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The non-commuting graph of a non-central hypergroup

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Abstract: The aim of this paper is to construct and study the properties of a certain graph associated with a non-central hypergroup, i.e. a hypergroup having non-commutative the associated fundamental group. The method of the construction of the graph is similar to that one proposed by Paul Erdős, when he defined a graph associated with a non-commutative group. We establish necessary and /or sufficient conditions for the associated graph to be Hamiltonian or planar.

Keywords: (semi)hypergroup, Hamiltonian graph, planar graph

MSC: 05C25, 05C10, 20N20

1 Introduction

It is much simpler to deal with a concrete problem, to find its solution if we can somehow graphically represent it, or at least part of it. As written in the book written by Bondy and Murty [1], "many real-world situations can conveniently be described by means of a diagram consisting of a set of points together with lines joining certain pairs of these points". This is the intuitive definition of the concept of *graph*, born in 1736 when Euler was asked to find a nice path across the seven Köningsberg bridges (for more details see Sudakov [2]). An Eulerian path crosses over each of the seven bridges exactly once. Since then graph theory has established itself as a fundamental branch of mathematics, as an important tool for solving theoretical and practical problems in combinatorics, operational research, chemistry, genetics, geography, architecture and engineering, etc. In recent years, several connections in both directions between graphs and algebraic hyperstructures (in particular hypergroupoids, hypergroups or hypergraphs) have been established and developed. On one side, paths in graphs define different hyperoperations (i.e. functions that associate with any pair of elements of a nonempty set a subset of the support set) on the set of their vertices, leading to the association of algebraic hyperstructures with graphs. This direction was considered, for example, by Massouros [3], Kalampakas and Spartalis [4–6], Rosenberg [7], Golmohamadian and Zahedi [8]. The aim of these works is to determine necessary and/or sufficient conditions such that the associated hyperstructure satisfies determined properties such as associativity, reproducibility, transposition, commutativity, etc. Another problem discussed by these articles concerns the computations of the number of the associated hyperstructures.

On the other side, several studies have been conducted in the other direction of the connection between graphs and hyperstructures. This time the starting object is a hypergroupoid (a hypergroup or a hypergraph) and the result is a graph. In a recently published article (Hamidi et al. [9]), the authors define and compute

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the number of graphs obtained from hypergraphs. In this note we propose a construction of a graph starting from a non-central hypergroup, i.e. a finite hypergroup having the associated fundamental group non-commutative. A similar method was considered in 1975 by Paul Erdős (as mentioned in Neumann [10]), who defined a graph in the following way. Let G be a non-abelian group and take the set of vertices $V = G \setminus Z(G)$, where $Z(G) = \{x \in G \mid xy = yx, \forall y \in G\}$ is the set of the elements of G commuting with all elements in G, i.e. the center of G. Then two vertices x and y are joined whenever $xy \neq yx$. Extending this construction to the hypergroups framework, instead of a non-abelian group, we will consider a *non-central hypergroup*, i.e. the associated fundamental group H/β is non-abelian. In other words, the set of the vertices is $T_H = \{x \in H \mid \exists y \in H : x \circ y \circ \omega_H \neq y \circ x \circ \omega_H\}$ and join two vertices x and y whenever $x \circ y \circ \omega_H \neq y \circ x \circ \omega_H$, where $\omega_H = \{x \in H \mid \beta(x)$ is the identity of the group $H/\beta\}$ is the core (heart) of the hypergroup $(H, \circ)$. We establish necessary and/or sufficient conditions for the associated graph to be Hamiltonian or planar.

2 Preliminaries

We recall here some basic notions of graph theory (connected with Hamiltonian and planar graphs) and hypergroup theory, and we fix the notations used in this note. For the first theory we referee the readers to the fundamental book by Bondy and Murty [1] (from which we stated all the notions and results in the first subsection), while surveys of the theory of hyperstructures can be found in the books written by Corsini [11], Davvaz and Leoreanu-Fotea [12], Corsini and Leoreanu [13] and Vougiouklis [14].

2.1 Hamiltonian and planar graphs

A *graph* G is a pair $G = (V, E)$, where V is a set of vertices and E is a (multi)set of unordered pairs of vertices, called edges. We write $V(G)$ for the set of vertices and $E(G)$ for the set of edges of a graph G. A *loop* is an edge (v, v) for some $v \in V$, so an edge that connects a vertex v to itself. An edge $e = (u, v)$ is a *multiple edge* if it appears multiple times in E. A graph is *simple* if it has no loops or multiple edges. If $e = (u, v)$ is an edge of a graph G, then we say that u and v are *adjacent* in G and that e joins u and v.

In this paper we will only consider finite simple graphs.

Definition 2.1. Given a graph $G = (V, E)$ and a vertex $v \in V$, we define the *degree* $deg(v)$ of v to be the number of all its adjacent vertices. A vertex v is *isolated* if $deg(v) = 0$. The minimum degree of G is denoted by $\delta(G)$.

