Open Mathematics

Research Article

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Dislocated quasi cone b-metric space over Banach algebra and contraction principles with application to functional equations

https://doi.org/10.1515/math-2019-0086
Received March 7, 2019; accepted August 9, 2019

Abstract: In this paper we introduce dislocated and dislocated quasi version of a cone b-metric space over a Banach algebra as well as weak semi α-admissible and α-identical pair of mappings and prove common fixed point theorems for a pair of α-identical and weak α-admissible mappings in the aforesaid spaces. Our results are supported with suitable examples and an application to a system of m-tupled functional equations.

Keywords: common fixed points, dislocated cone b-metric space, dislocated quasi cone b-metric space, weak semi α-admissible pair of mappings, generalised contraction mappings

MSC: 47H10; 47N99; 54H25

1 Introduction

The concept of cone b-metric space (in short $CbMS$) over a Banach algebra was introduced in [1] and the authors proved generalised contraction principles in this space which directly improved and extended many comparable results in b-metric spaces (See [2-4]). α-admissible mappings were introduced by Samet et al. [5] which further helped in weakening and generalizing many contractive conditions (See [6-8]). In this paper we have introduced dislocated quasi $CbMS$ over a Banach algebra as a generalisation of $CbMS$ over a Banach algebra. We have proved some generalised results of fixed points for a pair of generalised α-admissible contraction mappings in dislocated $CbMS$ and dislocated quasi $CbMS$ over a Banach algebra which are proper extension and generalisation of some recent interesting results and the references there in. We have given suitable examples and an application of our result to a system of m-tupled functional equations . In recent years the equivalence of a metric space and a cone metric space was announced in [9]. How ever equivalence of a metric space and a cone metric space over a Banach algebra does not exist and hence our results are significant in studies under fixed point theory.

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2 Preliminaries

Let $\mathcal{A}$ be a Banach algebra and $p \in \mathcal{A}$. By $r(p)$ we mean the spectral radius of p. For definition of a Banach algebra and more related results the reader may refer to [1, 10, 11]. However, below we give some important definitions and properties which will be used in our main results.

Definition 2.1. *A sequence p_n in $\mathcal{A}$ is a c-sequence provided for any $c \gg \theta$, we can find $n_0 \in \mathbb{N}$ satisfying $p_n \ll c$ for all $n \geq n_0$.*

Lemma 2.2. *[12] For any $p \in \mathcal{A}$, if*

$$r(p) = lim_{n\to\infty}\|p^n\|^{\frac{1}{n}} = inf_{n\geq 1}\|p^n\|^{\frac{1}{n}} < 1 \tag{2.1}$$

then

$$(e - p)^{-1} = \sum_{i=0}^{\infty} p^i \tag{2.2}$$

Lemma 2.3. *[12] For all $a, b \in \mathcal{A}$ with $ab = ba$, $r(a + b) \leq r(a) + r(b)$ and $r(ab) \leq r(a)r(b)$.*

Lemma 2.4. *[12] $\{u_n\}$ in $\mathcal{A}$ is a c-sequence provided $\{u_n\} \to \theta$ as $n \to \infty$.*

Lemma 2.5. *[13] $\{\alpha^n\}$ is a c-sequence provided $r(\alpha) < 1$.*

Lemma 2.6. *[1] For any c-sequence $\{u_n\}$ and $\beta \in \mathrm{P}$ $\{\beta u_n\}$ is also a c-sequence.*

3 Main results

3.1 Dislocated quasi cone b-metric space

Definition 3.1. *Let $\mathcal{A}$ be a Banach algebra, χ^q be a non empty set and $d : \chi^q \times \chi^q \to \mathcal{A}$. For all $p, q, r \in \chi^q$, consider the following conditions :*

(dqCM1) $\theta \preceq d(p, q)$ and $d(p, q) = d(q, p) = \theta$ implies $p = q$
(dqCM2) $d(p, q) \preceq s[d(p, r) + d(r, q)]$ for some $s \in \mathrm{P}$, $e \preceq s$.
(dqCM3) $d(p, q) = d(q, p)$

If d satisfies conditions $(dqCM1)$ and $(dqCM2)$, then d is a dislocated quasi cone b-metric and $(\chi^q, d, \mathcal{A})$ will be called a dislocated quasi CbMS over Banach algebra A (in short $dqCbMS-\mathcal{A}$). If d satisfies $(dqCM1)$, $(dqCM2)$ and $(dqCM3)$ then $(\chi^q, d, \mathcal{A})$ is a dislocated CbMS over Banach algebra A (in short $dCbMS-\mathcal{A}$). If $(dqCM1)$ is replaced with $\theta \preceq d(p, q)$ and $d(p, q) = d(q, p) = \theta$ if and only if $p = q$ then the above two definitions reduces to quasi CbMS over Banach algebra A (in short $qCbMS-\mathcal{A}$) and CbMS over Banach algebra A (in short $CbMS-\mathcal{A}$) respectively.

Remark 3.2. *In (Definition 2.1, [1]) the authors defined $CbMS - \mathcal{A}$ wherein they have taken $s \geq 1$ a scalar. However in our definition of $CbMS-\mathcal{A}$ we take $s \in \mathrm{P}$. Note that (Definition 2.1, [1]) implies definition 3.1(i) above in the sense that if $d(p, q) \preceq s[d(p, r) + d(r, q)]$ for some scalar $s \geq 1$ then $d(p, q) \preceq s'[d(p, r) + d(r, q)]$ with $s' = se \in P$ where e is the identity element of the Banach algebra A.*

Remark 3.3. *(An open problem) It is not known whether $CbMS-\mathcal{A}$ in the sense of Definition 3.1(i) above implies Definition (2.1) of [1].*

Remark 3.4. *In Definition 3.1 above, if we replace the co-domain of the metric function d with a Banach space instead of a Banach algebra then* $(\chi^q, d, \mathcal{A})$ *will be a dislocated quasi cone b-metric space which is a generalisation of a cone metric space introduced by Huang and Zhang [14]. Many interesting fixed point theorems have been proved in a cone metric space. In recent years using scalarization method, many authors proved the equivalence of a metric space and a cone metric space and showed that the fixed point results in cone metric spaces were the consequences of their usual metric versions (see [9],[15-17]). Consequently Liu and Xu[11] introduced the concept of a cone metric space over Banach algebra, and proved some fixed point results in such spaces. They showed that thefixed point results in this new setting cannot be derived from their usual metric versions. Thus the cocept of a dislocated quasi cone b-metric space over a Banach algebra becomes more general than that of a dislocated quasi cone b-metric space.*

Here after, throughout this paper, by $(\chi^q, d, \mathcal{A})$ *and* $(\chi^d, d, \mathcal{A})$ *we mean a* $dqCbMS - \mathcal{A}$ *and* $dCbMS - \mathcal{A}$ *respectively, with coefficient* $s \in P$ $(e \preceq s)$.

Example 3.5. *Let* $\chi^q = \mathbb{R}$. *Consider the Banach algebra* P_{n+1} *of all polynomials with complex coefficients and degree less than or equal to n, in which for any* $x(t) = \alpha_0+\alpha_1 t+\alpha_2 t^2+\cdots+\alpha_n t^n$ *and* $y(t) = \beta_0+\beta_1 t+\beta_2 t^2+\cdots+\beta_n t^n$ *the norm of* $x(t)$ *is given by* $\|x(t)\| = \sum_{i=0}^{n} \mid \alpha_i \mid$ *and the product by* $(xy)(t) = \sum_{i=0}^{n} a_k t^k$ *where* $a_k = \sum_{i+j=k} \alpha_i \beta_j$. *The unit elemant* $e = 1 + 0t + 0t^2 \cdots + 0t^n$, *zero element* $\theta = 0 + 0t + 0t^2 + \cdots + 0t^n$ *and* $P = \{x(t) \in P_{n+1} : a_i \geq 0, i = 0, 1, 2 \cdots n\}$ *is a non normal cone in* P_{n+1}. *Define the function* $d \colon \chi^q \times \chi^q \to P_{n+1}$ *by*

$$d(x, y)(t) =\mid x + y \mid^2 + \mid x + 2y \mid^2 t+ \mid x + 3y \mid^2 t^2 + \cdots + \mid x + (n + 1)y \mid^2 t^n$$

clearly (χ^q, d, P_{n+1}) *is a* $dqCbMS - P_{n+1}$.

Example 3.6. *Let* $\chi^d = \mathbb{R}^+ \cup \{0\}$ *and consider the Banach algebra* $\mathcal{A}$ *of all* 3×3 *matrices over the set* $\mathbb{R}$ *with* $\|a\| = \sum_{1\leq i,j\leq 3} |a_{i,j}|$ *and solid cone P of all* 3×3 *matrices over the set* $\mathbb{R}^+ \cup \{0\}$. *Let* $d \colon \chi^d \times \chi^d \to \mathcal{A}$ *be given by*

$$d(p, q) = \begin{pmatrix} (p+q)^2 & (p+q)^2 & (p+q)^2 \\ 2(p+q)^2 & 3(p+q)^2 & 4(p+q)^2 \\ 3(p+q)^2 & 4(p+q)^2 & 5(p+q)^2 \end{pmatrix}$$

then $(\chi^d, d, \mathcal{A})$ *is a* $dCbMS - \mathcal{A}$ *with* $s = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

Every $qCbMS - \mathcal{A}$ ($dqCbMS - \mathcal{A}$) induces a $CbMS - \mathcal{A}$ ($dCbMS - \mathcal{A}$). We give the following propositions in support of this claim.

