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Differential polynomials of L-functions with truncated shared values

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Abstract: In this paper, by using the idea of truncated counting functions, we study the uniqueness of transcendental meromorphic functions and L-functions whose certain nonlinear differential polynomials share one finite nonzero value. The result in this paper extends the theorem given by Liu, Li and Yi.

Keywords: L-function, meromorphic function, shared value, differential polynomial

MSC: 11M36, 30D35, 30D30

1 Introduction

L-functions in the Selberg class, with the Riemann zeta function as a prototype, are important objects in number theory, and value distribution of L-functions has been studied extensively. We refer the reader to the monograph for a detailed discussion on this topic and related works [1]. Throughout the paper, an L-function always means an L-function L in the Selberg class, which includes the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ and essentially those Dirichelet series where one might expect a Riemann hypothesis. Such an L-function is defined to be a Dirichelet series $L(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$ satisfying the following axioms:

(i) Ramanujan hypothesis: $a(n) \ll n^{\varepsilon}$ for every $\varepsilon > 0$.

(ii) Analytic continuation: There is a nonnegative integer m such that $(s-1)^m L(s)$ is an entire function of finite order.

(iii) Functional equation: L satisfies a functional equation of type

$$\Lambda_L(s) = \omega \overline{\Lambda_L(1-\overline{s})},$$

where

$$\Lambda_L(s) = L(s) Q^s \prod_{j=1}^{K} \Gamma(\lambda_j s + \nu_j),$$

with positive real numbers Q, λ_j, and complex numbers ν_j, ω with $Re\nu_j \geq 0$ and $|\omega| = 1$.

(iv) Euler product hypothesis: $\log L(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s}$, where $b(n) = 0$ unless n is a positive power of a prime and $b(n) \ll n^{\theta}$ for some $\theta < \frac{1}{2}$.

All of the L-functions in the paper are assumed to be the Dirichelet series from the extended Selberg class only satisfying the axioms (i)-(iii) [1, 2].

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Let f_1 be a nonconstant meromorphic function, and let k be a positive integer. We denote by $\overline{E}_{k)}(1, f_1)$ the set of zeros of $f_1 - 1$ with multiplicities at most k, where each zero is counted only once. In addition, we denote by $\lambda(f_1) = \limsup\limits_{r\to\infty} \frac{\log T(r,f_1)}{\log r}$ the order of f_1.

Definition 1.1. *Let $f_i (i = 1, 2)$ be two nonconstant meromorphic functions, and let $k \geq 2$ be a positive integer. If $\overline{E}_{k)}(1, f_1) = \overline{E}_{k)}(1, f_2)$, we say that 1 is a truncated sharing value of f_1 and f_2.*

Let f_1 be a nonconstant meromorphic function, let $k \geq 2$ be a positive integer and $a \in \mathbb{C}\bigcup\{\infty\}$. We denote by $N_{(k}\left(r, \frac{1}{f_1-a}\right)$ the counting function of those a-points of f_1 (counted with proper multiplicities) whose multiplicities are greater than k, denote by $N_{k)}\left(r, \frac{1}{f_1-a}\right)$ the counting function of those a-points of f_1 (counted with proper multiplicities) whose multiplicities are less than k. Naturally, denote by $\overline{N}_{(k}\left(r, \frac{1}{f_1-a}\right)$ and $\overline{N}_{k)}\left(r, \frac{1}{f_1-a}\right)$ the reduced forms respectively.

Definition 1.2. *Let f_1 be a nonconstant meromorphic function, and let k be a positive integer and $b \in \mathbb{C}\bigcup\{\infty\}$. We define*

$$N_k\left(r, \frac{1}{f_1-b}\right) = \overline{N}_{(1}\left(r, \frac{1}{f_1-b}\right) + \overline{N}_{(2}\left(r, \frac{1}{f_1-b}\right) + \ldots + \overline{N}_{(k}\left(r, \frac{1}{f_1-b}\right), \tag{1.1}$$

$$\Theta(b, f_1) = 1 - \limsup_{r\to\infty} \frac{\overline{N}\left(r, \frac{1}{f_1-b}\right)}{T(r, f_1)}, \quad \delta_k(b, f_1) = 1 - \limsup_{r\to\infty} \frac{N_k\left(r, \frac{1}{f_1-b}\right)}{T(r, f_1)}. \tag{1.2}$$

Remark 1. From (1.1) and (1.2), we can get that $0 \leq \delta_k(b, f_1) \leq \delta_{k-1}(b, f_1) \leq \delta_1(b, f_1) \leq \Theta(b, f_1) \leq 1$.

Definition 1.3. *Let $f_i (i = 1, 2)$ be two nonconstant meromorphic functions such that $\overline{E}_{k)}(1, f_1) = \overline{E}_{k)}(1, f_2)$, where $k \geq 2$ is a positive integer. Let z_0 be a common zero of $f_1 - 1$ and $f_2 - 1$ with multiplicity p and q respectively, we denote by $\overline{N}_L\left(r, \frac{1}{f_1-1}\right)$ the reduced counting function of those zeros of $f_1 - 1$ where $p > q$, by $\overline{N}^E_{(2}\left(r, \frac{1}{f_1-1}\right)$ the reduced counting function of those zeros $f_1 - 1$ where $p = q \geq 2$. Especially, we denote by $\overline{N}^1_L\left(r, \frac{1}{f_1-1}\right)$ the reduced counting function of those zeros of $f_1 - 1$ where $p > q = 1$, denote by $N^E_{1)}\left(r, \frac{1}{f_1-1}\right)$ the reduced counting function of those zeros of $f_1 - 1$ where $p = q = 1$. In the same way, we can define $\overline{N}_L\left(r, \frac{1}{f_2-1}\right)$, $\overline{N}^E_{(2}\left(r, \frac{1}{f_2-1}\right)$, $\overline{N}^1_L\left(r, \frac{1}{f_2-1}\right)$, $N^E_{1)}\left(r, \frac{1}{f_2-1}\right)$.*

Definition 1.4. *Let $f_i (i = 1, 2)$ be two nonconstant meromorphic functions such that $\overline{E}_{k)}(1, f_1) = \overline{E}_{k)}(1, f_2)$, where $k \geq 2$ is a positive integer. We denote by $\overline{N}_{(k+1}(r, 1; f_1|f_2 \neq 1)$ the reduced counting function of those zeros of $f_1 - 1$ where multiplicities are greater than $k + 1$, but not the zeros of $f_2 - 1$. In the same way, we can define $\overline{N}_{(k+1}(r, 1; f_2|f_1 \neq 1)$.*

Definition 1.5. *Let $f_i (i = 1, 2)$ be two nonconstant meromorphic functions. We denote by $N_0\left(r, \frac{1}{f'_1}\right)$ the counting function of those zeros of f'_1 which are not the zeros of f_1 and $f_1 - 1$, by $\overline{N}_0\left(r, \frac{1}{f'_1}\right)$ the corresponding reduced counting function. In the same way, we can define $N_0\left(r, \frac{1}{f'_2}\right)$ and $\overline{N}_0\left(r, \frac{1}{f'_2}\right)$.*

In 2008, Chen, Zhang, Lin and Chen [3] proved the following result.