Proposition 2.2. *For every graph* $G = (V, E)$, $\sum_{v\in V} deg(v) = 2|E|$.

Definition 2.3. A graph $H = (U, F)$ is a *subgraph* of a graph $G = (V, E)$ if $U \subseteq V$ and $F \subseteq E$.

Definition 2.4. Given $G = (V, E)$ and $\emptyset \neq U \subseteq V$, let $\ll U \gg$ denote the graph with vertex set U and edge set $E(\ll U \gg) = \{e \in E(G) \mid e \subseteq U\}$ (we include all the edges of G which have both endpoints in U). Then $\ll U \gg$ is called the *subgraph of G induced by U*.

Definition 2.5. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs. An *isomorphism* $\phi : G_1 \longrightarrow G_2$ is a bijection from V_1 to V_2 such that $(u, v) \in E_1$ if and only if $(\phi(u), \phi(v)) \in E_2$. We say that G_1 is isomorphic with G_2 if there is an isomorphism between them.

Definition 2.6. A *walk* in a graph G is a sequence of vertices $v_1, v_2, \dots, v_n$ and a sequence of edges $(v_i, v_{i+1}) \in E(G)$. A walk is a *path* if all vertices v_i are distinct. The *length* of a walk is the number of edges in it. A *connected graph* is a graph having a path between every pair of vertices, i.e. there are no unreachable (isolated) vertices.

Definition 2.7. A *circuit* in a graph G is a path that begins and ends at the same vertex, i.e. a closed path. A *Hamiltonian circuit* is a closed path that visits every vertex in the graph exactly once. A graph is Hamiltonian if it has a Hamiltonian circuit.

Definition 2.8. Let $G = (V, E)$ be a graph and $x = v_1, v_2, \dots, v_n = y$ be a path between two vertices x and y in G. We mean by $d(x, y)$ the minimum length of all paths from x to y. If there are no walks between x and y, let $d(x, y) = \infty$ by convention.

It is clear that a graph G is connected if $d(x, y) < \infty$ for all $x, y \in V$. For a connected simple finite graph G define the *diameter* of G as $diam(G) = \max\{d(x, y) \mid x, y \in V(G), x \neq y\}$.

Theorem 2.9. Dirac's theorem *If $G = (V, E)$ is a simple graph with $n \geq 3$ vertices and if $\delta(G) \geq \frac{n}{2}$, then G is a Hamiltonian graph.*

In the following we recall two types of graphs, that we will use in the next section. A simple graph that contains every possible edge between all the vertices is called a *complete graph*. A complete graph with n vertices is denoted as K_n.

A graph $G = (V, E)$ is *bipartite* if its vertex set V can be partitioned into two sets X and Y in such a way that every edge of G has one end vertex in X and the other one in Y. In this case, X and Y are called the partite sets. A bipartite graph with partite sets X and Y is called a *complete bipartite graph* if the graph contains exactly all edges that have one end vertex in X and the other end vertex in Y. If there are n vertices in X and m vertices in Y, we denote it as $K_{n,m}$. Usually $K_{n,m}$ and $K_{m,n}$ are considered to be the same.

It is obvious that each K_n is a Hamiltonian graph whenever $n \geq 3$, while $K_{n,m}$ is a Hamiltonian graph if and only if $n = m \geq 2$.

The last part of the short overview on graphs is dedicated to the *planar graphs*, i.e. those graphs isomorphic with a *plane graph*, that is a graph drawn on the plane without edge crossing. For example, K_5 and $K_{3,3}$ are not planar graphs, while K_4 is a planar graph.

Theorem 2.10. *For a simple connected planar graph with $n_v \geq 3$ vertices and n_e edges there is $n_e \leq 3n_v - 6$.*

As a corollary, we get the following result.

Proposition 2.11. *A simple connected planar graph with $n_v \geq 3$ vertices has a vertex of degree five or less, i.e. $\delta(G) \leq 5$.*

Definition 2.12. A planar graph $G = (V, E)$ is called *maximal planar* if, for every pair u, v of non-adjacent vertices of G, the graph $(V, E \cup \{(u, v)\})$ is nonplanar.

Such a graph is also called *triangulated* since all the faces are triangles. Every planar graph is a subgraph of a maximal planar graph.

Theorem 2.13. *If G is a maximal planar graph with n_v vertices and n_e edges, then $n_e = 3n_v - 6$.*

Theorem 2.14. *Let G be a maximal planar graph with $n_v \geq 4$ vertices and diameter $k = diam(G)$. Let n_i denote the number of vertices of degree i in G, for $i = 3, 4, \dots, k$. Then $3n_3 + 2n_4 + n_5 = 12 + n_7 + 2n_8 + \dots + (k-6)n_k$.*

One graph is *homeomorphic* to another one if we can turn one into the other by adding or removing degree-two vertices.