Proposition 3.7. *Let* $(\chi, d, \mathcal{A})$ *be a* $qCbMS-\mathcal{A}$ $(dqCbMS-\mathcal{A})$ *and for all* $x, y \in \chi$, *define* $d^\dagger(x, y) = \frac{d(x,y)+d(y,x)}{2}$. *Then* $(\chi, d^\dagger, \mathcal{A})$ *is a* $CbMS - \mathcal{A}$ $(dCbMS - \mathcal{A})$.

Let $a \in \chi^q$. The left-open ball with centre a and radius $u > \theta$ is

$$B_L(a, u) = \{b \in \chi^q : d(a, b) < u\}$$

and the right-open ball with centre a and radius $u > \theta$ is given by

$$B_R(a, u) = \{b \in \chi^q : d(b, a) < u\}.$$

The open ball with centre a and radius u is

$$B(a, u) = B_L(a, u) \bigcap B_R(a, u),$$

i.e

$$B(a,u) = \{b \in \chi^q : d(a,b) < u \text{ and } d(b,a) < u\}.$$

Let $\mathcal{U} = \{\Gamma \subseteq \chi^q : \forall x \in \Gamma, \exists u > \theta, \text{ such that } B_u(x) \subseteq \Gamma\}$. Then $\mathcal{U}$ defines the dislocated quasi cone b-metric topology for the $dqCbMS - BA$ (χ^q, d).

Definition 3.8. *Let $(\chi^q, d, \mathcal{A})$ be a $dqCbMS - \mathcal{A}$, $p \in \chi^q$ and $\{p_n\}$ be a sequence in χ^q.*

i) $\{p_n\}$ bi-converges to p if for each $\varepsilon \in \mathcal{A}$, with $\theta \ll \varepsilon$, there exist $n_0 \in \mathbb{N}$ such that $d(p_n, p) \ll \varepsilon$ and $d(p, p_n) \ll \varepsilon$ whenever $n \geq n_0$. We write it as $Lim_{n\to\infty} p_n = p$.

ii) $\{p_n\}$ is a $L - Cauchy$ sequence ($R - Cauchy$ sequence) provided for each $\varepsilon \in \mathcal{A}$, with $\theta \ll \varepsilon$, there exist $n_0 \in \mathbb{N}$ such that $d(p_n, p_m) \ll \varepsilon$ ($d(p_m, p_n) \ll \varepsilon$) for all $n > m \geq n_0$.

iii) $(\chi^q, d, \mathcal{A})$ is $L - complete$ ($R - complete$) $dqCbMS$ provided every $L - Cauchy$ sequence ($R - Cauchy$ sequence) in $(\chi^q, d, \mathcal{A})$ is bi-convergent.

Proposition 3.9. *Let $(\chi^q, d, \mathcal{A})$ be a $dCbMS - BA$ over $\mathcal{A}$, sequence $\{p_n\}$ in χ^q. If $\{p_n\}$ converges to $p \in \chi^q$ then*

i) $d(p_n, p)$ is a c-sequence.

ii) $d(p_n, p_{n+r})$ is a c-sequence.

Proof : Follows from definitions 2.1,3.1 and 3.8(i).

Proposition 3.10. *If $(\chi^q, d, \mathcal{A})$ is $L - complete$ or $R - complete$ $dqCbMS - \mathcal{A}$, then $(\chi^q, d^\dagger)$ is a complete $dCbMS - \mathcal{A}$.*

Proof : We will show that a $Cauchy$ sequence $\{p_n\}$ in $(\chi^q, d^\dagger, \mathcal{A})$ is always a $L - Cauchy$ sequence and $R - Cauchy$ sequence in $(\chi^q, d, \mathcal{A})$. Let $\varepsilon \in \mathcal{A}$ with $\theta \ll \varepsilon$, then $\theta \ll \frac{\varepsilon}{2}$. Then by hypothesis, we can find $n_0 \in \mathbb{N}$ such that $d^\dagger(p_n, p_m) = \frac{d(p_n,p_m)+d(p_m,p_n)}{2} \ll \frac{\varepsilon}{2}$ whenever $n, m \geq n_0$, that is $d(p_n, p_m) \ll \varepsilon$ and $d(p_m, p_n) \ll \varepsilon$ for all $n, m \geq n_0$ and thus $\{p_n\}$ is $L - Cauchy$ sequence and R-Cauchy sequence in $(\chi^q, d, \mathcal{A})$. Now if $(\chi^q, d, \mathcal{A})$ is $L - complete$ (or $R - complete$), the $L - Cauchy$ sequence (or $R - Cauchy$ sequence) $\{p_n\}$ bi-converges to some point p in χ^q and thus there exist $n_0 \in \mathbb{N}$ such that $d(p_n, p) \ll \varepsilon$ and $d(p, p_n) \ll \varepsilon$ whenever $n \geq n_0$. Thus $d^\dagger(p_n, p) = \frac{d(p_n,p)+d(p,p_n)}{2} \ll \varepsilon$ whenever $n \geq n_0$ and hence $\{p_n\}$ is convergent in $(\chi^q, d^\dagger, \mathcal{A})$. Hence $(\chi^q, d^\dagger, \mathcal{A})$ is a complete $dCbMS - \mathcal{A}$.

For proving the uniqueness of the fixed point under α-admissible conditions, different hypothesis were used by different authors. In the sequel Popescu [18] considered the following condition :

(**K**) Whenever $x \neq y \in X$, we can find $w \in X$ satisfying $\alpha(x, w) \geq 1$, $\alpha(y, w) \geq 1$ and $\alpha(w, Tw) \geq 1$.

We now introduce the following definitions and examples :

Definition 3.11. *Let (X^d, d) be a $dCbMS - \mathcal{A}$, $T, S : X^d \to X^d$ and $\alpha : X^d \times X^d \to [0, \infty)$ be mappings and $x \in X^d$. Then the pair (T, S) is*

(i) α-identical at x iff $\min\{\alpha(Tx, Sx), \alpha(Sx, Tx)\} \geq 1$. The pair (T, S) is α-identical on X^d iff (T, S) is α-identical at all $x \in X^d$.

(ii) Weak semi α-admissible iff $x, y \in X^d$ and $\alpha(x, y) \geq 1 \Rightarrow \min\{\alpha(x, TSy), \alpha(x, STy)\} \geq 1$.

(iii) α-dominated at x iff $\min\{\alpha(x, Tx), \alpha(x, Sx)\} \geq 1$. The pair (T, S) is α-dominated on X^d iff (T, S) is α-dominated at all $x \in X^d$.

(iv) Satisfies condition $(\mathbf{G}_\alpha)$ iff $\alpha(x, Tx) \geq 1$ and $\alpha(y, Sy) \geq 1$ implies $\alpha(x, y) \geq 1$ or $\alpha(Tx, Sy) \geq 1$ for any $x, y \in X^d$.

(v) Satisfies condition $(\mathbf{G}'_\alpha)$ iff whenever $x \neq y \in X$ with $\alpha(x, Tx) \geq 1$ and $\alpha(y, Sy) \geq 1$ there exists $w \in X^d$ satisfying $\alpha(x, w) \geq 1$, $\alpha(y, w) \geq 1$, $\alpha(w, w) \geq 1$, $\alpha(w, Tw) \geq 1$ and $\alpha(w, Sw) \geq 1$.

(vi) (X^d, d) is α-regular iff for any sequence $\{x_p\}$ in X^d with $\alpha(x_p, x_{p+1}) \geq 1$ and $x_p \to x_\star$ as $p \to \infty$, then $\alpha(x_p, x_\star) \geq 1$

Example 3.12. *Let* $\chi = [0, \infty]$,

$$Tx = x^2 \text{ for all } x \in \chi, \; Sx = \begin{cases} \frac{x^3}{3} & \text{if } x \in [0, 1] \\ x^2 & \text{if otherwise} \end{cases}$$

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in [0, 1] \text{ or } x = y \\ 0 & \text{otherwise} \end{cases}$$

then the pair (T, S) *is* α*-identical, satisfies condition* $(\mathbf{G}_\alpha)$ *and* $(\mathbf{G}'_\alpha)$*but the pair* (T, S) *does not satisfy condition* $(\mathbf{K})$ *and* T *is not* α*-dominated.*

Example 3.13. *Let* $\chi = [-n, n]$ *for some* $n \in \mathbb{N}$,

$$Tx = -x \text{ for all } x \in \chi, \; Sx = \begin{cases} x^2 & \text{if } x \in [-n, 0] \\ -x^2 & \text{if } x \in [0, n] \end{cases}$$

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x, y \in [-n, 0] \text{ or } x, y \in (0, n] \\ 0 & \text{otherwise} \end{cases}$$

then the pair (T, S) *is* α*-identical, satisfies condition* $(\mathbf{G}_\alpha)$ *and* $(\mathbf{G}'_\alpha)$*but the pair* (T, S) *does not satisfy condition* $(\mathbf{K})$ *and* T *is not* α*-dominated.*

Example 3.14. *Let* $A = [-n, 0]$, $B = [0, n]$ *and* $\chi = A \bigcup B$ *for some* $n \in \mathbb{N}$,

$$Tx = -x \text{ for all } x \in \chi, \; Sx = \begin{cases} \frac{x^2}{n} & \text{if } x \in A \\ -x^2 & \text{if } x \in B \end{cases}$$

$$\alpha(x, y) = \begin{cases} 1 & \text{if } (x, y) \in \{A \times B, B \times A\} \\ 0 & \text{otherwise} \end{cases}$$

then α *is not triangular and the pair* (T, S) *is not* α*-identical but* (T, S) *is weak semi* α*-admissible and* α*-dominted.* T *does not satisfy condition* $(\mathbf{G}_\alpha)$ *but satisfies condition* $(\mathbf{G}'_\alpha)$.