Theorem 1.1. (see [3]) *Let f and g be two nonconstant entire functions, and let n, k be two positive integers with $n > 5k + 13$. If $(f^n(f-1))^{(k)}$ and $(g^n(g-1))^{(k)}$ share 1 IM, then $f \equiv g$.*

In 2011, by using the idea of truncated counting functions, Lin and Lin [4] proved the following result.

Theorem 1.2. (see [4]) *Let f and g be two nonconstant entire functions, and let n, k and $m \geq 2$ be three positive integers with $n > k + (4k + 7)(1 - \Theta(0)) + 4(1 - \delta(1))$. If $\overline{E}_{m)}(1, (f^n(f-1))^{(k)}) = \overline{E}_{m)}(1, (g^n(g-1))^{(k)})$, then $f \equiv g$.*

In 2014, by using Zalcman's lemma, Li and Yi [5] considered the case of meromorphic functions and proved the following theorem.

Theorem 1.3. (see [5]) *Let f and g be two nonconstant meromorphic functions such that $\lambda(f) > 2$, and let n, k be two positive integers with $n > 9k + 18$. If $(f^n(f-1))^{(k)}$ and $(g^n(g-1))^{(k)}$ share 1 IM, and $\Theta(\infty, f) > 2/n$, then $f \equiv g$.*

In 2017, by using the idea of truncated counting functions of meromorphic functions, Chen and Cai [6] proved the following result.

Theorem 1.4. (see [6]) *Let f and g be two nonconstant meromorphic functions such that $\lambda(f) > 2$, and let n, k and $m \geq 2$ be three positive integers with $n > 9k + 18$. If $\overline{E}_{m)}(1, (f^n(f-1))^{(k)}) = \overline{E}_{m)}(1, (g^n(g-1))^{(k)})$, and $\Theta(\infty, f) > 2/n$, then $f \equiv g$.*

Recently, Liu, Li and Yi [7] further considered the case of L-functions and proved the following theorem.

Theorem 1.5. (see [7]) *Let f be a transcendental meromorphic function, L be an L-function, n, k be two positive integers with $n > 7k + 17$ and $k \geq 2$. If $(f^n(f-1))^{(k)}$ and $(L^n(L-1))^{(k)}$ share 1 IM, then $f \equiv L$.*

It's natural to ask whether Theorem 1.5 can be extended in the same way that Theorem 1.2 extends Theorem 1.1 or Theorem 1.4 extends Theorem 1.3. In this direction, we prove the following result.

Theorem 1.6. *Let f be a transcendental meromorphic function, L be an L-function, n, k, $l \geq 2$ be three positive integers with $n > 7k + 17$ and $k \geq 2$. If $\overline{E}_{l)}(1, (f^n(f-1))^{(k)}) = \overline{E}_{l)}(1, (L^n(L-1))^{(k)})$, then $f \equiv L$.*

2 Some lemmas

Lemma 2.1. (see [8]) *Let f_1 be a nonconstant meromorphic function, and let $\varphi(\neq 0, \infty)$ be a small function of f_1. Then*

$$T(r, f_1) \leq \overline{N}(r, f_1) + N\left(r, \frac{1}{f_1}\right) + N\left(r, \frac{1}{f_1^{(k)} - \varphi}\right) - N\left(r, \frac{1}{(\frac{f_1^{(k)}}{\varphi})'}\right) + S(r, f_1).$$

Lemma 2.2. (see [9]) *Let f_1 be a nonconstant meromorphic function, $k(\geq 1)$, $p(\geq 1)$ be two positive integers. Then*

$$N_p\left(r, \frac{1}{f_1^{(k)}}\right) \leq N_{p+k}\left(r, \frac{1}{f_1}\right) + k\overline{N}(r, f_1) + S(r, f_1).$$

Lemma 2.3. (see [4]) *Let $f_i (i = 1, 2)$ be two nonconstant meromorphic functions, $l(\geq 2)$ be positive integer. If f_1 and f_2 satisfy $\overline{E}_{l)}(1, f_1) = \overline{E}_{l)}(1, f_2)$, then*

$$l\overline{N}_{(l+1}(r, 1; f_1 | f_2 \neq 1) + \overline{N}_L\left(r, \frac{1}{f_1 - 1}\right) + \overline{N}_0\left(r, \frac{1}{f_1'}\right) \leq \overline{N}\left(r, \frac{1}{f_1}\right) + \overline{N}(r, f_1) + S(r, f_1).$$

Lemma 2.4. *Let $f_i (i = 1, 2)$ be two transcendental meromorphic functions such that $\overline{E}_{l)}(0, f_1^{(k)} - H) = \overline{E}_{l)}(0, f_2^{(k)} - H)$, where $k(\geq 1)$, $l(\geq 2)$ are two positive integers, H is a nonzero polynomial. If*

$$\iota_1 = (2k+4)\Theta(\infty, f_1) + (2k+3)\Theta(\infty, f_2) + 3\delta_{k+1}(0, f_1) + 2\delta_{k+1}(0, f_2) + \Theta(0, f_1) + \Theta(0, f_2) > 4k + 13, \quad (2.1)$$

$$\iota_2 = (2k+4)\Theta(\infty, f_2) + (2k+3)\Theta(\infty, f_1) + 3\delta_{k+1}(0, f_2) + 2\delta_{k+1}(0, f_1) + \Theta(0, f_2) + \Theta(0, f_1) > 4k + 13, \quad (2.2)$$

then $f_1^{(k)} f_2^{(k)} \equiv H^2$ or $f_1 \equiv f_2$.

Proof. *Let*

$$\widetilde{f_1} = \frac{f_1^{(k)}}{H}, \quad \widetilde{f_2} = \frac{f_2^{(k)}}{H}, \tag{2.3}$$

and

$$F = \frac{\widetilde{f_1}''}{\widetilde{f_1}'} - \frac{2\widetilde{f_1}'}{\widetilde{f_1} - 1} - \left(\frac{\widetilde{f_2}''}{\widetilde{f_2}'} - \frac{2\widetilde{f_2}'}{\widetilde{f_2} - 1} \right). \tag{2.4}$$

First of all, $f_1, f_1^{(k)}$ are two transcendental meromorphic functions; therefore, $T(r, H) = o\{T(r, f_1)\}$. By the lemma of logarithmic derivative and Nevanlinna first fundamental theorem, we have

$$\begin{aligned} T(r, \widetilde{f_1}) &= m\left(r, \frac{f_1^{(k)}}{H}\right) + N\left(r, \frac{f_1^{(k)}}{H}\right) \\ &\le m\left(r, \frac{f_1^{(k)}}{f_1}\right) + m\left(r, \frac{f_1}{H}\right) + N\left(r, \frac{f_1^{(k)}}{H}\right) \\ &\le m(r, f_1) + N(r, f_1^{(k)}) + S(r, f_1) \\ &\le (k+2)T(r, f_1) + S(r, f_1). \end{aligned}$$

Therefore, $S(r, \widetilde{f_1}) \le S(r, f_1)$. In the same way, $S(r, \widetilde{f_2}) \le S(r, f_2)$.