Theorem 2.15. *(Kuratowski's theorem). A graph is non-planar if and only if it contains a subgraph homeomorphic to K_5 or $K_{3,3}$.*

Planar graphs have many applications in real-world problems, for example the 4 color theorem states that it is possible to colour the faces of a planar graph with four or fewer colours so that no two adjacent faces are colored alike.

2.2 Hypergroups

Let us start this subsection with the definition of a *hypergroup*. It is a non-empty set H endowed with a hyperoperation $\circ : H \times H \longrightarrow \mathcal{P}^*(H)$, satisfying the associative property, i.e. for any $x, y, z \in H$, there is $(x \circ y) \circ z = x \circ (y \circ z)$, and the reproduction axiom, i.e. for any $x \in H$, $x \circ H = H \circ x = H$. One of the key element in the hypergroup theory is the concept of *heart of a hypergroup*, that we will briefly recall in this subsection. It has, somehow, a similar role as the center of a group, since it commutes with any non-empty subset of the hypergroup. For a particular type of hypergroups, i.e. the complete hypergroups, the heart is the set of all bilateral identities of the hypergroup, i.e. the set of the elements $e \in H$ satisfying the property $x \in x \circ e \cap e \circ x$ for any $x \in H$. In order to define the heart of a hypergroup, we need to introduce an equivalence relation, called also *fundamental relation* because its properties. This is the β relation. More details regarding its meaning and applications can be found, e.g. in Antampoufis et al. [15], Al Tahan et al. [16], Novák et al. [17], Hamidi [18].

Define first, for all $n \geq 1$, on a hypergroup $(H, \circ)$ the relation β_n as follows:

$$a \beta_n b \Leftrightarrow \exists (x_1, \ldots, x_n) \in H^n : \{a, b\} \subseteq \prod_{i=1}^{n} x_i$$

and take $\beta = \bigcup_{i=1}^{n} \beta_n$, where $\beta_1 = \{(x, x) \mid x \in H\}$ is the diagonal relation on H. Denote by $\beta^\star$ the transitive closure of β, so $\beta^\star$ is an equivalence relation on H, see Corsini [11]. It is well known that $\beta^\star$ is the smallest strongly regular relation on a hypergroup $(H, \circ)$, such that the quotient $H/\beta^\star$ is a group with respect to the following operation

$$\beta^\star(x) \otimes \beta^\star(y) = \beta^\star(z), \forall z \in x \circ y.$$

$H/\beta^\star$ is called the *fundamental group* associated with H. The *heart* ω_H of the hypergroup H is the set of all elements x of H, for which the equivalence class $\beta^\star(x)$ is the neutral element of the fundamental group $H/\beta^\star$. Moreover, in a hypergroup H we have

$$\beta(x) = \beta^\star(x) = x \circ \omega_H = \omega_H \circ x,$$

for all $x \in H$. Generalyzing, for any non-empty subset B of a hypergroup H, there is $\omega_H \circ B = B \circ \omega_H$. In other words, considering the canonical projection $\varphi_H : H \longrightarrow H/\beta^\star$, the heart ω_H is the kernel of the homomorphism φ_H, i.e. $\omega_H = Ker\varphi_H = \{x \in H \mid \varphi_H(x) = \beta^\star(x) = 1\}$, where 1 is the neutral element of the group $H/\beta^\star$. Therefore we have also $\omega_H \circ \omega_H = \omega_H$.

3 Non-commuting graph of a non-central finite hypergroup

This section is dedicated to the construction and study of the properties of a certain graph associated with a non-central hypergroup, i.e. a hypergroup having non-commutative the associated fundamental group. In particular we search for conditions under which this graph is Hamiltonian or planar. The method of the construction of the graph is similar to that one proposed by Paul Erdős, when he defined a graph associated with a non-commutative group G: the set of vertices is $V = G \setminus Z(G)$, where $Z(G) = \{x \in G \mid xy = yx, \forall y \in G\}$ and two vertices x and y are joined whenever $xy \neq yx$. This graph was called later on by Abdollahi et al. [19] the *non-commuting graph* of the group G.

First we will characterise all hypergroups having non-commutative fundamental group.

Definition 3.1. Let $(H, \circ)$ be a finite hypergroup and set

$$T_H = \{x \in H \mid \exists y \in H : y \circ x \circ \omega_H \neq x \circ y \circ \omega_H\}.$$

If $T_H \neq \emptyset$, then H is called a *non-central hypergroup*.

Based on the properties of the heart of a hypergroup, we immediately obtain the following characterisation of a non-central hypergroup.

Proposition 3.2. *$(H, \circ)$ is a non-central hypergroup if and only if its associated fundamental group $(H/\beta^\star, \otimes)$ is non-abelian.*

Proof. We know that, for any $x \in H$, we have $\beta(x) = x \circ \omega_H$. From here it results the equivalence

$$\beta(x) \otimes \beta(y) \neq \beta(y) \otimes \beta(x) \Longleftrightarrow x \circ y \circ \omega_H \neq y \circ x \circ \omega_H,$$

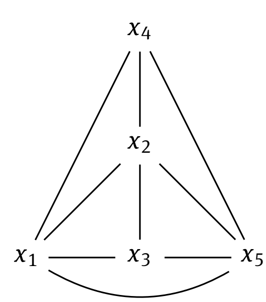
that concludes the proof. □

Since any non-abelian group contains more than 6 elements, it follows that, if a hypergroup H is non-central, i.e. the associated fundamental group $H/\beta^\star$ in non-abelian, then its cardinality is at least 6.

