec \psi$, then $L^\psi(X) \subset L^\varphi(X)$ and $\mathcal{T}_\varphi|_{L^\psi(X)} \subset \mathcal{T}_\psi$. Let

$$i_\psi : L^\psi(X) \to L^\varphi(X)$$

stand for the inclusion map and

$$B_{L^\psi(X)} = \{f \in L^\psi(X) : \|f\|_\psi \leq 1\}.$$

Recall that a bounded linear operator T from a Banach space Z to Y is said to be *weakly compact* (resp. *almost weakly compact*) if $T(B_Z)$ is a relatively weakly compact (resp. conditionally weakly compact) set in Y.

Theorem 3.3. *Assume that a Banach space X is reflexive and $T : L^\varphi(X) \to Y$ is a $(\mathcal{T}_\varphi^\wedge, \|\cdot\|_Y)$-continuous linear operator. Then for every Young function ψ with $\varphi \prec \psi$, the operator $T \circ i_\psi : L^\psi(X) \to Y$ is weakly compact.*

Proof. Let ψ be a Young function with $\varphi \prec \psi$. Then by Proposition 3.1 the set $\{\|f(\cdot)\|_X : f \in B_{L^\psi(X)}\}$ in L^φ is relatively $\sigma(L^\varphi, L^{\varphi^*})$-compact, and hence by Theorem 3.2 it is relatively $\sigma(L^\varphi, L^{\varphi^*})$-sequentially compact. By [21, Corollary 3.4 and Theorem 3.2] $B_{L^\psi(X)}$ is relatively $\sigma(L^\varphi(X), L^{\varphi^*}(X^*))$-compact. Since T is $(\sigma(L^\varphi(X), L^{\varphi^*}(X^*)), \sigma(Y, Y^*))$-continuous, $T(B_{L^\psi(X)})$ is relatively $\sigma(Y, Y^*)$-compact, and hence $T \circ i_\psi$ is weakly compact. □

Corollary 3.4. *Assume that a Banach space X is reflexive and Young function φ satisfies the ∇_2-condition. Then every $(\mathcal{T}_\varphi^\wedge, \|\cdot\|_Y)$-continuous linear operator $T : L^\varphi(X) \to Y$ is weakly compact.*

Theorem 3.5. *Assume that a Banach space X is almost reflexive and $T : L^\varphi(X) \to Y$ is a $(\mathcal{T}_\varphi^\wedge, \|\cdot\|_Y)$-continuous linear operator. Then for every Young function ψ with $\varphi \prec \psi$, the operator $T \circ i_\psi : L^\psi(X) \to Y$ is almost weakly compact.*

Proof. Let ψ be a Young function with $\varphi \prec \psi$. Then by Proposition 3.1 the set $\{\|f(\cdot)\|_X : f \in B_{L^\psi(X)}\}$ in L^φ is relatively $\sigma(L^\varphi, L^{\varphi^*})$-compact, and hence by Theorem 3.2 it is conditionally $\sigma(L^\varphi, L^{\varphi^*})$-compact. By [21, Corollary 2.5] $B_{L^\psi(X)}$ is conditionally $\sigma(L^\varphi(X), L^{\varphi^*}(X^*, X))$-compact. Since T is $(\sigma(L^\varphi(X), L^{\varphi^*}(X^*, X)), \sigma(Y, Y^*))$-continuous, $T(B_{L^\psi(X)})$ is conditionally $\sigma(Y, Y^*)$-compact, i.e., $T \circ i_\psi$ is almost weakly compact. □

Corollary 3.6. *Assume that a Banach space X is almost reflexive and a Young function φ satisfies the ∇_2-condition. Then every $(\mathcal{T}_\varphi^\wedge, \|\cdot\|_Y)$-continuous linear operator $T : L^\varphi(X) \to Y$ is almost weakly compact.*

4 Weakly completely continuous operators on $L^\varphi(X)$

Definition 4.1. Assume that (Z, ξ) is a locally convex Hausdorff space. A $(\xi, \|\cdot\|_Y)$-continuous linear operator $T : Z \to Y$ is said to be *weakly completely continuous* if T maps weakly-Cauchy sequences in Z onto weakly-convergent sequences in Y.