Next, we claim $F \equiv 0$. Suppose that $F \not\equiv 0$. Let $z_0 \notin \{z : H(z) = 0\}$ be a common simple zero of $f_1^{(k)} - H$ and $f_2^{(k)} - H$. From (2.3) we know that z_0 is a common simple zero of $\widetilde{f_1} - 1$ and $\widetilde{f_2} - 1$, which by calculating yields $F(z_0) = 0$. Thus, we have

$$N_{1)}^{E}\left(r, \frac{1}{\widetilde{f_1} - 1}\right) \le N\left(r, \frac{1}{F}\right) \le T(r, F) + O(1) \le N(r, F) + S(r, f_1) + S(r, f_2). \tag{2.5}$$

Let $z_1 \notin \{z : H(z) = 0\}$ be a simple pole of $\widetilde{f_1}$. Then by calculating we get that $\frac{\widetilde{f_1}''}{\widetilde{f_1}'} - \frac{2\widetilde{f_1}'}{\widetilde{f_1}-1}$ is analytic at z_1. Similarly, let $z_2 \notin \{z : H(z) = 0\}$ be a simple pole of $\widetilde{f_2}$. Then by calculating we get that $\frac{\widetilde{f_2}''}{\widetilde{f_2}'} - \frac{2\widetilde{f_2}'}{\widetilde{f_2}-1}$ is analytic at z_2. In addition, each common zero of $\widetilde{f_1} - 1$ and $\widetilde{f_2} - 1$ with the same multiplicities is not the pole of F. From (2.4), we can get

$$\begin{aligned} N(r, F) \le\, & \overline{N}_{(2}(r, \widetilde{f_1}) + \overline{N}_{(2}(r, \widetilde{f_2}) + \overline{N}\left(r, \frac{1}{f_1}\right) + \overline{N}\left(r, \frac{1}{f_2}\right) + \overline{N}_L\left(r, \frac{1}{\widetilde{f_1} - 1}\right) \\ & + \overline{N}_L\left(r, \frac{1}{\widetilde{f_2} - 1}\right) + \overline{N}_{(l+1}(r, 1; \widetilde{f_1} | \widetilde{f_2} \ne 1) + \overline{N}_{(l+1}(r, 1; \widetilde{f_2} | \widetilde{f_1} \ne 1) \\ & + \overline{N}_0\left(r, \frac{1}{\widetilde{f_1}'}\right) + \overline{N}_0\left(r, \frac{1}{\widetilde{f_2}'}\right) + O(\log r) \\ \le\, & \overline{N}(r, f_1) + \overline{N}(r, f_2) + \overline{N}\left(r, \frac{1}{f_1}\right) + \overline{N}\left(r, \frac{1}{f_2}\right) + \overline{N}_L\left(r, \frac{1}{\widetilde{f_1} - 1}\right) \\ & + \overline{N}_L\left(r, \frac{1}{\widetilde{f_2} - 1}\right) + \overline{N}_{(l+1}(r, 1; \widetilde{f_1} | \widetilde{f_2} \ne 1) + \overline{N}_{(l+1}(r, 1; \widetilde{f_2} | \widetilde{f_1} \ne 1) \\ & + \overline{N}_0\left(r, \frac{1}{\widetilde{f_1}'}\right) + \overline{N}_0\left(r, \frac{1}{\widetilde{f_2}'}\right) + O(\log r). \end{aligned} \tag{2.6}$$

We note that

$$\overline{N}\left(r,\frac{1}{\widetilde{f}_1-1}\right)=N_{1)}^{E}\left(r,\frac{1}{\widetilde{f}_1-1}\right)+\overline{N}_{(2}^{E}\left(r,\frac{1}{\widetilde{f}_1-1}\right)+\overline{N}_{L}\left(r,\frac{1}{\widetilde{f}_1-1}\right)+\overline{N}_{L}\left(r,\frac{1}{\widetilde{f}_2-1}\right)+\overline{N}_{(l+1}(r,1;\widetilde{f}_1|\widetilde{f}_2\neq 1),$$

$$\overline{N}_{(2}^{E}\left(r,\frac{1}{\widetilde{f}_2-1}\right)+\overline{N}_{L}\left(r,\frac{1}{\widetilde{f}_2-1}\right)+l\overline{N}_{(l+1}(r,1;\widetilde{f}_2|\widetilde{f}_1\neq 1)+\overline{N}_{L}\left(r,\frac{1}{\widetilde{f}_1-1}\right)$$
$$-\overline{N}_{L}^{1}\left(r,\frac{1}{\widetilde{f}_1-1}\right)\leq N\left(r,\frac{1}{\widetilde{f}_2-1}\right)-\overline{N}\left(r,\frac{1}{\widetilde{f}_2-1}\right).$$

It follows from the above two inequalities that

$$\overline{N}\left(r,\frac{1}{\widetilde{f}_1-1}\right)+\overline{N}\left(r,\frac{1}{\widetilde{f}_2-1}\right)\leq N_{1)}^{E}\left(r,\frac{1}{\widetilde{f}_1-1}\right)+\overline{N}_{(l+1}(r,1;\widetilde{f}_1|\widetilde{f}_2\neq 1)$$
$$+\overline{N}_{L}^{1}\left(r,\frac{1}{\widetilde{f}_1-1}\right)-l\overline{N}_{(l+1}(r,1;\widetilde{f}_2|\widetilde{f}_1\neq 1)+N\left(r,\frac{1}{\widetilde{f}_2-1}\right).$$

We deduce by the Nevanlinna first fundamental theorem that

$$\overline{N}\left(r,\frac{1}{\widetilde{f}_1-1}\right)+\overline{N}\left(r,\frac{1}{\widetilde{f}_2-1}\right)\leq N_{1)}^{E}\left(r,\frac{1}{\widetilde{f}_1-1}\right)+\overline{N}_{(l+1}(r,1;\widetilde{f}_1|\widetilde{f}_2\neq 1)$$
$$+\overline{N}_{L}^{1}\left(r,\frac{1}{\widetilde{f}_1-1}\right)-l\overline{N}_{(l+1}(r,1;\widetilde{f}_2|\widetilde{f}_1\neq 1)+T(r,\widetilde{f}_2)+O(1).$$