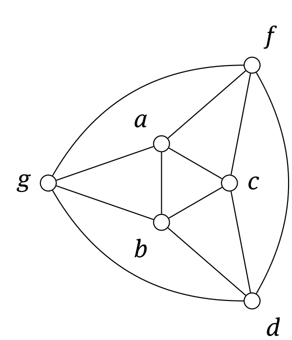
Definition 3.3. Let $(H, \circ)$ be a non-central hypergroup. We associate with H a graph G_H as follows: T_H is the set of vertices and join two distinct vertices x, y whenever $y \circ x \circ \omega_H \neq x \circ y \circ \omega_H$.

The most simple case of non-central hypergroup is a non-abelian group, where the hyperoperation on the hypergroup coincides with the operation of the group. In the following we will see some examples of the graphs associated with particular non-abelian groups, in the sense of Definition 3.3.

Example 3.4. *Let D_3 be the dihedral group of order 6, i.e. $D_3 = \langle a, b \mid a^2 = b^3 = e, bab = a\rangle = \{e, a, b, ab, ba, aba \mid a^2 = b^3 = e = (ab)^3\}$. We get that the set of vertices is $T_{D_3} = \{x_1 = a, x_2 = b, x_3 = ab, x_4 = ba, x_5 = aba\}$. Then the associated graph G_{D_3} is a maximal planar graph of order 5 as follows:*



Example 3.5. *Let D_4 and Q_4 be the dihedral group and the quaternion group both of order 8, respectively. They have isomorphic associated graphs, which are maximal planar graphs of order 6 as follows:*

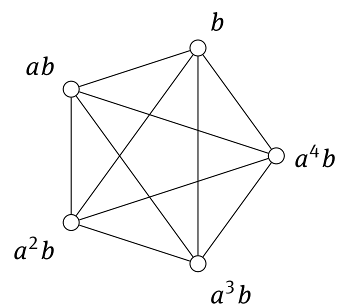


Example 3.6. *The associated graph G_{D_5} of the dihedral group D_5 of order 10 is not planar.*

Indeed, we can write $D_5 = \{e, a, a^2, a^3, a^4, b, ab, a^2b, a^3b, a^4b\}$ and then $T_{D_5} = D_5 \backslash \{e\}$. For any $x \in D_5$, denote by $C(x)$ the set of the elements of D_5 that commute with x. It is clear that

$$deg(x) = |D_5 \setminus C(x)|, \quad \textit{for any} \;\; x \in T_{D_5}.$$

Since the set $C(x)$ is a subgroup of D_5, it follows that $|C(x)|$ divides 10, so $|C(x)| \in \{1, 2, 5, 10\}$. Since $Z(D_5) = \{e\}$, we have $|C(x)| \neq 10$. Moreover $|C(x)| \neq 1$, for all $x \in T_{D_5}$ (since $e, x \in C(x)$). So $|C(x)| = 2$ or $|C(x)| = 5$, meaning that $deg(x) = 8$ or $deg(x) = 5$, for any $x \in T_{D_5}$. Using the operation table of the group D_5, we get $deg(a) = deg(a^2) = deg(a^3) = deg(a^4) = 5$ and $deg(b) = deg(ab) = deg(a^2b) = deg(a^3b) = deg(a^4b) = 8$. Thus we have the following induced subgraph of G_{D_5}:



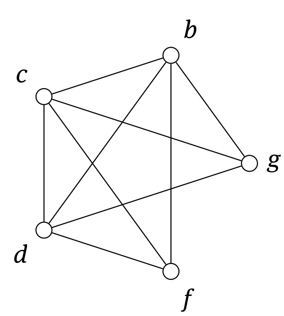
Therefore G_{D_5} has a subgraph isomorphic to K_5, meaning that G_{D_5} is not planar, by Kuratowski's Theorem.

Notice that all previous examples are for groups, while the following one is for a proper hypergroup.