Recall that a weakly completely continuous operator between Banach spaces is usually called a *Dieudonné operator*.

Theorem 4.1. *Let $T : L^\varphi(X) \to Y$ be a $(\mathcal{T}_\varphi^\wedge, \|\cdot\|_Y)$-continuous linear operator and $m : \Sigma_f(\mu) \to \mathcal{L}(X, Y)$ be its representing measure. If T weakly completely continuous, then for every $A \in \Sigma_f(\mu)$, $m(A)$ is a Dieudonné operator.*

Proof. Assume that (x_n) is a $\sigma(X, X^\star)$-Cauchy sequence in X and $A \in \Sigma_f(\mu)$. We shall show that $(\mathbb{1}_A \otimes x_n)$ is a $\sigma(L^\varphi(X), L^{\varphi^*}(X^\star, X))$-Cauchy sequence in $L^\varphi(X)$. Indeed, let $g \in L^{\varphi^*}(X^\star, X)$ be given. Then $g_{x_n}(\omega) \to v_g(\omega)$ μ-a.e., where $\vartheta(g) \in L^{\varphi^*}$. Since $|g_{x_n}(\omega)| \le \vartheta(g)(\omega)\|x_n\|_X \le a\vartheta(g)(\omega)$ μ-a.e., where $a = \sup_n \|x_n\|_X < \infty$, we get $v_g(\omega) \le a\vartheta(g)(\omega)$ μ-a.e. Hence $v_g \in L^{\varphi^*}$ and $\mathbb{1}_A v_g \in L^1$. Note that $\mathbb{1}_A(\omega)g_{x_n}(\omega) \to \mathbb{1}_A(\omega)v_g(\omega)$ μ-a.e. and $|\mathbb{1}_A(\omega)g_{x_n}(\omega)| \le a\mathbb{1}_A(\omega)\vartheta(g)(\omega)$ μ-a.e. for all $n \in \mathbb{N}$. Then by the Lebesgue dominated convergence theorem,

$$\int_\Omega \langle(\mathbb{1}_A \otimes x_n)(\omega), g(\omega)\rangle \, d\mu = \int_\Omega \mathbb{1}_A(\omega)g_{x_n}(\omega)\, d\mu \to \int_\Omega \mathbb{1}_A(\omega)v_g(\omega)\, d\mu.$$

This means that $(\mathbb{1}_A \otimes x_n)$ is a $\sigma(L^\varphi(X), L^{\varphi^*}(X^\star, X))$-Cauchy sequence in $L^\varphi(X)$. Since $m(A)(x_n) = T(\mathbb{1}_A \otimes x_n)$ for $n \in \mathbb{N}$, we obtain that $(m(A)(x_n))$ is a $\sigma(Y, Y^\star)$-convergent sequence in Y, as desired. □

Theorem 4.2. *Let $T : L^\varphi(X) \to Y$ be a $(\mathcal{T}_\varphi^\wedge, \|\cdot\|_Y)$-continuous linear operator. Assume that for every Young function ψ with $\varphi \prec \psi$, the operator $T \circ i_\psi : L^\psi(X) \to Y$ is weakly compact. Then T is weakly completely continuous.*

Proof. Assume that (f_n) is a $\sigma(L^\varphi(X), L^{\varphi^*}(X^\star, X))$-Cauchy sequence in $L^\varphi(X)$. Then the set $\{f_n : n \in \mathbb{N}\}$ is conditionally $\sigma(L^\varphi(X), L^{\varphi^*}(X^\star, X))$-compact in $L^\varphi(X)$, and it follows that $\{\|f_n(\cdot)\|_X : n \in \mathbb{N}\}$ is a conditionally $\sigma(L^\varphi, L^{\varphi^*})$-compact set in L^φ (see [21, Theorem 2.2]). Then in view of Theorem 3.2 there exists a Young function ψ with $\varphi \prec \psi$ such that $\sup_n \|f_n\|_\psi \le 1$. It follows that the set $\{T(f_n) : n \in \mathbb{N}\}$ is relatively $\sigma(Y, Y^\star)$-compact in Y. Then there exists a subsequence (f_{k_n}) of (f_n) such that $T(f_{k_n}) \to y_o$ in $\sigma(Y, Y^\star)$ for some $y_o \in Y$. On the other hand, since T is $(\sigma(L^\varphi(X), L^{\varphi^*}(X^\star, X)), \sigma(Y, Y^\star))$-continuous, $(T(f_n))$ is a $\sigma(Y, Y^\star)$-Cauchy sequence in Y. It follows that $T(f_n) \to y_o$ in $\sigma(Y, Y^\star)$. □