Example 3.15. *Let* $A = [-n, 0)$, $B = (0, n]$ *and* $\chi = A \bigcup \{0\} \bigcup B$ *for some* $n \in \mathbb{N}$,

$$Tx = \frac{x^2}{n} \text{ for all } x \in \chi, \; Sx = \begin{cases} \frac{-x}{n} & \text{if } x \in A \\ \frac{x}{n} & \text{if } x \in B \end{cases}$$

$$\alpha(x, y) = \begin{cases} 1 & \text{if } (x, y) \in \{A \times A, B \times B\} \\ 0 & \text{otherwise} \end{cases}$$

then α *is triangular and* (T, S) *is* α*-identical but* (T, S) *is not weak semi* α*-admissible and not* α*-dominted.* T *satisfy conditions* $(\mathbf{G}_\alpha)$ *and* $(\mathbf{G}'_\alpha)$ *but does not satisfy condition* $(\mathbf{K})$.

Lemma 3.16. *In any non empty set X consider the functions $T, S : X \to X$ and $\alpha : X \times X \to [0, \infty)$. For some $x_0 \in X$ consider the sequence $\{x_n\}$ given by*

$$x_{2n+1} = Tx_{2n} \text{ and } x_{2n+2} = Sx_{2n+1}, n \in N \tag{3.1}$$

with $\alpha(x_0, x_0) \geq 1$, $\alpha(x_0, Tx_0) \geq 1$. Then
(i) If α is a triangular function and (T, S) is α-admissible then whenever $n \geq 1$ and $0 \leq p \leq q \leq n$, we have $\alpha(x_p, x_q) \geq 1$.
(ii) If (T, S) is α-admissible and weak semi α-admissible, then for all $n \geq 1$ and $0 \leq p \leq q \leq n$, $\alpha(x_p, x_q) \geq 1$.

Proof : For proof we will make use of principle of mathematical induction. Let $P(n)$ denote the statement for all $0 \leq p \leq q \leq n$, we have $\alpha(x_p, x_q) \geq 1$.

(i) Since $\alpha(x_0, x_0) \geq 1$, $\alpha(x_0, x_1) \geq 1$ and (T, S) is α-admissible we have $\alpha(x_1, x_1) = \alpha(Tx_0, Tx_0) \geq 1$ and so $P(1)$ holds. Again by α-admissibility of (T, S) we get $\alpha(x_1, x_2) = \alpha(Tx_0, Sx_1) \geq 1$, $\alpha(x_2, x_2) = \alpha(Sx_1, Sx_1) \geq 1$ and then since α is triangular we get $\alpha(x_0, x_2) \geq 1$. Thus $P(2)$ holds. Suppose $P(r)$ holds, i.e. $\alpha(x_p, x_q) \geq 1$ for all $0 \leq p \leq q \leq r$. We will show that $P(r+1)$ holds. Its enough to consider the case $\alpha(x_p, x_{r+1})$, $0 \leq p \leq r+1$. By induction hypothesis since $\alpha(x_p, x_r) \geq 1$ for all $0 \leq p \leq r$ using α-admissibility of (T, S) we have $\alpha(x_p, x_{r+1}) \geq 1$ for all $1 \leq p \leq r+1$. Thus we have $\alpha(x_0, x_1) \geq 1$ and $\alpha(x_1, x_{r+1}) \geq 1$, hence by triangularity of function α we get $\alpha(x_0, x_{r+1}) \geq 1$ and thus $P(r+1)$ holds.

(ii) Since $\alpha(x_0, x_0) \geq 1$, $\alpha(x_0, x_1) \geq 1$ and (T, S) is α-admissible, $\alpha(x_1, x_1) = \alpha(Tx_0, Tx_0) \geq 1$ and so $P(1)$ holds. Again by α-admissibility of (T, S) we get $\alpha(x_1, x_2) = \alpha(Tx_0, Sx_1) \geq 1$ and $\alpha(x_2, x_2) = \alpha(Sx_1, Sx_1) \geq 1$. Since (T, S) is weak semi α-admissible and $\alpha(x_0, x_0) \geq 1$ we get $\alpha(x_0, x_2) = \alpha(x_0, STx_0) \geq 1$. Thus $P(2)$ holds. Suppose $P(r)$ holds, i.e. $\alpha(x_p, x_q) \geq 1$ for all $0 \leq p \leq q \leq r$. We will show that $P(r+1)$ holds. Its enough to consider the case $\alpha(x_p, x_{r+1})$, $0 \leq p \leq r+1$. By induction hypothesis and α-admissibility of (T, S) we have $\alpha(x_p, x_{r+1}) \geq 1$ for all $1 \leq p \leq r+1$. If r is even, then using $\alpha(x_0, x_1) \geq 1$ and repeated use of weak semi α-admissibility of T we get $\alpha(x_0, x_{r+1}) \geq 1$. If r is odd then using $\alpha(x_0, x_0) \geq 1$ and repeated use of weak semi α-admissibility of T we get $\alpha(x_0, x_{r+1}) \geq 1$. Thus $P(r+1)$ holds.

Lemma 3.17. *[1] For any sequence $\{p_n\}$ in a $CbMS-\mathcal{A}\,(\chi, d, \mathcal{A})$, if we can find $\alpha \in P$ with $r(\alpha) < \frac{1}{r(s)}$, satisfying $d(p_n, p_{n+1}) \preceq \alpha d(p_{n-1}, p_n)$ then $\{p_n\}$ is a Cauchy sequence.*

As a direct consequence of the above lemma we have the following :

Lemma 3.18. *For any sequence $\{p_n\}$ in a $CbMS - \mathcal{A}\,(\chi, d, \mathcal{A})$, if we can find $\alpha \in P$ with $r(\alpha) < \frac{1}{r(s)}$, satisfying $d(p_n, p_{n+1}) \preceq K\alpha^n$ then $\{p_n\}$ is a Cauchy sequence.*

Drawing inspiration from a recent result of Zoran Mitrović [19], we now give an improved version of the above lemma with an increased range of $r(\beta)$.

Lemma 3.19. *For any sequence $\{p_n\}$ in $(\chi^d, d, \mathcal{A})$, if we can find $\beta \in P$ with $r(\beta) < 1$, satisfying $d(p_n, p_{n+1}) \preceq \beta d(p_{n-1}, p_n)$ then $\{p_n\}$ is a Cauchy sequence.*

Proof : Let $k \in N$ and $k > \frac{\log r(s)^{-1}}{\log r(\beta)}$. Then we have

$$\begin{aligned} d(p_{kn}, p_{k(n+1)}) &\preceq s^k\{d(p_{kn}, p_{kn+1}) + d(p_{kn+1}, p_{kn+2}) + \ldots d(p_{k(n+1)-1}, p_{k(n+1)})\} \\ &\preceq s^k\{\beta^{kn} + \beta^{kn+1} + \ldots. + \beta^{k(n+1)-1}\}d(p_0, p_1) \\ &\preceq s^k\beta^{kn}(e-\beta)^{-1}d(p_0, p_1) \\ &\preceq K\alpha^n \end{aligned}$$

$K = s^k\beta^k(e-\beta)^{-1}d(p_0,p_1) \in \mathcal{A}$ and $\alpha = \beta^k$. $r(\alpha) = r(\beta^k) \le r(\beta)^k$ and since $k > \frac{\log r(s)^{-1}}{\log r(\beta)}$ we have $r(\beta)^k < \frac{1}{r(s)}$ and hence by Lemma 3.18, sequence $\{p_{kn}\}$ is a Cauchy sequence.
Now

$$\begin{aligned} d(p_n, p_{k[\frac{n}{k}]}) &\preceq s^k\{d(p_n,p_{n-1}) + d(p_{n-1},p_{n-2}) + \dots + d(p_{k[\frac{n}{k}]+1}, p_{k[\frac{n}{k}]})\\ &\preceq s^k\{\beta^{n-1} + \beta^{n-2} + \dots + \beta^{k[\frac{n}{k}]}\}d(p_0,p_1)\\ &\preceq s^k\beta^{k[\frac{n}{k}]}(e-\beta)^{-1}d(p_0,p_1) \end{aligned}$$

Now since $r(\beta) < 1$, by lemma 2.5 and 2.6 $s^k\beta^{k[\frac{n}{k}]}(e-\beta)^{-1}d(p_0,p_1)$ is a c-sequence and hence by lemma 2.4 $s^k\beta^{k[\frac{n}{k}]}(e-\beta)^{-1}d(p_0,p_1) \to 0$ as $n \to \infty$. Thus we have

$$d(p_n, p_{k[\frac{n}{k}]}) \to \infty \text{ as } n \to \infty. \tag{3.2}$$

Now we have

$$d(p_n,p_m) \preceq s^2\{d(p_n,p_{k[\frac{n}{k}]}) + d(p_{k[\frac{n}{k}]}, p_{k[\frac{m}{k}]}) + d(p_{k[\frac{m}{k}]}, p_m)$$

Now using (3.2) and the fact that sequence $\{p_{kn}\}$ is a Cauchy sequence, we conclude that sequence $\{p_n\}$ is a Cauchy sequence.