Example 3.7. *Let us consider the hypergroup represented by the following Cayley table:*

$(H, \circ)$	e	a	b	c	d	f	g
e	e	a	b	c	d	f	g
a	a	e	b	c	d	f	g
b	b	b	e, a	g	f	d	c
c	c	c	f	e, a	g	b	d
d	d	d	g	f	e, a	c	b
f	f	f	c	d	b	g	e, a
g	g	g	d	b	c	e, a	f

Based on the definition of the relation β, it is easy to see that $\beta^\star(e) = \beta^\star(a) = \{e, a\}$, while for any $x \in H \backslash \{e, a\}$, it holds $\beta^\star(x) = \{x\}$. Since, for any $x \in H$, there is $x \circ \omega_H = \beta^\star(x)$, it follows immediately that $\omega_H = \{e, a\}$ and therefore $T_H = \{b, c, d, f, g\}$. Now, for example, taking the elements b and c of H, we easily calculate that $b \circ c \circ \omega_H = g \circ \{e, a\} = \{g\}$ and $c \circ b \circ \omega_H = f \circ \{e, a\} = \{f\}$, meaning that the vertices b and c are joint. Similarly one finds all the vertices of the associated graph G_H, which results as below:



Let us fix now a notation. For any element x in a hypergroup $(H, \circ)$ denote by $\overline{x} = \beta(x)$ the equivalence class of x modulo the relation β.

Proposition 3.8. *Let $(H, \circ)$ be a non-central hypergroup.*
i) *If $x \in T_H$, then $x \circ \omega_H \subseteq T_H$.*
ii) $T_H = \bigcup_{\overline{x} \in T_{H/\beta}} x \circ \omega_H.$

Proof. *i*) Let $x \in T_H$. Then there exists $y \in T_H$ such that $x \circ y \circ \omega_H \neq y \circ x \circ \omega_H$. Now let $z \in x \circ \omega_H$. It follows that $z \circ \omega_H = x \circ \omega_H$ and so $z \circ y \circ \omega_H \neq y \circ z \circ \omega_H$. Hence $z \in T_H$ and consequently $x \circ \omega_H \subseteq T_H$.
ii) It follows immediately from part *i*) and equivalence $x \in T_H \Longleftrightarrow \overline{x} \in T_{H/\beta}$. □

Corollary 3.9. *If H is a non-central hypergroup, then $|T_H| \geq 3$. Moreover, G_H is always a connected graph.*

Proof. The proof is based on a fundamental property of groups: If G is a non-abelian group, then $|Z(G)| \leq \frac{1}{2}|G|$, where $Z(G)$ denotes the center of G. Now suppose that $|T_H| < 3$. We have

$$|Z(H/\beta)| \leq |H/\beta \setminus Z(H/\beta)| < |T_H| < 3.$$

Thus $|H/\beta| < 6$, that is H/β is an abelian group, which is a contradiction because H is a non-central hypergroup. Besides, since $|T_H| \geq 3$, there always exists an edge between two vertices of G_H, so the graph G_H is connected. □

The following result gives a sufficient condition to have an edge between two vertices.

Lemma 3.10. *If $(H, \circ)$ is a non-central hypergroup and $x \circ y \circ \omega_H \neq y \circ x \circ \omega_H$, for some $x, y \in T_H$, then, for any $a \in x \circ \omega_H$ and any $b \in y \circ \omega_H$, there exists an edge between a and b.*

Proof. Let $a \in x \circ \omega_H$ and $b \in y \circ \omega_H$. Then we have $a \circ \omega_H \circ b \circ \omega_H = x \circ \omega_H \circ y \circ \omega_H \neq y \circ \omega_H \circ x \circ \omega_H = b \circ \omega_H \circ a \circ \omega_H$. Consequently $a \circ b \circ \omega_H \neq b \circ a \circ \omega_H$, meaning that a and b are joined by an edge. □

Based on this and on the definition of a bipartite complete graph, we get the following result.

Theorem 3.11. *If H is a non-central hypergroup and $(x, y) \in E(G_H)$, then $K_{x,y} = \{(a, b) \in E(G_H) \mid a \in x \circ \omega_H, b \in y \circ \omega_H\}$ is a complete bipartite graph.*

Proof. It follows immediately from Lemma 3.10, since $K_{x,y}$ can be partitioned into the sets $X = x \circ \omega_H$ and $Y = y \circ \omega_H$ and each edge (a, b) has one end vertex in X and the other one in Y. □

Lemma 3.12. *Let H be a non-central hypergroup. Then the associated graph G_H is not complete.*