As a consequence of Theorem 4.2 we have:

Corollary 4.3. *Assume that $T : L^\varphi(X) \to Y$ is a $(\mathcal{T}_\varphi^\wedge, \|\cdot\|_Y)$-continuous. If T is weakly compact operator, then T is weakly completely continuous.*

Theorem 4.4. *Assume that a Banach space X is almost reflexive and $T : L^\varphi(X) \to Y$ is a $(\mathcal{T}_\varphi^\wedge, \|\cdot\|_Y)$-continuous linear operator. If T is weakly completely continuous, then for every Young function ψ with $\varphi \prec \psi$, the operator $T \circ i_\psi : L^\psi(X) \to Y$ is weakly compact.*

Proof. Let ψ be a Young function such that $\varphi \prec \psi$. Then by Proposition 3.1 the set $\{\|f(\cdot)\|_X : f \in B_{L^\psi(X)}\}$ in L^φ is relatively $\sigma(L^\varphi, L^{\varphi^*})$-compact, and hence it is conditionally $\sigma(L^\varphi, L^{\varphi^*})$-compact (see Theorem 3.2). In view of [21, Corollary 2.3] $B_{L^\psi(X)}$ is conditionally $\sigma(L^\varphi(X), L^{\varphi^*}(X^\star, X))$-compact. To show that $T(B_{L^\psi(X)})$ is relatively $\sigma(Y, Y^\star)$-compact, assume that (y_n) is a sequence in $T(B_{L^\psi(X)})$, i.e., $y_n = T(f_n)$, where $f_n \in B_{L^\psi(X)}$. Then there exists a $\sigma(L^\varphi(X), L^{\varphi^*}(X^\star, X))$-Cauchy subsequence (f_{k_n}) of (f_n). Hence $T(f_{k_n}) \to y_o$ in $\sigma(Y, Y^\star)$ for some $y_o \in Y$, and this means $T(B_{L^\psi(X)})$ is relatively $\sigma(Y, Y^\star)$-sequentially compact in Y. By the Eberlein-Šmulian theorem, $T(B_{L^\psi(X)})$ is relatively $\sigma(Y, Y^\star)$-compact, as desired. □

Corollary 4.5. *Assume that a Banach space X is almost reflexive and a Young function φ satisfies the ∇_2-condition. Then for a $(\mathcal{T}_\varphi^\wedge, \|\cdot\|_Y)$-continuous linear operator $T : L^\varphi(X) \to Y$ the following statements are equivalent:*

(i) *T is weakly completely continuous.*

(ii) *T is weakly compact.*

Proof. (i)⇒(ii) It follows from Theorem 4.3.

(ii)⇒(i) See Corollary 4.2. ■

Following [24, Section 9.4] we have:

Definition 4.2. A locally convex Hausdorff space (Z, ξ) is said to have the *Dieudonné property* if for every Banach space Y, every weakly completely continuous operator $T : Z \to Y$ maps ξ-bounded sets in Z onto relatively weakly compact sets in Y.

Corollary 4.6. *Assume that a Banach space X is almost reflexive and a Young function φ satisfies the ∇_2-condition. Then the space $(L^\varphi(X), \mathcal{T}_\varphi^\wedge)$ has the Dieudonné property.*

Proof. It follows from Corollary 4.5 because every $\mathcal{T}_\varphi^\wedge$-bounded set in $L^\varphi(X)$ is $\mathcal{T}_\varphi$-bounded (see Theorem 1.1). □

Definition 4.3. Assume that (Z, ξ) is a locally convex Hausdorff space. A $(\xi, \|\cdot\|_Y)$-contnuous linear operator $T : Z \to Y$ is said to be *unconditionally converging* if the series $\sum_{n=1}^\infty T(z_n)$ converges unconditionally in Y whenever $\sum_{n=1}^\infty |z^*(z_n)| < \infty$ for every $z^* \in (Z, \xi)^*$.

