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Research Article

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On the Gevrey ultradifferentiability of weak solutions of an abstract evolution equation with a scalar type spectral operator on the open semi-axis

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Abstract: Given the abstract evolution equation

$$y'(t) = Ay(t), \quad t \geq 0,$$

with *scalar type spectral operator* A in a complex Banach space, found are conditions *necessary and sufficient* for all *weak solutions* of the equation, which a priori need not be strongly differentiable, to be strongly Gevrey ultradifferentiable of order $\beta \geq 1$, in particular *analytic* or *entire*, on the open semi-axis $(0, \infty)$. Also, revealed is a certain interesting inherent smoothness improvement effect.

Keywords: weak solution, scalar type spectral operator, Gevrey classes**MSC:** Primary 34G10, 47B40, 30D60; Secondary 47B15, 47D06, 47D60, 30D15

1 Introduction

We find conditions on a *scalar type spectral operator* A in a complex Banach space necessary and sufficient for all *weak solutions* of the evolution equation

$$y'(t) = Ay(t), \quad t \geq 0, \tag{1.1}$$

which a priori need not be strongly differentiable, to be strongly Gevrey ultradifferentiable of order $\beta \geq 1$, in particular *analytic*, on the *open semi-axis* $(0, \infty)$ and reveal a certain interesting inherent smoothness improvement effect.

The found results generalize the corresponding ones of paper [1], where similar consideration is given to equation (1.1) with a *normal operator* A in a complex Hilbert space, and the characterizations of the generation of Gevrey ultradifferentiable C_0 -semigroups of Roumieu and Beurling types by scalar type spectral operators found in papers [2, 4] (see also [3]). They also develop the discourse of papers [5, 6], in which the strong differentiability of the weak solutions of equation (1.1) on $[0, \infty)$ and $(0, \infty)$ and their strong Gevrey ultradifferentiability of order $\beta \geq 1$ on the *closed semi-axis* $[0, \infty)$ are treated, respectively (cf. also [7]).

Definition 1.1 (Weak solution).

Let A be a densely defined closed linear operator in a Banach space $(X, \|\cdot\|)$. A strongly continuous vector

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function $y : [0, \infty) \rightarrow X$ is called a *weak solution* of equation (1.1) if, for any $g^* \in D(A^*)$,

$$\frac{d}{dt} \langle y(t), g^* \rangle = \langle y(t), A^* g^* \rangle, \quad t \geq 0,$$

where $D(\cdot)$ is the *domain* of an operator, A^* is the operator *adjoint* to A , and $\langle \cdot, \cdot \rangle$ is the *pairing* between the space X and its dual X^* (cf. [8]).

Remarks 1.1.

- Due to the *closedness* of A , the weak solution of (1.1) can be equivalently defined to be a strongly continuous vector function $y : [0, \infty) \rightarrow X$ such that, for all $t \geq 0$,

$$\int_0^t y(s) ds \in D(A) \text{ and } y(t) = y(0) + A \int_0^t y(s) ds$$

and is also called a *mild solution* (cf. [9, Ch. II, Definition 6.3], [10, Preliminaries]).

- Such a notion of *weak solution*, which need not be differentiable in the strong sense, generalizes that of *classical* one, strongly differentiable on $[0, \infty)$ and satisfying the equation in the traditional plug-in sense, the classical solutions being precisely the weak ones strongly differentiable on $[0, \infty)$.
- When a closed densely defined linear operator A in a complex Banach space X generates a C_0 -semigroup $\{T(t)\}_{t \geq 0}$ of bounded linear operators (see, e.g., [9, 11]), i.e., the associated *abstract Cauchy problem* (ACP)

$$\begin{cases} y'(t) = Ay(t), & t \geq 0, \\ y(0) = f \end{cases} \quad (1.2)$$

is *well-posed* (cf. [9, Ch. II, Definition 6.8]), the weak solutions of equation (1.1) are the orbits

$$y(t) = T(t)f, \quad t \geq 0, \quad (1.3)$$

with $f \in X$ [9, Ch. II, Proposition 6.4] (see also [8, Theorem]), whereas the classical ones are those with $f \in D(A)$ (see, e.g., [9, Ch. II, Proposition 6.3]).