3.2 Results in dislocated cone b-metric space

Our first new result of this section is the following :

Theorem 3.20. *Let $T, S : \chi^d \to \chi^d$. Let $\alpha : \chi^d \times \chi^d \to [0,\infty)$ be mappings such that*

(i) α is a triangular function or (T,S) is weak semi α-admissible.
(ii) (T,S) is α-admissible.
(iii) $\alpha(x_0,x_0) \ge 1$ and $\alpha(x_0,Tx_0) \ge 1$ for some $x_0 \in \chi^d$
(iv) (χ^d, d) is α-regular
(v) we can find $\lambda, \mu, \nu \in$ P such that $\lambda+\mu+s\nu$ commutes with $\mu+3s\nu$, $r(\lambda+\mu+s\nu) + r(\mu+3s\nu) < 1$ and for all $x, y \in \chi^d$ with $\alpha(x,y) \ge 1$ the following conditions are satisfied :

$$d(Sx,Ty) \preceq \lambda d(x,y) + \mu(d(x,Sx) + d(y,Ty)) + \nu(d(x,Ty) + d(y,Sx)) \tag{3.3}$$

$$d(Tx,Sy) \preceq \lambda d(x,y) + \mu(d(x,Tx) + d(y,Sy)) + \nu(d(x,Sy) + d(y,Tx)) \tag{3.4}$$

then Fix$\{T,S\} \ne \phi$. Further if (T,S) is α-identical or if (T,S) is α-dominated on χ^d , then $d(u,u) = \theta$ for any $u \in$ Fix$\{T,S\}$.

Proof : Consider the iterative sequence $\{x_p\}$ defined as in (3.1) and starting with x_0. Let $d(x_p,x_{p+1}) = d_p$ and $d(x_p,x_p) = d_{p,p}$. Note that $d_{p,p} \preceq 2sd_{p-1}$ and $d_{p,p} \preceq 2sd_{p+1}$. By Lemma 3.16, $\alpha(x_{2p},x_{2p+1}) \ge 1$ and $\alpha(x_{2p+1},x_{2p+2}) \ge 1$. Hence using (3.4) we have

$$\begin{aligned} d_{2p+1} &= d(x_{2p+1},x_{2p+2}) = d(Tx_{2p}, Sx_{2p+1})\\ &\preceq \lambda d(x_{2p},x_{2p+1}) + \mu(d(x_{2p},Tx_{2p}) + d(x_{2p+1},Sx_{2p+1})) + \nu(d(x_{2p},Sx_{2p+1}) + d(x_{2p+1},Tx_{2p}))\\ &= \lambda d_{2p} + \mu(d_{2p} + d_{2p+1}) + \nu[sd_{2p} + sd_{2p+1} + (2sd_{2p} \text{ or } 2sd_{2p+1})], \end{aligned}$$

i.e

$$d_{2p+1} \preceq \beta d_{2p} \text{ where } \beta = (\lambda+\mu+3s\nu)(e-\mu-s\nu)^{-1} \text{ or } \beta = (\lambda+\mu+s\nu)(e-\mu-3s\nu)^{-1}.$$

Similarly using (3.3) we have

$$d_{2p+2} \preceq \beta d_{2p+1} \text{ where } \beta = (\lambda + \mu + 3sv)(e - \mu - sv)^{-1} \text{ or } \beta = (\lambda + \mu + sv)(e - \mu - 3sv)^{-1}.$$

Thus for any integer p we have

$$d_{p+1} \preceq \beta d_p \text{ where } \beta = (\lambda + \mu + 3sv)(e - \mu - sv)^{-1} \text{ or } \beta = (\lambda + \mu + sv)(e - \mu - 3sv)^{-1}$$

or

$$d_{p+1} \preceq \beta^{p+1} d_0.$$

Since $\lambda + \mu + sv$ commutes with $\mu + 3sv$, we have

$$\begin{aligned}(\lambda + \mu + sv)(e - (\mu + 3sv))^{-1} &= (\lambda + \mu + sv)(1 + (\mu + 3sv) + (\mu + 3sv)^2 + \ldots)\\ &= (\lambda + \mu + sv) + (\lambda + \mu + sv)(\mu + 3sv) + (\lambda + \mu + sv)(\mu + 3sv)^2 + \ldots\\ &= (\lambda + \mu + sv) + (\mu + 3sv)(\lambda + \mu + sv) + (\mu + 3sv)^2(\lambda + \mu + sv) + \ldots\\ &= (1 + (\mu + 3sv) + (\mu + 3sv)^2 + \ldots)(\lambda + \mu + sv) = (e - (\mu + 3sv))^{-1}(\lambda + \mu + sv),\end{aligned}$$

i.e $\lambda + \mu + sv$ commutes with $(e - (\mu + 3sv))^{-1}$. Thus using Lemma 2.3 and Lemma 2.8 of [1], we have

$$r(\beta) = r(\lambda + \mu + sv)(e - \mu - 3sv)^{-1} \leq \frac{r(\lambda + \mu + sv)}{1 - r(\mu + 3sv)} < 1.$$

Thus using Lemma 3.19, $\{x_p\}$ is Cauchy and since (χ^d, d) is complete we have $u \in \chi^d$ such that

$$\lim_{n\to\infty} x_p = u. \tag{3.5}$$

Then since (χ^d, d) is α-regular, we get $\alpha(x_{2p}, u) \geq 1$ and $\alpha(x_{2p-1}, u) \geq 1$. Since $d_p \neq d_q$ whenever $p \neq q$ there exist $k \in \mathbb{N}$ such that $d(u, Tu) \neq \{d_k, d_{k+1}, \ldots\}$. Thus by (3.3) for any $p > k$

$$\begin{aligned}d(u, Tu) &\preceq s[d(u, x_{2p}) + d(x_{2p}, Tu)]\\ &= s[d(u, x_{2p}) + d(Sx_{2p-1}, Tu)]\\ &\preceq s[d(u, x_{2p}) + \lambda d(x_{2p-1}, u) + \mu(d(x_{2p-1}, Sx_{2p-1}) + d(u, Tu))\\ &\quad +v(d(x_{2p-1}, Tu) + d(u, Sx_{2p-1})]\\ &\preceq s[d(u, x_{2p}) + \lambda d(x_{2p-1}, u) + \mu(d(x_{2p-1}, x_{2p}) + d(u, Tu))\\ &\quad +v(sd(x_{2p-1}, u) + sd(u, Tu) + d(u, x_{2p})],\end{aligned}$$

i.e.

$$\begin{aligned}d(u, Tu) &\preceq (s + sv)(e - s(\mu + sv))^{-1} d(u, x_{2p})\\ &\quad + (e - s(\mu + sv))^{-1}(s(\lambda + sv))d(x_{2p-1}, u) + (e - s(\mu - sv))^{-1} s\mu d(x_{2p-1}, x_{2p}).\end{aligned}$$

By Proposition 3.9 and Lemma 2.4, $d(x_{2p}, u) \to \theta$, $d(x_{2p-1}, u) \to \theta$, $d(x_{2p}, x_{2p+1}) \to \theta$ and $d(x_{2p-1}, x_{2p}) \to \theta$ and hence $d(u, Tu) \to \theta$ that is $Tu = u$.
Similarly, by (3.4) for any $p > k$

$$\begin{aligned}d(u, Su) &\preceq s[d(u, x_{2p+1}) + d(x_{2p+1}, Tu)]\\ &= s(d(u, x_{2p+1}) + d(Tx_{2p}, Su))\\ &\preceq s[d(u, x_{2p+1}) + \lambda d(x_{2p}, u) + \mu(d(x_{2p}, Tx_{2p}) + d(u, Su))\\ &\quad +v(d(x_{2p}, Su) + d(u, Tx_{2p})]\\ &\preceq s[d(u, x_{2p+1}) + \lambda d(x_{2p}, u) + \mu(d(x_{2p}, x_{2p+1}) + d(u, Su))\\ &\quad +v(sd(x_{2p}, u) + sd(u, Su) + d(u, x_{2p+1}))],\end{aligned}$$

i.e.

$$d(u, Su) \preceq (s + sv)(e - s(\mu + sv))^{-1} d(u, x_{2p+1}) + (e - s(\mu + sv))^{-1}(s(\lambda + sv))d(x_{2p}, u) + (e - s(\mu - sv))^{-1} s\mu d(x_{2p+1}, x_{2p}).$$

By Proposition 3.9 and Lemma 2.4, $d(x_{2p}, u) \to \theta$, $d(x_{2p-1}, u) \to \theta$, $d(x_{2p}, x_{2p+1}) \to \theta$ and $d(x_{2p-1}, x_{2p}) \to \theta$ and hence $d(u, Tu) \to \theta$ and $Tu = u$. Thus we have $Tu = u = Su$.