Proposition 4.7. *Let $T : L^\varphi(X) \to Y$ be a $(\mathcal{T}_\varphi^\wedge, \|\cdot\|_Y)$-continuous linear operator. If T is weakly completely continuous, then T is unconditionally converging.*

Proof. Assume that (f_n) is a sequence in $L^\varphi(X)$ such that $\sum_{n=1}^\infty |\int_\Omega \langle f_n(\omega), g(\omega)\rangle d\mu| < \infty$ for all $g \in L^{\varphi^*}(X^*, X)$. For a subsequence (f_{k_n}) of (f_n), let $S_n = \sum_{i=1}^n f_{k_i}$. Then (S_n) is a $\sigma(L^\varphi(X), L^{\varphi^*}(X^*, X))$-Cauchy sequence in $L^\varphi(X)$. It follows that the series $\sum_{n=1}^\infty T(f_{k_n})$ is $\sigma(Y, Y^*)$-convergent in Y and in view of the Orlicz-Pettis theorem (see [20, p. 22]), the series $\sum_{n=1}^\infty T(f_n)$ is unconditionally convergent. This means that T is unconditionally converging. □

5 Completely continuous operators on $L^\varphi(X)$

Definition 5.1. Assume that (Z, ξ) is a locally convex Hausdorff space. A $(\xi, \|\cdot\|_Y)$-continuous linear operator $T : Z \to Y$ is said to be *completely continuous* if $\|T(z_n)\|_Y \to 0$ whenever (z_n) converges weakly to 0 in Z.

Recall that a completely continuous operator between Banach spaces is usually called a *Dunford-Pettis operator*.

Theorem 5.1. *Let $T : L^\varphi(X) \to Y$ be a $(\mathcal{T}_\varphi^\wedge, \|\cdot\|_Y)$-continuous linear operator and $m : \Sigma_f(\mu) \to \mathcal{L}(X, Y)$ be its representing measure. If T completely continuous, then for every $A \in \Sigma_f(\mu)$, $m(A)$ is a Dunford-Pettis operator.*

Proof. Assume that $x_n \to 0$ in $\sigma(X, X^*)$ and $A \in \Sigma_f(\mu)$. We shall show that $\mathbb{1}_A \otimes x_n \to 0$ in $\sigma(L^\varphi(X), L^{\varphi^*}(X^*, X))$. Indeed, let $g \in L^{\varphi^*}(X^*, X)$ be given. Note that $\mathbb{1}_A(\omega) g_{x_n}(\omega) \to 0$ μ-a.e. and $|\mathbb{1}_A(\omega) g_{x_n}(\omega)| \le \mathbb{1}_A(\omega)\vartheta(g)(\omega)\|x_n\|_X \le a\mathbb{1}_A(\omega)\vartheta(g)(\omega)$ μ-a.e. for all $n \in \mathbb{N}$, where $a = \sup_n \|x_n\|_X < \infty$. Since $\vartheta(g) \in L^{\varphi^*}$, we get $\mathbb{1}_A\vartheta(g) \in L^1$. Hence by the Lebesgue dominated convergence theorem

$$\int_\Omega \langle (\mathbb{1}_A \otimes x_n)(\omega), g(\omega)\rangle\, d\mu = \int_\Omega \mathbb{1}_A(\omega) g_{x_n}(\omega)\, d\mu \to 0.$$

It folows that $\|m(A)(x_n)\|_Y = \|T(\mathbb{1}_A \otimes x_n)\|_Y \to 0$. □

Bourgain [30, Proposition 1] showed that a bounded linear operator $T : L^1 \to Y$ ($\mu(\Omega) < \infty$) is Dunford-Pettis if and only if T restricted to L^p for some $p \in (1, \infty]$ is compact. Now we extend this result to operator $T : L^\varphi(X) \to Y$. We study the relationships between completely continuous operators $T : L^\varphi(X) \to Y$ and the compactness properties of T restricted to $L^\psi(X)$, where $\varphi \prec \psi$.