- In our consideration, the associated ACP need not be *well-posed*, i.e., the scalar type spectral operator A need not generate a C_0 -semigroup (cf. [12]).

2 Preliminaries

Here, for the reader's convenience, we outline certain essential preliminaries.

2.1 Scalar type spectral operators

Henceforth, unless specified otherwise, A is supposed to be a *scalar type spectral operator* in a complex Banach space $(X, \|\cdot\|)$ with strongly σ -additive *spectral measure* (the *resolution of the identity*) $E_A(\cdot)$ assigning to each Borel set δ of the complex plane \mathbb{C} a projection operator $E_A(\delta)$ on X and having the operator's *spectrum* $\sigma(A)$ as its *support* [13–15].

Observe that, on a complex finite-dimensional space, the scalar type spectral operators are all linear operators that furnish an *eigenbasis* for the space (see, e.g., [14, 15]) and, in a complex Hilbert space, the scalar type spectral operators are precisely all those that are similar to the *normal* ones [16].

Associated with a scalar type spectral operator in a complex Banach space is the *Borel operational calculus* analogous to that for a *normal operator* in a complex Hilbert space [17, 18], which assigns to any

Borel measurable function $F : \sigma(A) \rightarrow \mathbb{C}$ a scalar type spectral operator

$$F(A) := \int_{\sigma(A)} F(\lambda) dE_A(\lambda)$$

(see [14, 15]).

In particular,

$$A^n = \int_{\sigma(A)} \lambda^n dE_A(\lambda), \quad n \in \mathbb{Z}_+,$$

($\mathbb{Z}_+ := \{0, 1, 2, \dots\}$ is the set of nonnegative integers, $A^0 := I$, I is the identity operator on X) and

$$e^{zA} := \int_{\sigma(A)} e^{z\lambda} dE_A(\lambda), \quad z \in \mathbb{C}. \quad (2.4)$$

The properties of the *spectral measure* and *operational calculus*, exhaustively delineated in [14, 15], underlie the subsequent discourse. Here, we touch upon a few facts of particular importance.

Due to its *strong countable additivity*, the spectral measure $E_A(\cdot)$ is *bounded* [15, 19], i.e., there is such an $M \geq 1$ that, for any Borel set $\delta \subseteq \mathbb{C}$,

$$\|E_A(\delta)\| \leq M. \quad (2.5)$$

Observe that the notation $\|\cdot\|$ is used here to designate the norm in the space $L(X)$ of all bounded linear operators on X . We adhere to this rather conventional economy of symbols in what follows also adopting the same notation for the norm in the dual space X^* .

For any $f \in X$ and $g^* \in X^*$, the *total variation measure* $\nu(f, g^*, \cdot)$ of the complex-valued Borel measure $\langle E_A(\cdot)f, g^* \rangle$ is a *finite* positive Borel measure with

$$\nu(f, g^*, \mathbb{C}) = \nu(f, g^*, \sigma(A)) \leq 4M\|f\|\|g^*\| \quad (2.6)$$

(see, e.g., [2, 20]).

Also (Ibid.), for a Borel measurable function $F : \mathbb{C} \rightarrow \mathbb{C}$, $f \in D(F(A))$, $g^* \in X^*$, and a Borel set $\delta \subseteq \mathbb{C}$,

$$\int_{\delta} |F(\lambda)| d\nu(f, g^*, \lambda) \leq 4M\|E_A(\delta)F(A)f\|\|g^*\|. \quad (2.7)$$

In particular, for $\delta = \sigma(A)$,

$$\int_{\sigma(A)} |F(\lambda)| d\nu(f, g^*, \lambda) \leq 4M\|F(A)f\|\|g^*\|. \quad (2.8)$$

Observe that the constant $M \geq 1$ in (2.6)–(2.8) is from (2.5).