If (T, S) is α-identical and $Tu = u = Su$, then $\alpha(u, u) = \alpha(Tu, Su) \geq 1$. If (T, S) is α-dominated then $\alpha(u, u) = \alpha(u, Tu) \geq 1$. Then from (3.4) we have $d(u, u) = d(Tu, Su) \preceq \lambda d(u, u) + \mu(d(u, u) + d(u, u)) + v(d(u, u) + d(u, u)) = (\lambda + 2\mu + 2v)d(u, u) \preceq (\lambda + 2\mu + 4sv)d(u, u)$. Note that $r(\lambda + 2\mu + 4sv) \leq r(\lambda + \mu + sv) + r(\lambda + \mu + 3sv) < 1$. Thus $(e - \lambda - 2\mu - 4sv)$ is invertible and so we get $(e - \lambda - 2\mu - 4sv)^{-1} d(u, u) \preceq \theta$, i.e $d(u, u) = \theta$.

Our second new result is the following :

Theorem 3.21. *Let $T, S\colon \chi^d \to \chi^q$ and $\alpha : \chi^d \times \chi^d \to [0, \infty)$ be mappings satisfying conditions (i), (ii), (iii), (iv), (v) of Theorem 3.20 and either of the following two conditions*

(a) T or S satisfy condition ($\mathbf{G}_\alpha$)
(b) T or S satisfiy condition ($\mathbf{G}'_\alpha$)

then $Fix\{T, S\}$ is a singleton set and the iterative sequence (3.1) *converges to the unique $u \in Fix\{T, S\}$.*

Proof : From Theorem 3.20, we see that $Fix(T, S) \neq \phi$ and the iterative sequence (3.1) converges to some $u \in Fix\{T, S\}$. Suppose $u, w \in Fix(T, S)$. Then again from Theorem 3.20 we have $\alpha(u, u) \geq 1$, $\alpha(w, w) \geq 1$, $d(w, w) = \theta$ and $d(w, w) = \theta$. If T or S satisfy condition ($\mathbf{G}_\alpha$), we have $\alpha(u, w) \geq 1$ and then by (3.4)

$$d(u, w) = d(Tu, Sw) \preceq \lambda d(u, w) + \mu(d(u, u) + d(w, w)) + v(d(u, w) + d(w, u)).$$

Thus $d(u, w) \preceq \theta$ and so $u = w$. If T or S satisfy condition ($\mathbf{G}'_\alpha$), the proof follows in a similar way.

Taking $\lambda = \theta$ and $v = \theta$ in Theorems 3.20 and 3.21 we have the following :

Corollary 3.22. *[Generalised α-admissible Kannan type contraction] Let $T, S\colon \chi^d \to \chi^d$ and $\alpha : \chi^d \times \chi^q \to [0, \infty)$ be mappings such that conditions (i), (ii), (iii), (iv) of Theorem 3.20, (iia) or (iib) of Theorem 3.21 and the following hold :*
there exist $\mu, \in$ P such that μ commutes s, $2r(\mu) < 1$ and for all $x, y \in \chi^d$ with $\alpha(x, y) \geq 1$

$$d(Sx, Ty) \preceq \mu(d(x, Sx) + d(y, Ty)) \tag{3.6}$$

$$d(Tx, Sy) \preceq \mu(d(x, Tx) + d(y, Sy)). \tag{3.7}$$

Then the iterative sequence (3.1) *converges to a unique common fixed point of S and T.*

Taking $\lambda = \theta$ and $\mu = \theta$ in Theorems 3.20 and 3.21 we have :

Corollary 3.23. *[Generalised α-admissible Chatterjee type contraction] Let $T, S\colon \chi^d \to \chi^d$ and $\alpha : \chi^d \times \chi^d \to [0, \infty)$ be mappings such that conditions (i), (ii), (iii), (iv) of Theorem 3.20, (iia) or (iib) of Theorem 3.21 and the following hold :*
there exist $v \in$ P such that v commutes s, $4r(sv) < 1$ and for all $x, y \in \chi^d$ with $\alpha(x, y) \geq 1$

$$d(Sx, Ty) \preceq v(d(x, Ty) + d(y, Sx)) \tag{3.8}$$

$$d(Tx, Sy) \preceq \mu(d(x, Sy) + d(y, Tx)). \tag{3.9}$$

Then the iterative sequence (3.1) *converges to the unique $u \in Fix\{T, S\}$.*

Taking $v = \theta$ in Theorems 3.20 and 3.21 we have the following :

Corollary 3.24. *[Generalised α-admissible Riech type contraction] Let $T, S\colon \chi^d \to \chi^d$ and $\alpha : \chi^d \times \chi^d \to [0, \infty)$ be mappings such that conditions (i), (ii), (iii), (iv) of Theorem 3.20, (iia) or (iib) of Theorem 3.21 and the following hold :*
there exist $\lambda, \mu \in \mathrm{P}$ such that v commutes s, $r(\lambda) + 2r(\mu) < 1$ and for all $x, y \in \chi^d$ with $\alpha(x, y) \geq 1$

$$d(Sx, Ty) \preceq \lambda d(x, y) + \mu(d(x, Sx) + d(y, Ty)) \tag{3.10}$$

$$d(Tx, Sy) \preceq \lambda d(x, y) + \mu(d(x, Tx) + d(y, Sy)). \tag{3.11}$$

Then the iterative sequence (3.1) converges to the unique $u \in Fix\{T, S\}$.

Remark 3.25. *If α is a symmetric function, that is $\alpha(x, y) = \alpha(y, x)$ for all $x, y \in \chi^q$, then we require only either of the conditions (3.3) or (3.4), (3.6) or (3.7), (3.8) or (3.9) and (3.10) or (3.11) in Theorems 3.20, 3.21, 3.22 and 3.23 respectively.*

Corollary 3.26. *Let $T, S\colon \chi^d \to \chi^d$. If there exist $\lambda, \mu, v \in \mathrm{P}$ such that s, λ , μ and v commutes pairwise with each other, $r(\lambda + \mu + sv) + r(\mu + 3sv) < 1$ and for all $x, y \in \chi^d$ with*

$$d(Sx, Ty) \preceq \lambda d(x, y) + \mu(d(x, Sx) + d(y, Ty)) + v(d(x, Ty) + d(y, Sx)). \tag{3.12}$$

Then the iterative sequence (3.1) converges to the unique $u \in Fix\{T, S\}$.

Proof : The proof follows from Theorems 3.20 and 3.21 by taking $\alpha(x, y) = 1$ for $x, y \in \chi^q$.

Our next result in $dCbMS - BA$ is an extension and proper generalisations of some recent results in $CbMS - BA$.

Theorem 3.27. *Let $R, Q\colon \chi^d \to \chi^d$. Let $k_1, k_2, k_3, k_4, k_5 \in P$ with $r(k_1)+r(sk_2+sk_3+sk_4+sk_5)+r(2sk_4+2sk_5) < 1$. Suppose that k_1 and $sk_4 + sk_5$ commutes with $sk_2 + sk_3 + sk_4 + sk_5$ and $f, g : \chi^d \to \chi^d$ satisfy*

$$d(Rx, Ry) \preceq k_1 d(Qx, Qy) + k_2 d(Qx, Rx) + k_3 d(Qy, Ry) + k_4 d(Qx, Ry) + k_5 d(Qy, Rx) \tag{3.13}$$

for all $x, y \in \chi^d$. If $R(\chi^d) \subset Q(\chi^d)$, $Q(\chi^d)$ is a complete subspace of χ^d, and (R, Q) are weakly compatible pair of mappings, then $Fix\{R, Q\}$ is a singleton set.

Proof Let $x_0 \in \chi^d$ be an arbitrary point. Since $R(\chi^d) \subset Q(\chi^q)$, there exists an $x_1 \in \chi^q$ such that $Rx_0 = Qx_1$. By induction, a sequence Rx_n can be chosen such that $Rx_n = Qx_{n+1} (n = 0, 1, 2, \ldots)$. Thus, by (3.13), for any natural number n, on one hand, we have

$$\begin{aligned} d(Qx_{n+1}, Qx_n) =& d(Rx_n, Rx_{n-1}) \\ \preceq& k_1 d(Qx_n, Qx_{n-1}) + k_2 d(Rx_n, Qx_n) + k_3 d(Rx_{n-1}, Qx_{n-1}) \\ &+ k_4 d(Qx_n, Rx_{n-1}) + k_5 d(Rx_n, Qx_{n-1}) \\ \preceq& (k_1 + sk_3 + 2sk_4 + sk_5) d(Qx_n, Qx_{n-1}) + (sk_2 + sk_5) d(Qx_{n+1}, Qx_n), \end{aligned}$$

which implies that

$$(e - sk_2 - sk_5) d(Qx_{n+1}, Qx_n) \preceq (k_1 + sk_3 + 2sk_4 + sk_5) d(Qx_n, Qx_{n-1}). \tag{3.14}$$

On the other hand, we have

$$\begin{aligned} d(Qx_n, Qx_{n+1}) =& d(Rx_{n-1}, Rx_n) \\ \preceq& k_1 d(Qx_{n-1}, Qx_n) + k_2 d(Rx_{n-1}, Rx_{n-1}) + k_3 d(Rx_n, Qx_n) \end{aligned}$$

$$+ k_4 d(Qx_{n-1}, Rx_n) + k_5 d(Rx_{n-1}, Qx_n)$$
$$\preceq (k_1 + sk_2 + sk_4 + 2sk_5) d(Qx_{n-1}, Qx_n) + (sk_3 + sk_4) d(Qx_n, Qx_{n+1}),$$

which means that

$$(e - sk_3 - sk_4) d(Qx_{n+1}, Qx_n) \preceq (k_1 + sk_2 + sk_4 + 2sk_5) d(Qx_n, Qx_{n-1}). \tag{3.15}$$