Theorem 5.2. *Let $T : L^\varphi(X) \to Y$ be a $(\mathcal{T}_\varphi^\wedge, \|\cdot\|_Y)$-continuous linear operator. Assume that for every Young function ψ with $\varphi \prec \psi$, the operator $T \circ i_\psi : L^\psi(X) \to Y$ is compact. Then T is completely continuous.*

Proof. Assume that $f_n \to 0$ in $\sigma(L^\varphi(X), L^{\varphi^*}(X^*, X))$. Then the set $\{f_n : n \in \mathbb{N}\}$ is relatively $\sigma(L^\varphi(X), L^{\varphi^*}(X^*, X))$-sequentially compact in $L^\varphi(X)$, and it follows that $\{\|f_n(\cdot)\|_X : n \in \mathbb{N}\}$ is relatively $\sigma(L^\varphi, L^{\varphi^*})$-sequentially compact set in L^φ (see [21, Theorem 3.3]). Then by Theorem 3.2 there exists a Young function ψ with $\varphi \prec \psi$ such that $\sup_n \|f_n\|_\psi \le 1$. It follows that $\{T(f_n) : n \in \mathbb{N}\}$ is a relatively norm compact set in Y. Hence there exists a subsequence (f_{k_n}) of (f_n) and $y_o \in Y$ such that $\|T(f_{k_n}) - y_o\|_Y \to 0$. On the other hand, since T is $(\sigma(L^\varphi(X), L^{\varphi^*}(X^*, X)), \sigma(Y, Y^*))$-continuous, we get $T(f_n) \to 0$ in $\sigma(Y, Y^*)$. Hence $y_o = 0$ and $\|T(f_{k_n})\|_Y \to 0$. This means that $\|T(f_n)\|_Y \to 0$. □

As a consequence of Theorem 5.2, we have:

Corollary 5.3. *Assume that $T : L^\varphi(X) \to Y$ is a $(\mathcal{T}_\varphi^\wedge, \|\cdot\|_Y)$-continuous linear operator. If T is compact, then T is completely continuous.*

Theorem 5.4. *Assume that X is a reflexive Banach space and $T : L^\varphi(X) \to Y$ is a $(\mathcal{T}_\varphi^\wedge, \|\cdot\|_Y)$-continuous linear operator. If T is completely continuous, then for every Young function ψ with $\varphi \prec \psi$, the operator $T \circ i_\psi : L^\psi(X) \to Y$ is compact.*

Proof. Let ψ be a Young function such that $\varphi \prec \psi$. Then by Proposition 3.1 the set $\{\|f(\cdot)\|_X : f \in B_{L^\psi(X)}\}$ in L^φ is relatively $\sigma(L^\varphi, L^{\varphi^*})$-compact and hence it is relatively $\sigma(L^\varphi, L^{\varphi^*})$-sequentially compact (see Theorem 3.2). In view of [21, Corollary 3.4] $B_{L^\psi(X)}$ is a relatively $\sigma(L^\varphi(X), L^{\varphi^*}(X^*))$-sequentially compact set in $L^\varphi(X)$. To show that $T(B_{L^\psi(X)})$ is a relatively norm compact subset of Y, assume that (y_n) is a sequence in $T(B_{L^\psi(X)})$, i.e., $y_n = T(f_n)$, where $f_n \in B_{L^\psi(X)}$. Then there exists a subsequence (f_{k_n}) of (f_n) such that $f_{k_n} \to f_o$ in $\sigma(L^\varphi(X), L^{\varphi^*}(X^*))$ for some $f_o \in L^\varphi(X)$. Hence $\|T(f_{k_n}) - T(f_o)\|_Y \to 0$ and this means that $T(B_{L^\psi(X)})$ is relatively compact in Y. □

Corollary 5.5. *Assume that X is a reflexive Banach space and a Young function φ satisfies the ∇_2-condition. Then for a $(\mathcal{T}_\varphi^\wedge, \|\cdot\|_Y)$-continuous linear operator $T : L^\varphi(X) \to Y$ the following statements are equivalent:*

(i) *T is completely continuous.*

(ii) *T is compact.*

Proof. (i)⇒(ii) This follows from Theorem 5.4.

(ii)⇒(i) See Corollary 5.3. □

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