Further, for a Borel measurable function $F : \mathbb{C} \rightarrow [0, \infty)$, a Borel set $\delta \subseteq \mathbb{C}$, a sequence $\{\Delta_n\}_{n=1}^\infty$ of pairwise disjoint Borel sets in \mathbb{C} , and $f \in X$, $g^* \in X^*$,

$$\int_{\delta} F(\lambda) d\nu(E_A(\cup_{n=1}^\infty \Delta_n)f, g^*, \lambda) = \sum_{n=1}^\infty \int_{\delta \cap \Delta_n} F(\lambda) d\nu(E_A(\Delta_n)f, g^*, \lambda). \quad (2.9)$$

Indeed, since, for any Borel sets $\delta, \sigma \subseteq \mathbb{C}$,

$$E_A(\delta)E_A(\sigma) = E_A(\delta \cap \sigma)$$

[14, 15], for the total variation,

$$\nu(E_A(\delta)f, g^*, \sigma) = \nu(f, g^*, \delta \cap \sigma).$$

Whence, due to the *nonnegativity* of $F(\cdot)$ (see, e.g., [21, 22]),

$$\int_{\delta} F(\lambda) d\nu(E_A(\cup_{n=1}^\infty \Delta_n)f, g^*, \lambda) = \int_{\delta \cap \cup_{n=1}^\infty \Delta_n} F(\lambda) d\nu(f, g^*, \lambda) = \sum_{n=1}^\infty \int_{\delta \cap \Delta_n} F(\lambda) d\nu(f, g^*, \lambda)$$

$$= \sum_{n=1}^{\infty} \int_{\delta \cap \Delta_n} F(\lambda) d\nu(E_A(\Delta_n)f, g^*, \lambda).$$

The following statement, allowing to characterize the domains of Borel measurable functions of a scalar type spectral operator in terms of positive Borel measures, is fundamental for our consideration.

Proposition 2.1 ([23, Proposition 3.1]).

Let A be a scalar type spectral operator in a complex Banach space $(X, \|\cdot\|)$ with spectral measure $E_A(\cdot)$ and $F : \sigma(A) \rightarrow \mathbb{C}$ be a Borel measurable function. Then $f \in D(F(A))$ iff

$$(i) \text{ for each } g^* \in X^*, \int_{\sigma(A)} |F(\lambda)| d\nu(f, g^*, \lambda) < \infty \text{ and}$$

$$(ii) \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \mid |F(\lambda)| > n\}} |F(\lambda)| d\nu(f, g^*, \lambda) \rightarrow 0, \quad n \rightarrow \infty,$$

where $\nu(f, g^*, \cdot)$ is the total variation measure of $\langle E_A(\cdot)f, g^* \rangle$.

The succeeding key theorem provides a description of the weak solutions of equation (1.1) with a scalar type spectral operator A in a complex Banach space.

Theorem 2.1 ([23, Theorem 4.2]).

Let A be a scalar type spectral operator in a complex Banach space $(X, \|\cdot\|)$. A vector function $y : [0, \infty) \rightarrow X$ is a weak solution of equation (1.1) iff there is an $f \in \bigcap_{t \geq 0} D(e^{tA})$ such that

$$y(t) = e^{tA}f, \quad t \geq 0, \quad (2.10)$$

the operator exponentials understood in the sense of the Borel operational calculus (see (2.4)).

Remarks 2.1.

- Theorem 2.1 generalizing [24, Theorem 3.1], its counterpart for a normal operator A in a complex Hilbert space, in particular, implies
 - that the subspace $\bigcap_{t \geq 0} D(e^{tA})$ of all possible initial values of the weak solutions of equation (1.1) is the largest permissible for the exponential form given by (2.10), which highlights the naturalness of the notion of weak solution, and
 - that associated ACP (1.2), whenever solvable, is solvable *uniquely*.
- Observe that the initial-value subspace $\bigcap_{t \geq 0} D(e^{tA})$ of equation (1.1) is *dense* in X since it contains the subspace

$$\bigcup_{\alpha > 0} E_A(\Delta_\alpha)X, \quad \text{where } \Delta_\alpha := \{\lambda \in \mathbb{C} \mid |\lambda| \leq \alpha\}, \quad \alpha > 0,$$

which is dense in X and coincides with the class $\mathcal{E}^{\{0\}}(A)$ of *entire* vectors of A of *exponential type* [25, 26].