Adding (3.14) and (3.15) we see that

$$(2e - sk_2 - ks_3 - sk_4 - sk_5) d(Qx_n, Qx_{n+1}) \preceq (2k_1 + sk_2 + sk_3 + sk_4 + sk_5 + 2(sk_4 + sk_5)) d(Qx_{n-1}, Qx_n)$$

Put $k = sk_2 + sk_3 + sk_4 + sk_5$, $k' = sk_4 + sk_5$. Then we get

$$(2e - k) d(Qx_n, Qx_{n+1}) \preceq (2k_1 + k + 2k') d(Qx_{n-1}, Qx_n). \tag{3.16}$$

Note that $r(k) < 1 < 2$ and so $(2e - k)$ is invertible and $(2e - k)^{-1} = \sum_{i=0}^{\infty} \frac{k^i}{2^{i+1}}$. Again since k_1 and k'commutes with k we have

$$(2e - k)^{-1}(2k_1 + k + 2k') = \Big(\sum_{i=0}^{\infty} \frac{k^i}{2^{i+1}}\Big)(2k_1 + k + 2k') = 2\Big(\sum_{i=0}^{\infty} \frac{k^i}{2^{i+1}}\Big)k_1 + \sum_{i=0}^{\infty} \frac{k^{i+1}}{2^{i+1}} + 2\Big(\sum_{i=0}^{\infty} \frac{k^i}{2^{i+1}}\Big)k'$$
$$= 2k_1\Big(\sum_{i=0}^{\infty} \frac{k^i}{2^{i+1}}\Big) + k\sum_{i=0}^{\infty} \frac{k^i}{2^{i+1}} + 2k'\Big(\sum_{i=0}^{\infty} \frac{k^i}{2^{i+1}}\Big) = (2k_1 + k + 2k')\Big(\sum_{i=0}^{\infty} \frac{k^i}{2^{i+1}}\Big) = (2k_1 + k)(2e - k)^{-1},$$

that is , $(2e - k)^{-1}$ commutes with $2k_1 + k + 2k'$. From (3.16) we have

$$d(Qx_n, Qx_{n+1}) \preceq \beta d(Qx_{n-1}, Qx_n) \tag{3.17}$$

where $\beta = (2k_1 + k + 2k')(2e - k)^{-1}$. Using Lemma 1.6 of [1] and Lemma 2.3 we have

$$r(\beta) = r\Big((2k_1 + k + 2k')(2e - k)^{-1}\Big) \le \frac{2r(k_1) + r(k) + 2r(k')}{2 - r(k)} < 1$$

hence by Lemma 3.19, $\{Qx_n\}$ is a Cauchy sequence in (χ^d, d). By completeness of $Q(\chi^d)$ we get $q \in Q(\chi^q)$ such that $Qx_n \to q$ $(n \to \infty)$ or in other words, there is a $p \in \chi^d$ satisfying $Qp = q$.

$$d(Qx_n, Rp) = d(Rx_{n-1}, Rp)$$
$$\preceq k_1 d(Qx_{n-1}, Qp) + k_2 d(Rx_{n-1}, Qx_{n-1}) + k_3 d(Rp, Qp)$$
$$+ k_4 d(Qx_{n-1}, Rp) + k_5 d(Rx_{n-1}, Qp)$$
$$\preceq k_1 d(Qx_{n-1}, q) + k_2 d(Qx_n, Qx_{n-1}) + sk_3 d(Qx_n, Rp)$$
$$+ sk_3 d(Qx_n, q) + sk_4[d(Qx_{n-1}, Qx_n) + d(Qx_n, Rp)] + k_5 d(Qx_n, q)$$

which implies that

$$(e - sk_3 - sk_4) d(Qx_n, Rp) \preceq k_1 d(Qx_{n-1}, q) + (sk_3 + k_5) d(Qx_n, q) + (k_2 + sk_4) d(Qx_n, Qx_{n-1}). \tag{3.18}$$

Also we have ,

$$d(Qx_n, Rp) = d(Rx_{n-1}, Rp) = d(Rp, Rx_{n-1})$$
$$\preceq k_1 d(Qp, Qx_{n-1}) + k_2 d(Rp, Qp) + k_3 d(Rx_{n-1}, Qx_{n-1})$$
$$+ k_4 d(Qp, Rx_{n-1}) + k_5 d(Rp, Qx_{n-1})$$
$$\preceq k_1 d(Qx_{n-1}, q) + sk_2 d(Qx_n, Rp) + sk_2 d(Qx_n, q)$$
$$+ k_3 d(Qx_n, Qx_{n-1}) + k_4 d(Qx_n, q)$$
$$+ sk_5[d(Rp, Qx_n) + d(Qx_n, Qx_{n-1})]$$

which implies that

$$(e - sk_2 - sk_5)d(Qx_n, Rp) \preceq k_1 d(Qx_{n-1}, q) + (sk_2 + k_4)d(Qx_n, q) + (k_3 + sk_5)d(Qx_n, Qx_{n-1}). \tag{3.19}$$

Adding (3.18) and (3.19) we have

$$(2e - k)d(Qx_n, Rp) \preceq 2k_1 d(Qx_{n-1}, q) + (sk_2 + sk_3 + k_4 + k_5)d(Qx_n, q) + (k_2 + k_3 + sk_4 + sk_5)d(Qx_n, Qx_{n-1}). \tag{3.20}$$

Again $r(k) < 1 < 2$ and so $2e - k$ is invertible. Thus we have from (3.20)

$$d(Qx_n, Rp) \preceq 2k_1(2e - k)^{-1}d(Qx_{n-1}, q) + (sk_2 + sk_3 + k_4 + k_5)(2e - k)^{-1}d(Qx_n, q) + (k_2 + k_3 + sk_4 + sk_5)(2e - k)^{-1}d(Qx_n, Qx_{n-1}). \tag{3.21}$$

Using Proposition 3.9 and Lemma 2.5 in (3.21) we get $Qx_n \to Rp$ $(as\ n \to \infty)$. Next we will prove that $Rp = Qp$. Using (3.13) we have

$$\begin{aligned} d(Qp, Rp) &\preceq sd(Qp, Qx_{n+1}) + sd(Rx_n, Rp) \\ &\preceq d(Qp, Qx_{n+1}) + sk_1 d(Qx_n, Qp) + sk_2 d(Rx_n, Rx_{n-1}) + s^2 k_3 d(Rp, Qx_n) \\ &\quad + s^2 k_3 d(Qx_n, Qp) + sk_4 d(Qx_n, Rp) + k_5 d(Qx_{n+1}, Qp). \end{aligned}$$

Using Proposition 3.9 and Lemma 2.5 we get $d(Qp, Rp) = \theta$. Hence $Rp = Qp = q$.
Next we show that if $p \in CP\{R, Q\}$ then $d(Rp, Qp) = \theta$. We have

$$\begin{aligned} d(Qp, Rp) &\preceq d(Rp, Rp) \\ &\preceq k_1 d(Qp, Qp) + k_2 d(Rp, Qp) + k_3 d(Rp, Qp) + k_4 d(Qp, Rp) + k_5 d(Rp, Qp) \\ &\preceq Ld(Qp, Rp) \preceq \cdots L^n d(Qp, Rp), \end{aligned}$$

where $L = k_1 + sk_2 + sk_3 + sk_4 + sk_5$. Note that $r(L) < 1$ and so by Lemma 2.5 and 2.6, we see that $L^n.d(Qp, Rp)$ is a c-sequence and thus $d(Qp, Rp) = \theta$.
Now we will show that if $q, q' \in POC\{R, Q\}$ then $q = q'$. suppose $Rp = Qp = q$ and $Rp' = Qp' = q'$. Then we have

$$\begin{aligned} d(q', q) &= d(Rp', Rp) \\ &\preceq k_1 d(Qp', Qp) + k_2 d(Rp', Qp') + k_3 d(Rp, Qp) + k_4 d(Qp', Rp) + k_5 d(Rp', Qp) \\ &= (k_1 + 2sk_2 + 2sk_3 + k_4 + k_5)d(Qp', Qp) \preceq (k_1 + 2sk_2 + 2sk_3 + 2sk_4 + 2sk_5)d(q', q). \end{aligned}$$

Let $\alpha = k_1 + 2sk_2 + 2sk_3 + 2sk_4 + 2sk_5$, then it follows that

$$d(q', q) \preceq \alpha d(q', q) \preceq \cdots \preceq \alpha^n d(q', q). \tag{3.22}$$

Note that $r(\alpha) \leq r(k_1) + 2r(k') < 1$. Thus by Lemma 2.5 and 2.6, we see that $\alpha^n.d(q', q)$ is a c-sequence and thus $d(q', q) = \theta$, that is, $q' = q$ and so R and Q has a unique point of coincidence.

Existence of the unique common fixed point follows from Lemma 1.8 of [1].