From this, (2.3), and (2.5), we obtain

$$\overline{N}\left(r,\frac{1}{\widetilde{f}_1-1}\right)+\overline{N}\left(r,\frac{1}{\widetilde{f}_2-1}\right)\leq N\left(r,\frac{1}{F}\right)+\overline{N}_{(l+1}(r,1;\widetilde{f}_1|\widetilde{f}_2\neq 1)+\overline{N}_{L}^{1}\left(r,\frac{1}{\widetilde{f}_1-1}\right)$$
$$-l\overline{N}_{(l+1}(r,1;\widetilde{f}_2|\widetilde{f}_1\neq 1)+T(r,f_2^{(k)})+O(\log r)$$
$$\leq T(r,F)+\overline{N}_{(l+1}(r,1;\widetilde{f}_1|\widetilde{f}_2\neq 1)+\overline{N}_{L}^{1}\left(r,\frac{1}{\widetilde{f}_1-1}\right)$$
$$-l\overline{N}_{(l+1}(r,1;\widetilde{f}_2|\widetilde{f}_1\neq 1)+T(r,f_2)+k\overline{N}(r,f_2)+S(r,f_2)+O(\log r).$$

It follows from the lemma of logarithmic derivative and (2.6) that

$$\overline{N}\left(r,\frac{1}{\widetilde{f}_1-1}\right)+\overline{N}\left(r,\frac{1}{\widetilde{f}_2-1}\right)\leq N(r,F)+\overline{N}_{(l+1}(r,1;\widetilde{f}_1|\widetilde{f}_2\neq 1)+\overline{N}_{L}^{1}\left(r,\frac{1}{\widetilde{f}_1-1}\right)$$
$$-l\overline{N}_{(l+1}(r,1;\widetilde{f}_2|\widetilde{f}_1\neq 1)+T(r,f_2)+k\overline{N}(r,f_2)+S(r,f_1)+S(r,f_2)$$
$$\leq\overline{N}(r,f_1)+\overline{N}(r,f_2)+\overline{N}\left(r,\frac{1}{f_1}\right)+\overline{N}\left(r,\frac{1}{f_2}\right)+\overline{N}_{L}\left(r,\frac{1}{\widetilde{f}_1-1}\right)$$
$$+\overline{N}_{L}\left(r,\frac{1}{\widetilde{f}_2-1}\right)+\overline{N}_{(l+1}(r,1;\widetilde{f}_1|\widetilde{f}_2\neq 1)+\overline{N}_{(l+1}(r,1;\widetilde{f}_2|\widetilde{f}_1\neq 1)$$
$$+\overline{N}_{0}\left(r,\frac{1}{\widetilde{f}_1'}\right)+\overline{N}_{0}\left(r,\frac{1}{\widetilde{f}_2'}\right)+\overline{N}_{(l+1}(r,1;\widetilde{f}_1|\widetilde{f}_2\neq 1)$$

$$
\begin{aligned}
&+\overline{N}_L^1\left(r,\frac{1}{\widetilde{f}_1-1}\right)-l\overline{N}_{(l+1}(r,1;\widetilde{f}_2|\widetilde{f}_1\neq 1)+T(r,f_2)\\
&+k\overline{N}(r,f_2)+S(r,f_1)+S(r,f_2)\\
&\le\overline{N}(r,f_1)+(k+1)\overline{N}(r,f_2)+\overline{N}\left(r,\frac{1}{f_1}\right)+\overline{N}\left(r,\frac{1}{f_2}\right)+\overline{N}_L\left(r,\frac{1}{\widetilde{f}_1-1}\right)\\
&+\overline{N}_L\left(r,\frac{1}{\widetilde{f}_2-1}\right)+2\overline{N}_{(l+1}(r,1;\widetilde{f}_1|\widetilde{f}_2\neq 1)+\overline{N}_{(l+1}(r,1;\widetilde{f}_2|\widetilde{f}_1\neq 1)\\
&+\overline{N}_L^1\left(r,\frac{1}{\widetilde{f}_1-1}\right)+N_0\left(r,\frac{1}{\widetilde{f}_1'}\right)+N_0\left(r,\frac{1}{\widetilde{f}_2'}\right)\\
&+T(r,f_2)+S(r,f_1)+S(r,f_2).
\end{aligned}
\tag{2.7}
$$

Applying Lemma 2.1, we have

$$
\begin{aligned}
T(r,f_1)&\le\overline{N}(r,f_1)+N\left(r,\frac{1}{f_1}\right)+N\left(r,\frac{1}{\widetilde{f}_1-1}\right)-N\left(r,\frac{1}{\widetilde{f}_1'}\right)+S(r,f_1)\\
&\le\overline{N}(r,f_1)+N_{k+1}\left(r,\frac{1}{f_1}\right)+\overline{N}\left(r,\frac{1}{\widetilde{f}_1-1}\right)-N_0\left(r,\frac{1}{\widetilde{f}_1'}\right)+O(\log r)+S(r,f_1)\\
&\le\overline{N}(r,f_1)+N_{k+1}\left(r,\frac{1}{f_1}\right)+\overline{N}\left(r,\frac{1}{\widetilde{f}_1-1}\right)-N_0\left(r,\frac{1}{\widetilde{f}_1'}\right)+S(r,f_1).
\end{aligned}
\tag{2.8}
$$

In the same way, we have

$$
T(r,f_2)\le\overline{N}(r,f_2)+N_{k+1}\left(r,\frac{1}{f_2}\right)+\overline{N}\left(r,\frac{1}{\widetilde{f}_2-1}\right)-N_0\left(r,\frac{1}{\widetilde{f}_2'}\right)+S(r,f_2).
\tag{2.9}
$$

By Lemma 2.3, we get

$$
\overline{N}_{(l+1}(r,1;\widetilde{f}_1|\widetilde{f}_2\neq 1)+\overline{N}_L\left(r,\frac{1}{\widetilde{f}_1-1}\right)\le\overline{N}\left(r,\frac{1}{\widetilde{f}_1}\right)+\overline{N}(r,\widetilde{f}_1)+S(r,\widetilde{f}_1).
\tag{2.10}
$$

Similarly,

$$
\overline{N}_{(l+1}(r,1;\widetilde{f}_2|\widetilde{f}_1\neq 1)+\overline{N}_L\left(r,\frac{1}{\widetilde{f}_2-1}\right)\le\overline{N}\left(r,\frac{1}{\widetilde{f}_2}\right)+\overline{N}(r,\widetilde{f}_2)+S(r,\widetilde{f}_2).
\tag{2.11}
$$