- When a scalar type spectral operator A in a complex Banach space generates a C_0 -semigroup $\{T(t)\}_{t \geq 0}$,

$$T(t) = e^{tA} \text{ and } D(e^{tA}) = X, \quad t \geq 0,$$

[12], and hence, Theorem 2.1 is consistent with the well-known description of the weak solutions for this setup (see (1.3)).

Subsequently, the frequent terms “spectral measure” and “operational calculus” are abbreviated to *s.m.* and *o.c.*, respectively.

2.2 Gevrey classes of functions

Definition 2.1 (Gevrey classes of functions).

Let $(X, \|\cdot\|)$ be a (real or complex) Banach space, $C^\infty(I, X)$ be the space of all X -valued functions strongly infinite differentiable on an interval $I \subseteq \mathbb{R}$, and $0 \leq \beta < \infty$.

The following subspaces of $C^\infty(I, X)$

$$\begin{aligned}\mathcal{E}^{\{\beta\}}(I, X) &:= \{g(\cdot) \in C^\infty(I, X) \mid \forall [a, b] \subseteq I \exists \alpha > 0 \exists c > 0 : \max_{a \leq t \leq b} \|g^{(n)}(t)\| \leq c\alpha^n [n!]^\beta, n \in \mathbb{Z}_+\}, \\ \mathcal{E}^{(\beta)}(I, X) &:= \{g(\cdot) \in C^\infty(I, X) \mid \forall [a, b] \subseteq I \forall \alpha > 0 \exists c > 0 : \max_{a \leq t \leq b} \|g^{(n)}(t)\| \leq c\alpha^n [n!]^\beta, n \in \mathbb{Z}_+\},\end{aligned}$$

are called the β th-order Gevrey classes of strongly ultradifferentiable vector functions on I of Roumieu and Beurling type, respectively (see, e.g., [27–30]).

Remarks 2.2.

- In view of Stirling's formula, the sequence $\{[n!]^\beta\}_{n=0}^\infty$ can be replaced with $\{n^{\beta n}\}_{n=0}^\infty$.
- For $0 \leq \beta < \beta' < \infty$, the inclusions

$$\mathcal{E}^{(\beta)}(I, X) \subseteq \mathcal{E}^{\{\beta\}}(I, X) \subseteq \mathcal{E}^{(\beta')}(I, X) \subseteq \mathcal{E}^{\{\beta'\}}(I, X) \subseteq C^\infty(I, X)$$

hold.

- For $1 < \beta < \infty$, the Gevrey classes are *non-quasianalytic* (see, e.g., [29]).
- For $\beta = 1$, $\mathcal{E}^{\{1\}}(I, X)$ is the class of all *analytic* on I , i.e., *analytically continuable* into complex neighborhoods of I , vector functions and $\mathcal{E}^{(1)}(I, X)$ is the class of all *entire*, i.e., allowing *entire* continuations, vector functions [31].
- For $0 \leq \beta < 1$, the Gevrey class $\mathcal{E}^{\{\beta\}}(I, X)$ ($\mathcal{E}^{(\beta)}(I, X)$) consists of all functions $g(\cdot) \in \mathcal{E}^{(1)}(I, X)$ such that, for some (any) $\gamma > 0$, there is an $M > 0$ for which

$$\|g(z)\| \leq M e^{\gamma|z|^{1/(1-\beta)}}, \quad z \in \mathbb{C}, \quad (2.11)$$

[32] (see also [33]). In particular, for $\beta = 0$, $\mathcal{E}^{\{0\}}(I, X)$ and $\mathcal{E}^{(0)}(I, X)$ are the classes of entire vector functions of *exponential* and *minimal exponential type*, respectively (see, e.g., [34]).

2.3 Gevrey classes of vectors

One can consider the Gevrey classes in a more general sense.

Definition 2.2 (Gevrey classes of vectors).