Proof. Suppose that H is a non-central hypergroup. Let us assume by contradiction that G_H is a complete graph, then $G_{H/\beta}$ is a complete graph, too. Thus $diam(G_{H/\beta}) = 1$. Now let $\overline{x} \in T_{H/\beta}$. Then $\overline{x}^{-1} \in T_{H/\beta}$. Since $diam(G_{H/\beta}) = 1$, we have $\overline{x} = \overline{x}^{-1}$, because otherwise $\overline{x}$ and $\overline{x}^{-1}$ are joined, meaning that $\overline{x} \otimes \overline{x}^{-1} \neq \overline{x}^{-1} \otimes \overline{x}$, which is false. Moreover, if $\overline{y} \notin T_{H/\beta}$, then $\overline{x} \otimes \overline{y} \in T_{H/\beta}$ because, otherwise, we would get that $\overline{x} \otimes \overline{y}$ commutes with every element in H/β, i.e. $\overline{x} \otimes \overline{y} \otimes \overline{z} = \overline{z} \otimes \overline{x} \otimes \overline{y}$, for all $\overline{z} \in H/\beta$. Since $\overline{y}$ commutes with all elements of H/β, it results that $\overline{x} \otimes \overline{z} \otimes \overline{y} = \overline{z} \otimes \overline{x} \otimes \overline{y}$, so $\overline{x} \otimes \overline{z} = \overline{z} \otimes \overline{x}$, for all $\overline{z} \in H/\beta$, which contradicts the fact that $\overline{x} \in T_{H/\beta}$. Thus $\overline{x} \otimes \overline{y} \in T_{H/\beta}$ and so $(\overline{x} \otimes \overline{y})^{-1} = \overline{x} \otimes \overline{y}$, equivalently with $\overline{y}^{-1} \otimes \overline{x}^{-1} = \overline{x} \otimes \overline{y} = \overline{y} \otimes \overline{x}$ (since $\overline{y} \in T_{H/\beta}$), thereby $\overline{y} = \overline{y}^{-1}$. We conclude that $\overline{x} = \overline{x}^{-1}$, for all $\overline{x} \in H/\beta$, meaning that H/β is an abelian group, which is a contradiction. □

Theorem 3.13. *If H is a non-central hypergroup, then $diam(G_H) = 2$.*

Proof. Using Lemma 3.12, we know that the associated graph G_H is not complete, so there exist $x, y \in T_H$ such that $d(x, y) \neq 1$. Therefore $y \circ x \circ \omega_H = x \circ y \circ \omega_H$, while there exist $x', y' \in T_H$ such that $x \circ x' \circ \omega_H \neq x \circ x' \circ \omega_H$ and similarly $y \circ y' \circ \omega_H \neq y \circ y' \circ \omega_H$. We must consider the following two cases.

1. If $x' \circ y \circ \omega_H \neq y \circ x' \circ \omega_H$ or $x \circ y' \circ \omega_H \neq y' \circ x\omega_H$, then we have $d(x, y) = 2$.

2. Now let $x' \circ y \circ \omega_H = y \circ x' \circ \omega_H$ and $x \circ y' \circ \omega_H = y' \circ x \circ \omega_H$. If $t \in x' \circ y'$ and $x \circ t \circ \omega_H = t \circ x \circ \omega_H$, we have $x \circ (x' \circ y') \circ \omega_H = (x' \circ y') \circ x \circ \omega_H = (x' \circ x) \circ y' \circ \omega_H$ and so $x \circ x' \circ \omega_H = x' \circ x \circ \omega_H$, which is a contradiction. Hence $x \circ t \circ \omega_H \neq t \circ x \circ \omega_H$. Similarly we have $y \circ t \circ \omega_H \neq t \circ y \circ \omega_H$. Consequently $d(x, y) = 2$.

□

Definition 3.14. A hypergroup H is called *Hamiltonian* if the associated graph G_H is a Hamiltonian graph.

Theorem 3.15. *Let $(H, \circ)$ be a non-central hypergroup and $n = |x \circ \omega_H|$ for all $x \in T_H$. Then H is a Hamiltonian hypergroup.*

Proof. For any $\overline{x} \in T_{H/\beta} = H/\beta \setminus Z(H/\beta)$, let $C(\overline{x})$ be the set of elements that commute with $\overline{x}$ in H/β. Now we have $|C(\overline{x})| \leq \frac{|H/\beta|}{2}$. It follows that

$$deg(\overline{x}) = |H/\beta \setminus C(\overline{x})| > \frac{|H/\beta \setminus Z(H/\beta)|}{2} = \frac{|T_{H/\beta}|}{2}.$$

Indeed, if $deg(\overline{x}) \leq \frac{|H/\beta \setminus Z(H/\beta)|}{2}$, then more than half elements of $T_{H/\beta}$ can commute with $\overline{x}$, therefore $|C(\overline{x})| \geq \frac{|H/\beta \setminus Z(H/\beta)|}{2} + |Z(H/\beta)|$, following that $|C(\overline{x})| > \frac{|H/\beta|}{2}$, which is a contradiction.

Morever, using Lemma 3.10, we have

$$deg(x) = deg(\overline{x}) \cdot n = deg(\overline{x}) \cdot \frac{|T_H|}{|T_{H/\beta}|} > \frac{|T_{H/\beta}|}{2} \cdot \frac{|T_H|}{|T_{H/\beta}|} = \frac{|T_H|}{2}.$$

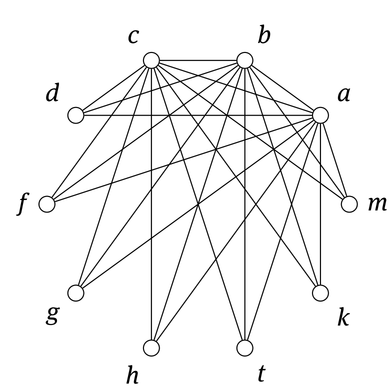
Hence, by Dirac's theorem, it follows that G_H is a Hamiltonian graph. □