Combining (2.8) with (2.9) we have

$$
\begin{aligned}
T(r,f_1)+T(r,f_2)\le&\overline{N}(r,f_1)+\overline{N}(r,f_2)+N_{k+1}\left(r,\frac{1}{f_1}\right)+N_{k+1}\left(r,\frac{1}{f_2}\right)+\overline{N}\left(r,\frac{1}{\widetilde{f}_1-1}\right)\\
&+\overline{N}\left(r,\frac{1}{\widetilde{f}_2-1}\right)-N_0\left(r,\frac{1}{\widetilde{f}_1'}\right)-N_0\left(r,\frac{1}{\widetilde{f}_2'}\right)+S(r,f_1)+S(r,f_2),
\end{aligned}
$$

which together with (2.7) yields

$$
\begin{aligned}
T(r,f_1)\le&2\overline{N}(r,f_1)+(k+2)\overline{N}(r,f_2)+N_{k+1}\left(r,\frac{1}{f_1}\right)+N_{k+1}\left(r,\frac{1}{f_2}\right)+\overline{N}\left(r,\frac{1}{f_1}\right)\\
&+\overline{N}\left(r,\frac{1}{f_2}\right)+\overline{N}_L\left(r,\frac{1}{\widetilde{f}_1-1}\right)+\overline{N}_L\left(r,\frac{1}{\widetilde{f}_2-1}\right)+2\overline{N}_{(l+1}(r,1;\widetilde{f}_1|\widetilde{f}_2\neq 1)\\
&+\overline{N}_{(l+1}(r,1;\widetilde{f}_2|\widetilde{f}_1\neq 1)+\overline{N}_L^1\left(r,\frac{1}{\widetilde{f}_1-1}\right)+S(r,f_1)+S(r,f_2)\\
\le&2\overline{N}(r,f_1)+(k+2)\overline{N}(r,f_2)+N_{k+1}\left(r,\frac{1}{f_1}\right)+N_{k+1}\left(r,\frac{1}{f_2}\right)
\end{aligned}
$$

$$+\overline{N}\left(r,\frac{1}{f_1}\right)+\overline{N}\left(r,\frac{1}{f_2}\right)+2\left[\overline{N}_L\left(r,\frac{1}{\widetilde{f}_1-1}\right)+\overline{N}_{(l+1}(r,1;\widetilde{f}_1|\widetilde{f}_2\neq 1)\right]$$
$$+\left[\overline{N}_L\left(r,\frac{1}{\widetilde{f}_2-1}\right)+\overline{N}_{(l+1}(r,1;\widetilde{f}_2|\widetilde{f}_1\neq 1)\right]+S(r,f_1)+S(r,f_2).$$

By (2.10) and (2.11), we get

$$\begin{aligned}T(r,f_1)\le&2\overline{N}(r,f_1)+(k+2)\overline{N}(r,f_2)+N_{k+1}\left(r,\frac{1}{f_1}\right)+N_{k+1}\left(r,\frac{1}{f_2}\right)\\&+\overline{N}\left(r,\frac{1}{f_1}\right)+\overline{N}\left(r,\frac{1}{f_2}\right)+2\left[\overline{N}(r,\frac{1}{\widetilde{f}_1})+\overline{N}(r,\widetilde{f}_1)\right]\\&+\left[\overline{N}\left(r,\frac{1}{\widetilde{f}_2}\right)+\overline{N}(r,\widetilde{f}_2)\right]+S(r,f_1)+S(r,f_2).\end{aligned}$$

Applying Lemma 2.2, we can get

$$\begin{aligned}T(r,f_1)\le&2\overline{N}(r,f_1)+(k+2)\overline{N}(r,f_2)+N_{k+1}\left(r,\frac{1}{f_1}\right)+N_{k+1}\left(r,\frac{1}{f_2}\right)\\&+\overline{N}\left(r,\frac{1}{f_1}\right)+\overline{N}\left(r,\frac{1}{f_2}\right)+2\left[N_{k+1}\left(r,\frac{1}{f_1}\right)+k\overline{N}(r,f_1)+\overline{N}(r,f_1)\right]\\&+\left[N_{k+1}\left(r,\frac{1}{f_2}\right)+k\overline{N}(r,f_2)+\overline{N}(r,f_2)\right]+S(r,f_1)+S(r,f_2)\\=&(2k+4)\overline{N}(r,f_1)+(2k+3)\overline{N}(r,f_2)+3N_{k+1}\left(r,\frac{1}{f_1}\right)+2N_{k+1}\left(r,\frac{1}{f_2}\right)\\&+\overline{N}\left(r,\frac{1}{f_1}\right)+\overline{N}\left(r,\frac{1}{f_2}\right)+S(r,f_1)+S(r,f_2).\end{aligned}\tag{2.12}$$

In the same way, we have

$$\begin{aligned}T(r,f_2)\le&(2k+4)\overline{N}(r,f_2)+(2k+3)\overline{N}(r,f_1)+3N_{k+1}\left(r,\frac{1}{f_2}\right)+2N_{k+1}\left(r,\frac{1}{f_1}\right)\\&+\overline{N}\left(r,\frac{1}{f_2}\right)+\overline{N}\left(r,\frac{1}{f_1}\right)+S(r,f_1)+S(r,f_2).\end{aligned}\tag{2.13}$$

Suppose that there exists some subset $I\subseteq\mathbf{R}^+$, $mesI=\infty$ such that

$$T(r,f_2)\le T(r,f_1),\tag{2.14}$$

as $r\in I$ and $r\to\infty$. Then by (2.12) and (2.14) we have

$$\iota_1=(2k+4)\Theta(\infty,f_1)+(2k+3)\Theta(\infty,f_2)+3\delta_{k+1}(0,f_1)+2\delta_{k+1}(0,f_2)+\Theta(0,f_1)+\Theta(0,f_2)\le 4k+13,$$

which contradicts the condition $\iota_1>4k+13$.

Similarly, if there exists some subset $I\subseteq\mathbf{R}^+$, $mesI=\infty$ such that

$$T(r,f_1)\le T(r,f_2),\tag{2.15}$$

as $r\in I$ and $r\to\infty$, then by (2.13) and (2.15) we have

$$\iota_2=(2k+4)\Theta(\infty,f_2)+(2k+3)\Theta(\infty,f_1)+3\delta_{k+1}(0,f_2)+2\delta_{k+1}(0,f_1)+\Theta(0,f_2)+\Theta(0,f_1)\le 4k+13,$$

which contradicts the condition $\iota_2>4k+13$.