Remark 3.28. *Theorem 3.27 is an improved version of Theorem 2.9 of [1], in the sense that we have given an increased range for the Lipschitz constants.*

Taking $k_1 = k$ and $k_2 = k_3 = k_4 = k_5 = \theta$ in Theorem 3.27 we get

Corollary 3.29. *(Jungck contraction principle in dCbMS-BA) Theorem 3.27 with $k \in P$, $r(k) < 1$ and*

$$d(Rx, Ry) \preceq kd(Qx, Qy)$$

Remark 3.30. *Corollary 3.29 is a proper extension of Theorem of [10] and Corollary 2.10 of [1].*

Taking Q to be the identity mapping in Corollary 3.29 we get :

Corollary 3.31. *(Banach contraction principle in dCbMS-BA) Corollary 3.29 with $k \in P$, $r(k) < 1$*

$$d(Rx, Ry) \preceq kd(x, y)$$

Taking $k_2 = k_3 = k$ and $k_1 = k_4 = k_5 = \theta$ in Theorem 3.27 we get

Corollary 3.32. *Theorem 3.27 with $k \in P$, $2r(sk) < 1$. and*

$$d(Rx, Ry) \preceq k\Big(d(Rx, Qx) + d(Ry, Qy)\Big)$$

Taking $k_4 = k_5 = k$ and $k_1 = k_3 = k_4 = \theta$ in Theorem 3.27 we get

Corollary 3.33. *Theorem 3.27 with $k \in P$, $3r(sk) < 1$ and*

$$d(Rx, Ry) \preceq k\Big(d(Qx, Ry) + d(Rx, Qy)\Big)$$

3.3 Fixed point theorems in dislocated quasi cone b-metric space

Now we further give two more new results as follows :

Theorem 3.34. *Let $(\chi^q, d, \mathcal{A})$ be a $L-complete$ or $R-complete$ $dqCbMS-\mathcal{A}$ with coefficient s ($e \preceq s$). Let $R, Q\colon \chi \to \chi$ and $\alpha : \chi^q \times \chi^q \to [0, \infty)$ be mappings satisfying conditions (i), (ii), (iii) and (iv) of Theorem 3.20. If there exist $\lambda, \mu, \nu \in$ P such that $\lambda + \mu + s\nu$ commutes with $\mu + 3s\nu$, $(r(\lambda) + r(2\mu + 2s\nu)) + r(2\mu + 6s\nu) < 1$ and for all $x, y \in \chi^q$ with $\alpha(x, y) \geq 1$*

$$d(Qx, Ry) \preceq \lambda d(x, y) + \mu(d(x, Qx) + d(y, Ry)) + \nu(d(x, Ry) + d(y, Qx)) \tag{3.23}$$

and

$$d(Rx, Qy) \preceq \lambda d(x, y) + \mu(d(x, Rx) + d(y, Qy)) + \nu(d(x, Qy) + d(y, Rx)) \tag{3.24}$$

then $Fix\{R, Q\} \neq \phi$. Further if (R, Q) is α-identical or if (R, Q) is α-dominated on χ^q , then $d(u, u) = \theta$ for any $u \in Fix\{R, Q\}$.

Proof : Define $d^{\dagger}(x, y) = \frac{d(x,y)+d(y,x)}{2}$. Then by propositions 3.7 and 3.10, $(\chi, d^{\dagger})$ is a complete $dCbMS-\mathcal{A}$. We will show that the pair (R, Q) satisfies equation 3.4. Let $\alpha(x, y) \geq 1$. Then

$$\begin{aligned}
d^{\dagger}(Qx, Ry) =& \frac{d(Qx, Ry) + d(Ry, Qx)}{2}\\
\preceq& \frac{\lambda(d(x, y) + d(y, x)) + 2\mu(d(x, Qx) + d(y, Ry)) + 2\nu(d(x, Ry) + d(y, Qx))}{2}\\
\preceq& \frac{\lambda(d(x, y) + d(y, x)) + 2\mu(d(x, Qx) + d(Qx, x) + d(y, Ry) + d(Ry, y))}{2}\\
&+ \frac{2\nu(d(x, Ry) + d(Ry, x) + d(y, Qx) + d(Qx, y))}{2}\\
=&\lambda d^{\dagger}(x, y) + 2\mu(d^{\dagger}(x, Qx) + d^{\dagger}(y, Ry)) + 2\nu(d^{\dagger}(x, Ry) + d^{\dagger}(y, Qx)).
\end{aligned}$$

Thus, the pair (R, Q) satifies the conditions of Theorem 3.20 and hence the result follows.

Theorem 3.35. *Let $(\chi^q, d, \mathcal{A})$ be a $dqCbMS-\mathcal{A}$ with coefficient s ($e \preceq s$) and $R, Q\colon \chi^q \to \chi^q$. Let $\alpha, \beta, \gamma \in P$ with $r(\alpha)+r(2s\beta+2s\gamma)+r(4s\gamma) < 1$. Suppose that α and $s\gamma$ commutes with $s\beta+s\gamma$ and the mappings $R, Q : X \to X$ satisfy*

$$d(Rx, Ry) \preceq \alpha d(Qx, Qy) + \beta(d(Rx, Qx) + d(Qy, Ry)) + \gamma(d(Qx, Ry) + d(Rx, Qy)) \tag{3.25}$$

for all $x, y \in \chi^q$. If $R(\chi^q) \subset Q(\chi^q)$, $Q(\chi^q)$ is a $L-complete$ or $R-complete$ subspace of χ^q, and (R, Q) is weakly compatible pair of mappings, then $Fix\{R, Q\}$ is a singleton set.

Proof : Proceeding on the same lines as in the proof of Theorem 3.34 we see that the pair satisfies all conditions of Theorem 3.27 with $k_1 = \alpha$, $k_2 = k_3 = \beta$ and $k_4 = k_5 = \gamma$ and hence the result follows.

Example 3.36. *Let $\mathcal{A}$ be the Banach algebra and (χ, d_{lc}) be the dqCbMS over $\mathcal{A}$ given in example 3.5. Let $T : \chi \to \chi$ be given by*

$$Tx = \begin{cases} log(1 + \frac{x}{3}) & if\, x \in [0, 1] \\ log(1 + 10x) & otherwise \end{cases}$$

$$Sx = \begin{cases} log(1 + \frac{x}{3}) & if\, x \in [0, 1] \\ log(1 + 100x) & otherwise \end{cases}$$

and

$$\alpha(x, y) = \begin{cases} 1 & (x, y) \in [0, 1] \; or\; x = y \\ 0 & otherwise \end{cases}$$

*Then α is a triangular function, (T, S) is α-admissible, weak semi α-admissible, α-identical and satisfy condition (**G**). Also for all $\alpha(x, y) \geq 1$ we see that $d_{lc}(Tx, Sy) \preceq \lambda.d_{lc}(x, y)$ where $\lambda = \frac{1}{9} + 0.t + 0.t^2 + \cdots 0.t^n$, i.e. inequality 3.3 and 3.4 is satsfied with $\lambda = \frac{1}{9} + 0.t + 0.t^2 + \cdots 0.t^n$, $\mu = \nu = (0, 0)$. Thus T and S satisfy all conditions of Theorems 3.21 and 3.22. Further 0 is a unique common fixed point of T and S.*

Example 3.37. *Let $X = [0, 1]$ and $\mathcal{A}$ be as in example 3.5. Define a mapping $d : X \times X \to \mathcal{A}$ by*

$$d(x, y)(t) = |x - y|^2 + |x - y|^2.t + |x - y|^2.t^2 + \cdots |x - y|^2.t^n.$$

Then (X, d) is a complete CbMS–BA over Banach algebra $\mathcal{A}$ with the coefficient $s = 2$ and hence a dCbMS–BA. Consider the mappings $f, g : X \to X$ by

$$f(x) = \frac{2x}{5}, \quad g(x) = \frac{x}{2}.$$

It easy to verify that f and g satisfy the conditions of Theorem 3.27 and Corollary 3.29 with $k_1 = k = \frac{3}{4}$ and $k_2 = k_3 = k_4 = k_5 = 0$. It is also easy to check that f and g do not satisfy the conditins of Theorem 2.9 and Corollary 2.10 of [1] at $x = 0, y = 1$. However, 0 is the unique common fixed point of f and g.

Remark 3.38. *(An open problem) In [20] the authors introduced generalised $\alpha - \eta - \psi$ Geraghty contractive mappings and proved fixed point theorems for such mappings in a partial b-metric space. Note that every partial b-metric space is a dislocated quasi cone b-metric space over a Banach algebra but the converse is not necessarily true. Thus it will be interesting to define generalised $\alpha - \eta - \psi$ contractive pair of mappings and prove common fixed point theorems for such mappings in a dislocated quasi cone b-metric space over a Banach algebra.*

4 Applications

In this section, we shall apply the obtained results to deal with the existence and uniqueness of solution for some equations.

Consider the following system of m-tupled equations:

$$\begin{cases} F_1(x_1, x_2, ...x_m) = 0 \\ F_2(x_1, x_2, ...x_m) = 0 \\ . \\ . \\ . \\ F_m(x_1, x_2, ...x_m) = 0 \end{cases} \tag{4.0}$$

where $F_i : \mathbb{R}^m \to \mathbb{R}$, $i = 1 \ldots m$ are mappings.