Corollary 3.16. *Every non-abelian group is a Hamiltonian hypergroup.*

Example 3.17. *Consider the hypergroup in Example 3.7. It is easy to see that he Hamiltonian circuit is f, b, g, d, c, f, so the hypergroup is Hamiltonian.*

Example 3.18. *Let $S = S_3$ be the symmetric group of order 6 and take $A_{(1)} = \{e\}, A_{(1,2)} = \{a\}, A_{(1,3)} = \{b\}, A_{(2,3)} = \{c\}, A_{(1,3,2)} = \{d\}, A_{(1,2,3)} = \{f, g, h, t, k, m\}$ and $H = \cup_{\sigma \in S} A_\sigma$. Notice that, for each $x \in H$, there exists a unique $\sigma_x \in S_3$, such that $x \in A_{\sigma_x}$. Then we define on H the hyperoperation $\circ$ as follows: $x \circ y = A_{\sigma_x \cdot \sigma_y}$. This is the standard method to obtain a complete hypergroup, starting from a group [20]. Besides, H is a non-central hypergroup, which is not a Hamiltonian hypergroup.*

Indeed, in this case $\omega_H = \{e\}$ and $T_H = H \setminus \{e\}$. In the graph of G_H there are not edges between the vertices of the set $A_{(1,2,3)}$, so if G_H was Hamiltonian, then the vertices of $A_{(1,2,3)}$ would be among the other vertices, which is impossible. The associated graph G_H is as below:



Proposition 3.19. *Let $(H, \circ)$ be a Hamiltonian hypergroup. Then we have $|y \circ \omega_H| < \sum_{x \neq y} |x \circ \omega_H|$, for all $x, y \in T_H$.*

Proof. Suppose that H is a Hamiltonian hypergroup. In the associated graph G_H there is no edge between the vertices of $y \circ \omega_H$ and thus, in a Hamiltonian circuit, the vertices of $y \circ \omega_H$ must be among the vertices of $\cup_{x \neq y} x \circ \omega_H$. Thus we have $|y \circ \omega_H| < \sum_{x \neq y} |x \circ \omega_H|$, for all $x, y \in T_H$. □

In the following we will find conditions such that the associated graph of a non-central hypergroup is not planar.

Proposition 3.20. *Let H be a non-central hypergroup and $|x \circ \omega_H| \geq 2$, for all $x \in T_H$. Then G_H is not planar.*

Proof. Let H be a non-central hypergroup and suppose that $|x \circ \omega_H| \geq 2$, for all $x \in T_H$. Since H is a non-central hypergroup, it follows that for each $x \in T_H$ there exists $y \in T_H$ such that $x \circ y \circ \omega_H \neq y \circ x \circ \omega_H$. Suppose that $z \in x \circ y \circ \omega_H$ and $t \in y \circ x \circ \omega_H$, we have $z \circ \omega_H = x \circ y \circ \omega_H$ and $t \circ \omega_H = y \circ x \circ \omega_H$. Using Corollary 3.11, we have that $K_{i,j}$ is a complete bipartite graph for all $i \neq j$, and $i, j \in \{x, y, z, t\}$. If there exists $i \neq j$ such that $i, j \in \{x, y, z, t\}$ and $3 \leq |i \circ \omega_H|$ and $3 \leq |j \circ \omega_H|$ then $K_{3,3}$ is a subgraph of $K_{i,j}$ and so G_H is not a planar graph. Otherwise there exists $i \in \{x, y, z, t\}$ such that $|j \circ \omega_H| = 2$ for all $j \neq i$ and $j \in \{x, y, z, t\}$. Now consider the induced subgraph $S = \ll x \circ \omega_H \cup y \circ \omega_H \cup z \circ \omega_H \cup t \circ \omega_H \gg$. Without loss of generality, suppose that $|x \circ \omega_H| = n \geq 2$ and $|y \circ \omega_H| = |z \circ \omega_H| = |t \circ \omega_H| = 2$, then the number of edges of S is $n_e = \frac{(n+6)(n+5)}{2} - \frac{n(n-1)+6}{2} = 6n + 12$ and the number of vertices of S is $n_v = n + 6$. Because $3n_v - 6 < n_e$, by using Theorem 2.10, it follows that S is not planar and thereby G_H is not planar. □

Proposition 3.21. *Let H be a non-central hypergroup and $|x \circ \omega_H \cup y \circ \omega_H| \geq 4$, for all $(x, y) \in E(G_H)$. Then G_H is not planar.*

Proof. If $|x \circ \omega_H| \geq 2$, for all $x \in T_H$, then by Proposition 3.20, it follows that the graph G_H is not planar. Now suppose that there exists $x \in T_H$ such that $|x \circ \omega_H| = 1$. Since $|x \circ \omega_H \cup y \circ \omega_H| \geq 4$, for all $(x, y) \in E(G_H)$, it results that $|y \circ \omega_H| \geq 3$. Hence there exists $(x, y) \in E(G_H)$ such that $|x \circ \omega_H| = 1$ and $|y \circ \omega_H| \geq 3$. Because $x \circ y \circ \omega_H \neq y \circ x \circ \omega_H$, we conclude that $x \circ u \circ \omega_H \neq u \circ x \circ \omega_H$ and $y \circ u \circ \omega_H \neq u \circ y \circ \omega_H$, for all $u \in x \circ y \cup y \circ x$. Thus $deg(v) \geq 6$ for all $v \in T_H$. According with Proposition 2.11, the graph G_H is not planar since it does not consist a vertex of degree less than or equal to 5. □

We conclude the study with two interesting properties of non-central hypergroups having a planar associated graph.