Therefore, we get $F \equiv 0$ and so by (2.4) it follows that

$$\frac{\widetilde{f_1}''}{\widetilde{f_1}'} - \frac{2\widetilde{f_1}'}{\widetilde{f_1} - 1} \equiv \frac{\widetilde{f_2}''}{\widetilde{f_2}'} - \frac{2\widetilde{f_2}'}{\widetilde{f_2} - 1}.$$

Integrating this equation, we obtain

$$\frac{1}{\widetilde{f_1} - 1} \equiv \frac{\tau_2 \widetilde{f_2} + \tau_1 - \tau_2}{\widetilde{f_2} - 1}, \tag{2.16}$$

where τ_1, τ_2 are two constants which are not equal to zero at the same time.

We discuss the following three cases.

Case 1. If $\tau_2 = 0$, by calculating, we can get from (2.3) and (2.16)

$$f_2 = \tau_1 f_1 + (1 - \tau_1) H_1, \tag{2.17}$$

where H_1 is a polynomial of degree $deg(H_1) \leq deg(H) + k$. From (2.17) we get $T(r, f_1) = T(r, f_2) + O(\log r)$. If $\tau_1 \neq 1$, combining the Nevanlinna's three small functions theorem with (2.17), we have

$$\begin{aligned} T(r, f_2) &\leq \overline{N}(r, f_2) + \overline{N}\left(r, \frac{1}{f_2}\right) + \overline{N}\left(r, \frac{1}{f_2 - (1 - \tau_1)H_1}\right) + S(r, f_2) \\ &= \overline{N}(r, f_2) + \overline{N}\left(r, \frac{1}{f_2}\right) + \overline{N}\left(r, \frac{1}{f_1}\right) + S(r, f_2). \end{aligned} \tag{2.18}$$

Therefore, we can obtain

$$\Theta(\infty, f_2) + \Theta(0, f_2) + \Theta(0, f_1) \leq 2. \tag{2.19}$$

By (2.1) we have

$$\iota_1 = (2k+4)\Theta(\infty, f_1) + (2k+3)\Theta(\infty, f_2) + 3\delta_{k+1}(0, f_1) + 2\delta_{k+1}(0, f_2) + \Theta(0, f_1) + \Theta(0, f_2) \leq 4k + 13,$$

which contradicts the condition $\iota_1 > 4k + 13$. Hence, $\tau_1 = 1$. Substituting this into (2.17) we get $f_1 \equiv f_2$. The second conclusion of Lemma 2.4 holds.

Case 2. If $\tau_2 \neq 0$ and $\tau_1 = \tau_2$. We consider the following two subcases.
Subcase 2.1. If $\tau_1 = \tau_2 = -1$, from (2.3) and (2.16) we get $f_1^{(k)} f_2^{(k)} \equiv 1$, the first conclusion of Lemma 2.4 holds.
Subcase 2.2. If $\tau_1 = \tau_2 \neq -1$, then, by (2.16) we have

$$\widetilde{f_1} \equiv \frac{(1 + \tau_2)\widetilde{f_2} - 1}{\tau_2 \widetilde{f_2}}, \quad \widetilde{f_2} \equiv \frac{-1}{\tau_2} \frac{1}{\widetilde{f_1} - 1 - \frac{1}{\tau_2}}. \tag{2.20}$$

From the right equality of (2.20) and (2.3) we get

$$\overline{N}\left(r, \frac{1}{\widetilde{f_1} - 1 - \frac{1}{\tau_2}}\right) = \overline{N}(r, f_2) + O(\log r).$$

From Lemma 2.1 and the above equality, we obtain

$$\begin{aligned} T(r, f_1) &\leq \overline{N}(r, f_1) + N\left(r, \frac{1}{f_1}\right) + N\left(r, \frac{1}{\widetilde{f_1} - 1 - \frac{1}{\tau_2}}\right) - N\left(r, \frac{1}{\widetilde{f_1}'}\right) + S(r, f_1) \\ &\leq \overline{N}(r, f_1) + N_{k+1}\left(r, \frac{1}{f_1}\right) + \overline{N}\left(r, \frac{1}{\widetilde{f_1} - 1 - \frac{1}{\tau_2}}\right) - N_0\left(r, \frac{1}{\widetilde{f_1}'}\right) + O(\log r) + S(r, f_1) \\ &\leq \overline{N}(r, f_1) + N_{k+1}\left(r, \frac{1}{f_1}\right) + \overline{N}(r, f_2) + S(r, f_1). \end{aligned} \tag{2.21}$$

On the other hand, from the left equality of (2.20) and (2.3) we get

$$\overline{N}\left(r,\frac{1}{\widetilde{f_2}-\frac{1}{\tau_2+1}}\right)=\overline{N}\left(r,\frac{1}{f_1^{(k)}}\right)+O(\log r).$$

From Lemma 2.1, Lemma 2.2 and the above equality, we have

$$\begin{aligned}T(r,f_2)&\le\overline{N}(r,f_2)+N\left(r,\frac{1}{f_2}\right)+N\left(r,\frac{1}{\widetilde{f_2}-\frac{1}{\tau_2+1}}\right)-N\left(r,\frac{1}{\widetilde{f_2}'}\right)+S(r,f_2)\\&\le\overline{N}(r,f_2)+N_{k+1}\left(r,\frac{1}{f_2}\right)+\overline{N}\left(r,\frac{1}{\widetilde{f_2}-\frac{1}{\tau_2+1}}\right)-N_0\left(r,\frac{1}{\widetilde{f_2}'}\right)+O(\log r)+S(r,f_2)\\&\le\overline{N}(r,f_2)+N_{k+1}\left(r,\frac{1}{f_2}\right)+\overline{N}\left(r,\frac{1}{f_1^{(k)}}\right)+S(r,f_2)\\&\le\overline{N}(r,f_2)+N_{k+1}\left(r,\frac{1}{f_2}\right)+N_{k+1}\left(r,\frac{1}{f_1}\right)+k\overline{N}(r,f_1)+S(r,f_1)+S(r,f_2).\end{aligned}\tag{2.22}$$

If (2.15) holds, from (2.22) we get

$$\Theta(\infty,f_2)+\delta_{k+1}(0,f_2)+\delta_{k+1}(0,f_1)+k\Theta(\infty,f_1)\le k+2.\tag{2.23}$$

Substituting this into (2.2) we obtain

$$\iota_2=(2k+4)\Theta(\infty,f_2)+(2k+3)\Theta(\infty,f_1)+3\delta_{k+1}(0,f_2)+2\delta_{k+1}(0,f_1)+\Theta(0,f_2)+\Theta(0,f_1)\le 4k+13,$$

which contradicts the condition $\iota_2>4k+13$.

If (2.14) holds, from (2.21) we get

$$\Theta(\infty,f_1)+\delta_{k+1}(0,f_1)+\Theta(\infty,f_2)\le 2.\tag{2.24}$$

Substituting this into (2.1) we obtain

$$\iota_1=(2k+4)\Theta(\infty,f_1)+(2k+3)\Theta(\infty,f_2)+3\delta_{k+1}(0,f_1)+2\delta_{k+1}(0,f_2)+\Theta(0,f_1)+\Theta(0,f_2)\le 4k+13,$$

which contradicts the condition $\iota_1>4k+13$.