We will analyse (4.0) under the following conditions :

For $k \in \mathbb{R}$ and for all $x = (x_1, x_2, \cdots x_m)$, $y = (y_1, y_2, \cdots y_m) \in \mathbb{R}^m$

$$|F_i(x) - F_i(y) + x_i - y_i| + |F_i(x) + x_i| + |F_i(y) + y_i| \leq k[|x_i - y_i| + |x_i| + |y_i|], i = 1 \cdots m \text{ and } 0 < k < 1, \tag{4.1}$$

$$|F_i(x) + x_i - y_i| + |F_i(x) + x_i| + |y_i| \leq k[|x_i - y_i| + |x_i| + |y_i|], i = 1 \cdots m \text{ and } 0 < k < 1, \tag{4.2}$$

$$|F_i(x) + x_i - y_i| + |F_i(x) + x_i| + |y_i| \leq k[|F_i(x)| + |F_i(x) + x_i| + |x_i| + 2|y_i|], i = 1 \cdots m \text{ and } 0 < k < \frac{1}{2}, \tag{4.3}$$

$$|F_i(x) + x_i - y_i| + |F_i(x) + x_i| + |y_i| \leq k[|F_i(x) + x_i - y_i| + |F_i(x) + x_i| + |x_i - y_i| + |y_i|], i = 1 \cdots m \text{ and } 0 < k < \frac{1}{4}, \tag{4.4}$$

Theorem 4.1. *If there exists $k \in R^+$ such that (4.1) or (4.2) or (4.3) or (4.4) is satisfied, then the system of n-tupled equations (4.0) has a unique common solution in $\mathbb{R}^m$.*

Proof Let $\mathcal{A} = \mathbb{R}^m$ with the norm $\|(u_1, u_2, \cdots u_m)\| = |u_1| + |u_2| + \cdots |u_m|$ and the multiplication given by

$$uv = (u_1, u_2 \cdots u_m)(v_1, v_2 \cdots v_m) = (u_1 v_1, u_1 v_2 + u_2 v_1, \cdots, u_1 v_m + u_2 v_{m-1} + \cdots u_m v_1).$$

Let $P = \{u = (u_1, u_2 \cdots u_m) \in \mathcal{A} : u_1, u_2 \cdots u_m \geq 0\}$. Let $X = \mathbb{R}^m$ and $d : X \times X \to \mathcal{A}$ be given by

$$d((x_1, x_2 \cdots x_m), (y_1, y_2 \cdots y_m)) = (|x_1 - y_1| + |x_1| + |y_1|, |x_2 - y_2| + |x_2| + |y_2|, \cdots |x_m - y_m| + |x_m| + |y_m|).$$

Then (X, d) is a complete $dCbMS - BA$ over $\mathcal{A}$ with the coefficient $s = (1, 0, \ldots, 0)$.

Consider $S, T : X \to X$ given by

$$S(x_1, x_2, \cdots, x_m) = (x_1, x_2, \cdots, x_m)$$

$$T((x_1, x_2, \cdots, x_m) = (F_1((x_1, x_2, \cdots, x_m) + x_1, F_2((x_1, x_2, \cdots, x_m) + x_2, \cdots, F_m((x_1, x_2, \cdots, x_m) + x_m).$$

Then for all $x = (x_1, x_2, \cdots, x_m)$ and $y = (y_1, y_2, \cdots, y_m)$ we have

$$\begin{aligned} d(T(x), T(y)) &= d\big((F_1((x) + x_1, F_2((x) + x_2, \cdots F_m((x) + x_m, (F_1((y) + y_1, F_2((y) + y_2, \cdots, F_m((y) + y_m)\big) \\ &= \big(|F_1(x) - F_1(y) + x_1 - y_1| + |F_1(x) + x_1| + |F_1(y) + y_1|, \\ &\quad |F_2(x) - F_2(y) + x_2 - y_2| + |F_2(x) + x_2| + |F_2(y) + y_2|, \\ &\quad \cdots, |F_m(x) - F_m(y) + x_m - y_m| + |F_m(x) + x_m| + |F_m(y) + y_m|\big). \end{aligned}$$

If (4.1) is satisfied we get

$$\begin{aligned} d(T(x), T(y)) &\preceq \big(k(|x_1 - y_1| + |x_1| + |y_1|), k(|x_2 - y_2| + |x_2| + |y_2|), \ldots, k(|x_m - y_m| + |x_m| + |y_m|)\big) \\ &\preceq (k, 1, \ldots, 1)(|x_1 - y_1| + |x_1| + |y_1|, |x_2 - y_2| + |x_2| + |y_2|, \ldots, |x_m - y_m| + |x_m| + |y_m|) \\ &= (k, 1, \ldots, 1) d(Sx, Sy). \end{aligned}$$

Note that for any $k \in R^+$ and $(k, 1, \ldots, 1) \in R^m$

$$\begin{aligned} \|(k, 1, \ldots, 1)^n\|^{\frac{1}{n}} &= \|(k^n, P_1(n)k^n, P_2(n)k^n, \ldots, P_{m-1}k^n)\|^{\frac{1}{n}}, \quad \text{where } P_i(n) \text{is a polynomial of degree } i \text{ in } n \\ &= (k^n + P_1(n)k^n + P_2(n)k^n + \ldots + P_{m-1}k^n)^{\frac{1}{n}} \to k \ (as \ n \to \infty), \end{aligned}$$

i.e $r(k, 1, \ldots, 1) = k < 1$. Now choose $\lambda = (k, 1, \ldots, 1)$, then all conditions of Corollary 3.29 are satisfied. Hence, by Corollary 3.29 T has a unique common fixed point in X. In other words, the m-tupled equations

(4.0) has a unique common solution in $\mathbb{R}^m$.
Again we have

$$\begin{aligned} d(T(x), S(y)) &= d\big((F_1((x) + x_1, F_2((x) + x_2, \cdots, F_m((x) + x_m), (y_1, y_2, \cdots, y_m)\big) \\ &= \big(|F_1(x) + x_1 - y_1| + |F_1(x) + x_1| + |y_1|, |F_2(x) + x_2 - y_2| + |F_2(x) + x_2| + |y_2|, \\ &\quad \cdots, |F_m(x) + x_m - y_m| + |F_m(x) + x_n| + |y_m|\big). \end{aligned}$$

If (4.2) is satisfied we get

$$\begin{aligned} d(T(x), S(y)) &\preceq \big(k(|x_1 - y_1| + |x_1| + |y_1|), k(|x_2 - y_2| + |x_2| + |y_2|), \dots, k(|x_m - y_m| + |x_m| + |y_m|)\big) \\ &\preceq (k, 1, \dots, 1)(|x_1 - y_1| + |x_1| + |y_1|, |x_2 - y_2| + |x_2| + |y_2|, \dots, |x_m - y_m| + |x_m| + |y_m|) \\ &= (k, 1, \dots, 1)d(x, y). \end{aligned}$$

Now $r((k, 1, \dots, 1)) = k < 1$. Choose $\lambda = (k, 1, \dots, 1)$, $\mu = \theta$ and $\nu = \theta$. Then all conditions of Theorem 3.20 are satisfied. Hence, by Theorem 3.20, T has a unique common fixed point in X.

If (4.3) is satisfied we get

$$\begin{aligned} d(T(x), S(y)) &\preceq \big(k(|F_1(x)| + |F_1(x) + x_1| + |x_1| + 2|y_1|), k(|F_2(x)| + |F_2(x) + x_2| + |x_2| + 2|y_2|), \\ &\quad \cdots, k(|F_m(x)| + |F_m(x) + x_m| + |x_m| + 2|y_m|)\big) \\ &\preceq (k, 1, \dots, 1)(|F_1(x)| + |F_1(x) + x_1| + |x_1| + 2|y_1|, |F_2(x)| + |F_2(x) + x_2| + |x_2| + 2|y_2|, \\ &\quad \cdots, |F_m(x)| + |F_m(x) + x_m| + |x_m| + 2|y_m|) \\ &= (k, 1, \dots, 1)\big(d(x, Tx) + d(y, Sy)\big). \end{aligned}$$

Now $r((k, 1, \dots, 1)) = k < \frac{1}{2}$. Choose $\mu = (k, 1, \dots, 1)$. Then all conditions of Theorem 3.22 are satisfied. Hence, by Theorem 3.22, T has a unique common fixed point in X. If (4.4) is satisfied we get

$$\begin{aligned} d(T(x), S(y)) &\preceq \big(k(|F_1(x) + x_1 - y_1| + |F_1(x) + x_1| + |x_1 - y_1| + |y_1|), \\ &\quad k(|F_2(x) + x_2 - y_2| + |F_2(x) + x_2| + |x_2 - y_2| + |y_2|), \\ &\quad \cdots k(|F_m(x) + x_m - y_m| + |F_m(x) + x_m| + |x_m - y_m| + |y_m|)\big) \\ &\preceq (k, 1, \dots, 1)(|F_1(x) + x_1 - y_1| + |F_1(x) + x_1| + |x_1 - y_1| + |y_1|, \\ &\quad |F_2(x) + x_2 - y_2| + |F_2(x) + x_2| + |x_2 - y_2| + |y_2|, \\ &\quad \cdots, |F_m(x) + x_m - y_m| + |F_m(x) + x_m| + |x_m - y_m| + |y_m|) \\ &= (k, 1, \dots, 1)\big(d(x, Sy) + d(y, Tx)\big). \end{aligned}$$

Now $r((k, 1, \dots, 1))k < \frac{1}{4}$. Choose $\nu = (k, 1, \dots, 1)$. Then all conditions of Theorem 3.23 are satisfied. Hence, by Theorem 3.23, T has a unique common fixed point in X.

Acknowledgement: This project is supported by Deanship of Scientific Research at Prince Sattam bin Abdulaziz University, Al kharj, Kingdom of Saudi Arabia, under International Project Grant No. 2016/01/6714.
The authors are thankful to the learned reviewers for their valuable comments which helped in improving this paper to its present form.

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