Proposition 3.22. *Let H be a non-central hypergroup. If the associated graph G_H is planar, then the quotient group H/β is a non-abelian group of order less than or equal to 8, and $K_{x,y} \ncong K_{3,3}$, for all $(x, y) \in E(G_H)$.*

Proof. Suppose that H is a non-central hypergroup such that G_H is a planar graph. According with Proposition 2.11, there exists $x \in T_H$ such that $deg(x) \leq 5$. Therefore in the graph $G_{H/\beta}$, we have $deg(\overline{x}) \leq 5$. Consequently $|H/\beta \setminus C(\overline{x})| \leq 5$, where $C(\overline{x})$ is the set of elements of H/β commuting with $\overline{x}$. Because $|C(\overline{x})| \leq \frac{|H/\beta|}{2}$, it results that $|H/\beta| \leq 10$. If $|H/\beta| = 10$ then H/β must be the dihedral group of order 10, but its associated graph is not planar (as shown in Example 3.6). Hence $|H/\beta| \leq 8$ (notice that every group of order 9 is abelian). Moreover, according with Kuratowski's theorem, we have $K_{x,y} \ncong K_{3,3}$, for all $(x, y) \in E(G_H)$. □

Theorem 3.23. *Let H be a non-central hypergroup such that the associated graph G_H is planar. Then $H \simeq H/\beta$.*

Proof. Consider H be a non-central hypergroup such that G_H is a planar graph. According with Proposition 3.22, the quotient group H/β is of order 8 or 6, thus $G_{H/\beta} \simeq G_{D_3}$ or $G_{H/\beta} \simeq G_{D_4}$. We consider the two following cases.

1. Suppose that $G_{H/\beta} \simeq G_{D_3}$. Since $Z(D_3) = \{1\}$, it follows that $|T_{D_3}| = 5$. Besides, for any $x \in T_{D_3}$, we have $deg(x) = |D_3 \setminus C(x)|$, with $|C(x)|$ dividing $6 = |D_3|$, so $|C(x)| \in \{1, 2, 3, 6\}$. Obviously $|C(x)| \notin \{1, 6\}$, meaning that $|C(x)| \in \{2, 3\}$, equivalently $deg(x) \in \{3, 4\}$. For any $x \in V(G_H)$ and any $y \in x \circ \omega_H$, we have $deg(x) = deg(y) \in \{3, 4\}$ (see also Example 3.4). If $y \neq x$ and $deg(y) = 4$, then $|E(G_H)| \geq 13$, which contradicts Theorem

2.10. On the other hand, if $y \neq x$ and $deg(y) = 3$, then Theorem 2.14 drives us into a contradiction. Therefore we conclude that $y = x$ and thus $|x \circ \omega_H| = 1$. Hence $H \simeq D_3$.

2. Suppose that $G_{H/\beta} \simeq G_{D_4}$. For any $x \in V(G_H)$ and $y \in x \circ \omega_H$ we have $deg(y) = deg(x) = 4$ (see Example 3.5). In this case, if $y \neq x$, then we have $|E(G_H)| \geq 14$. At the same time, according with Theorem 2.10, we get $|E(G_H)| \leq 3 \times 6 - 6 = 12$, leading to a contradiction. Hence $|x \circ \omega_H| = 1$ and so H is a non-abelian group such that $H \simeq D_4$ or $H \simeq Q_4$.

□

4 Conclusions

Paul Erdős defined a graph having the set of vertices $V = G \setminus Z(G)$, where $Z(G)$ is the set of the elements of a non-abelian group G commuting with all elements in G, and joining two vertices x and y whenever $xy \neq yx$. This construction has been extended in this note to the hypergroups framework, considering a non-central hypergroup H, and considering the set of the vertices as $T_H = \{x \in H \mid \exists y \in H : x \circ y \circ \omega_H \neq y \circ x \circ \omega_H\}$ and join two vertices x and y whenever $x \circ y \circ \omega_H \neq y \circ x \circ \omega_H$. We have established necessary and /or sufficient conditions for the associated graph to be Hamiltonian or planar. In a future work other similar constructions will be investigated, in the sense that we will construct new graphs or hypergraphs associated with hypergroups, and vice-versa we will study the properties of the hyperstructures associated with some particular graphs.

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