Case 3. If $\tau_2\ne 0$ *and* $\tau_1\ne\tau_2$. *We consider the following two subcases.*

Subcase 3.1. If $\tau_2=-1$, *then* $\tau_1\ne 0$. *Otherwise, from (2.16) we have* $\frac{1}{\widetilde{f_1}-1}=\frac{-\widetilde{f_2}+1}{\widetilde{f_2}-1}$. *Thus* $\widetilde{f_1}\equiv 0$, *which contradicts that* f_1 *is a transcendental meromorphic function.*

From (2.16) we have

$$\widetilde{f_1}\equiv\frac{\tau_1}{-\widetilde{f_2}+\tau_1+1},\tag{2.25}$$

$$\widetilde{f_2}\equiv\frac{(\tau_1+1)\widetilde{f_1}-\tau_1}{\widetilde{f_1}}.\tag{2.26}$$

By (2.3) and (2.25) we obtain

$$\overline{N}\left(r,\frac{\tau_1}{-\widetilde{f_2}+\tau_1+1}\right)=\overline{N}(r,f_1)+O(\log r).$$

From Lemma 2.1, we have

$$\begin{aligned}T(r,f_2)&\le\overline{N}(r,f_2)+N\left(r,\frac{1}{f_2}\right)+N\left(r,\frac{1}{\widetilde{f_2}-(\tau_1+1)}\right)-N\left(r,\frac{1}{\widetilde{f_2}'}\right)+S(r,f_2)\\&\le\overline{N}(r,f_2)+N_{k+1}\left(r,\frac{1}{f_2}\right)+\overline{N}\left(r,\frac{1}{\widetilde{f_2}-(\tau_1+1)}\right)-N_0\left(r,\frac{1}{\widetilde{f_2}'}\right)+O(\log r)+S(r,f_2)\end{aligned}$$

$$\leq \overline{N}(r,f_2)+N_{k+1}\left(r,\frac{1}{f_2}\right)+\overline{N}(r,f_1)+S(r,f_2).$$

If (2.15) holds, using the same argument as in Case 2, we have $\iota_2 \leq 4k+13$, *a contradiction.*

On the other hand, from (2.3) and (2.26) we have

$$\overline{N}\left(r,\frac{1}{\widetilde{f}_1-\tau_1(\tau_1+1)^{-1}}\right)=\overline{N}\left(r,\frac{1}{\widetilde{\widetilde{f}}_2}\right)=\overline{N}\left(r,\frac{1}{f_2^{(k)}}\right)+O(\log r).$$

Similarly, from Lemma 2.1, Lemma 2.2, we have

$$T(r,f_1)\leq \overline{N}(r,f_1)+N_{k+1}\left(r,\frac{1}{f_1}\right)+N_{k+1}\left(r,\frac{1}{f_2}\right)+k\overline{N}(r,f_2)+S(r,f_1)+S(r,f_2).$$

If (2.14) holds, using the same argument as in Case 2, we have $\iota_1 \leq 4k+13$, *a contradiction.*

Subcase 3.2. If $\tau_2 \neq -1$, *then (2.16) can be rewritten as*

$$\widetilde{f}_1-\frac{\tau_2+1}{\tau_2}\equiv\frac{-\tau_1}{\tau_2^2}\cdot\frac{1}{\widetilde{f}_2+(\tau_1-\tau_2)\tau_2^{-1}},$$

and

$$\widetilde{f}_2+\frac{\tau_1-\tau_2}{\tau_2}\equiv\frac{-\tau_1}{\tau_2^2}\cdot\frac{1}{\widetilde{f}_1-(\tau_2+1)\tau_2^{-1}}.$$

Hence, combining these above equalities with (2.3), we have

$$\overline{N}\left(r,\frac{1}{\widetilde{f}_2+(\tau_1-\tau_2)\tau_2^{-1}}\right)=\overline{N}(r,f_1)+O(\log r),$$

and

$$\overline{N}\left(r,\frac{1}{\widetilde{f}_1-(\tau_2+1)\tau_2^{-1}}\right)=\overline{N}(r,f_2)+O(\log r).$$

Using the same argument as in Case 2, we have $\iota_2 \leq 4k+13$ *and* $\iota_1 \leq 4k+13$, *a contradiction. Lemma 2.4 is proved.*

Lemma 2.5. (see [10]) *Let f be a nonconstant meromorphic function, and*

$$R(f)=\frac{\sum_{\mu=0}^{s}a_\mu f^\mu}{\sum_{\nu=0}^{p}b_\nu f^\nu}$$

be an irreducible rational function in f with constant coefficients $\{a_\mu\}$ *and* $\{b_\nu\}$, *where* $a_s \neq 0$ *and* $b_p \neq 0$. *Then* $T(r,R(f))=max\{s,p\}T(r,f)+O(1)$.

Lemma 2.6. (see [11]) *Let* $p>0$ *and q be relatively prime integers, and let t be a finite complex number such that* $t^p=1$. *Then there exists one and only one common zero of* ω^p-t *and* ω^q-t.

Lemma 2.7. (see [12]) *Suppose that f is a meromorphic of finite order in the plane, and that* $f^{(k)}$ *has finitely many zeros for some* $k\geq 2$. *Then f has finitely many poles in the complex plane.*

3 Proof of Theorem 1.6

First of all, we denote by d the degree of L. Then by Steuding [1] we have

$$T(r,L)=\frac{d}{\pi}r\log r+O(r). \tag{3.1}$$

From this equality, we can get that $\lim_{r\to\infty} \frac{T(r,L)}{\log r} = \infty$; therefore, L is a transcendental meromorphic function.

Next, we set

$$f_1 = f^n(f-1), \quad f_2 = L^n(L-1). \tag{3.2}$$

By Lemma 2.5 and (3.2), we obtain

$$\begin{aligned}\Theta(\infty, f_1) &= 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,f_1)}{T(r,f_1)} = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r,f)}{(n+1)T(r,f)+O(1)}\\ &\geq 1 - \limsup_{r\to\infty} \frac{T(r,f)}{(n+1)T(r,f)+O(1)} \geq 1 - \frac{1}{n+1},\end{aligned} \tag{3.3}$$

$$\begin{aligned}\Theta(0, f_1) &= 1 - \limsup_{r\to\infty} \frac{\overline{N}\left(r,\frac{1}{f_1}\right)}{T(r,f_1)} = 1 - \limsup_{r\to\infty} \frac{\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f-1}\right)}{(n+1)T(r,f)+O(1)}\\ &\geq 1 - \limsup_{r\to\infty} \frac{2T(r,f)}{(n+1)T(r,f)+O(1)} \geq 1 - \frac{2}{n+1},\end{aligned} \tag{3.4}$$

$$\begin{aligned}\delta_{k+1}(0, f_1) &= 1 - \limsup_{r\to\infty} \frac{N_{k+1}\left(r,\frac{1}{f_1}\right)}{T(r,f_1)} \geq 1 - \limsup_{r\to\infty} \frac{(k+1)\overline{N}\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{f-1}\right)}{(n+1)T(r,f)+O(1)}\\ &\geq 1 - \frac{k+2}{n+1}.\end{aligned} \tag{3.5}$$

In the same way,

$$\Theta(0, f_2) \geq 1 - \frac{2}{n+1}, \quad \delta_{k+1}(0, f_2) \geq 1 - \frac{k+2}{n+1}. \tag{3.6}$$

Since an L-function at most has one pole $z = 1$ in the complex plane, by (3.1) we have

$$\Theta(\infty, f_2) = 1. \tag{3.7}$$

Now we let

$$\iota_1 = (2k+4)\Theta(\infty, f_1) + (2k+3)\Theta(\infty, f_2) + 3\delta_{k+1}(0, f_1) + 2\delta_{k+1}(0, f_2) + \Theta(0, f_1) + \Theta(0, f_2), \tag{3.8}$$

$$\iota_2 = (2k+4)\Theta(\infty, f_2) + (2k+3)\Theta(\infty, f_1) + 3\delta_{k+1}(0, f_2) + 2\delta_{k+1}(0, f_1) + \Theta(0, f_2) + \Theta(0, f_1). \tag{3.9}$$

Substituting (3.3)-(3.7) into (3.8) and (3.9) we have

$$\iota_1 \geq 4k + 14 - \frac{7k+18}{n+1}, \quad \iota_2 \geq 4k + 14 - \frac{7k+17}{n+1}. \tag{3.10}$$

By the assumption $n > 7k + 17$, we have $\iota_1 > 4k + 13$ and $\iota_2 > 4k + 13$. From Lemma 2.4, we get $f_1^{(k)} f_2^{(k)} \equiv 1$ or $f_1 \equiv f_2$.

We consider the following two cases.

Case 1. $f_1 \equiv f_2$, that is,

$$f^n(f-1) \equiv L^n(L-1). \tag{3.11}$$

Let

$$P = \frac{L}{f}. \tag{3.12}$$

By (3.11) and (3.12) we deduce

$$(P^{n+1} - 1)f = P^n - 1. \tag{3.13}$$

We consider the following two subcases.

Subcase 1.1. Suppose that P is a nonconstant meromorphic function. Then, by (3.13) we have

$$f = \frac{1 - P^n}{1 - P^{n+1}}. \tag{3.14}$$

By noting that n and $n+1$ are two relatively prime positive integers, from Lemma 2.6 we know that $P = 1$ is the only one common zero of $1 - P^n$ and $1 - P^{n+1}$. Thus, (3.14) can be rewritten as

$$f = \frac{1 + P + P^2 + \cdots + P^{n-1}}{1 + P + P^2 + \cdots + P^n}. \tag{3.15}$$

By (3.15) and Lemma 2.5 we obtain

$$T(r, f) = T\left(r, \frac{1 + P + P^2 + \cdots + P^{n-1}}{1 + P + P^2 + \cdots + P^n}\right) = nT(r, f) + O(1). \tag{3.16}$$

On the other hand, by (3.12) and (3.15) we have $L = Pf = 1 - \frac{1}{1+P+P^2\cdots+P^n}$. From the Nevanlinna second fundamental theorem we have

$$\overline{N}(r, L) = \sum_{j=1}^{n} \overline{N}\left(r, \frac{1}{P - \sigma_j}\right) \geq (n-2)T(r, P) - S(r, P), \tag{3.17}$$

where $\sigma_1, \sigma_2, \cdots, \sigma_n$ are finite values with $\sigma_j \neq 1$ and $\sigma_j^{n+1} = 1$ for $1 \leq j \leq n$. Noting that L has at most one pole $z = 1$ in the complex plane, we deduce by (3.17) that there exists some small positive number ε_1 satisfying $0 < \varepsilon_1 < 1$, such that

$$(n - 2 - \varepsilon_1)T(r, P) \leq \overline{N}(r, L) = \log r + O(1). \tag{3.18}$$

By (3.18) and the assumption $n > 7k + 17$ and $k \geq 2$, we get $\lim\limits_{r\to\infty} \frac{T(r,P)}{\log r} \neq \infty$. Thus, P is a nonconstant rational function such that

$$T(r, P) \geq \log r + O(1), \tag{3.19}$$

which is a contradiction to (3.18).

Subcase 1.2. Suppose that P is a constant. If $P^{n+1} \neq 1$, by (3.13) we get (3.14), which contradicts the assumption that f is a transcendental meromorphic function. Hence, $P^{n+1} = 1$. Combining this with (3.13), we have $P^{n+1} - 1 = P^n - 1 = 0$, which implies that $P = 1$. By (3.12) we obtain $f \equiv L$.

Case 2. $f_1^{(k)} f_2^{(k)} \equiv 1$, that is,

$$(f^n(f-1))^{(k)}(L^n(L-1))^{(k)} \equiv 1. \tag{3.20}$$

By (3.1) we know that $\lambda(L) = 1$. Combining this, (3.20), and Lemma 2.5, we have

$$\lambda(f) = \lambda(f^n(f-1)) = \lambda((f^n(f-1))^{(k)}) = \lambda((L^n(L-1))^{(k)}) = \lambda(L^n(L-1)) = \lambda(L) = 1. \tag{3.21}$$

Since an L-function at most has one pole $z = 1$ in the complex plane, we deduce by (3.20) that $(f^n(f-1))^{(k)}$ at most has one zero $z = 1$ in the complex plane. Combining this, (3.21), Lemma 2.7, and the assumption $k \geq 2$, we know that $f^n(f-1)$ has at most finitely many poles in the complex plane, that is f has at most finitely many poles in the complex plane. Hence, by (3.20) we get $(L^n(L-1))^{(k)}$ has at most finitely many zeros in the complex plane. Furthermore, by the assumption $n > 7k + 17$ we get L has at most finitely many zeros in the complex plane. Thus, there exists a nonconstant rational function Q such that L has no zeros and poles, that is,

$$L = Qe^{\alpha z + \beta}, \tag{3.22}$$

where $\alpha \neq 0$ and β are constants.

By (3.22) and Hayman [13, p. 7] we get

$$T(r, L) = T(r, Qe^{\alpha z+\beta}) \leq \frac{|\alpha|\, r}{\pi}(1 + O(1)) + O(\log r),$$

which contradicts (3.1). Theorem 1.6 is proved.

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