Let $(A, D(A))$ be a densely defined closed linear operator in a (real or complex) Banach space $(X, \|\cdot\|)$, $0 \leq \beta < \infty$, and

$$C^\infty(A) := \bigcap_{n=0}^\infty D(A^n)$$

be the subspace of *infinite differentiable vectors* of A .

The following subspaces of $C^\infty(A)$

$$\begin{aligned}\mathcal{E}^{\{\beta\}}(A) &:= \left\{x \in C^\infty(A) \mid \exists \alpha > 0 \exists c > 0 : \|A^n x\| \leq c\alpha^n [n!]^\beta, n \in \mathbb{Z}_+\right\}, \\ \mathcal{E}^{(\beta)}(A) &:= \left\{x \in C^\infty(A) \mid \forall \alpha > 0 \exists c > 0 : \|A^n x\| \leq c\alpha^n [n!]^\beta, n \in \mathbb{Z}_+\right\}\end{aligned}$$

are called the β th-order Gevrey classes of ultradifferentiable vectors of A of Roumieu and Beurling type, respectively (see, e.g., [35–37]).

Remarks 2.3.

- In view of *Stirling's formula*, the sequence $\{[n!]^\beta\}_{n=0}^\infty$ can be replaced with $\{n^{\beta n}\}_{n=0}^\infty$.
- For $0 \leq \beta < \beta' < \infty$, the inclusions

$$\mathcal{E}^{(\beta)}(A) \subseteq \mathcal{E}^{\{\beta\}}(A) \subseteq \mathcal{E}^{(\beta')}(A) \subseteq \mathcal{E}^{\{\beta'\}}(A) \subseteq C^\infty(A)$$

hold.

- In particular, $\mathcal{E}^{\{1\}}(A)$ and $\mathcal{E}^{(1)}(A)$ are the classes of *analytic* and *entire* vectors of A , respectively [38, 39] and $\mathcal{E}^{\{0\}}(A)$ and $\mathcal{E}^{(0)}(A)$ are the classes of *entire* vectors of A of *exponential* and *minimal exponential type*, respectively (see, e.g., [26, 36]).
- In view of the *closedness* of A , it is easily seen that the class $\mathcal{E}^{(1)}(A)$ forms the subspace of the initial values $f \in X$ generating the (classical) solutions of (1.1), which are entire vector functions represented by the power series

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} A^n f, \quad t \geq 0,$$

the classes $\mathcal{E}^{\{\beta\}}(A)$ and $\mathcal{E}^{(\beta)}(A)$ with $0 \leq \beta < 1$ being the subspaces of such initial values for which the solutions satisfy growth estimate (2.11) with some (any) $\gamma > 0$ and some $M > 0$, respectively (cf. [34]).

As is shown in [35] (see also [36, 37]), if $0 < \beta < \infty$, for a *normal operator* A in a complex Hilbert space,

$$\mathcal{E}^{\{\beta\}}(A) = \bigcup_{t>0} D(e^{t|A|^{1/\beta}}) \text{ and } \mathcal{E}^{(\beta)}(A) = \bigcap_{t>0} D(e^{t|A|^{1/\beta}}), \quad (2.12)$$

the operator exponentials $e^{t|A|^{1/\beta}}$, $t > 0$, understood in the sense of the Borel operational calculus (see, e.g., [17, 18]).

In [20, 25], descriptions (2.12) are extended to *scalar type spectral operators* in a complex Banach space, in which form they are basic for our discourse. In [25], similar nature descriptions of the classes $\mathcal{E}^{\{0\}}(A)$ and $\mathcal{E}^{(0)}(A)$ ($\beta = 0$), known for a normal operator A in a complex Hilbert space (see, e.g., [36]), are also generalized to scalar type spectral operators in a complex Banach space. In particular [25, Theorem 5.1],

$$\mathcal{E}^{\{0\}}(A) = \bigcup_{\alpha>0} E_A(\Delta_\alpha)X,$$

where

$$\Delta_\alpha := \{\lambda \in \mathbb{C} \mid |\lambda| \leq \alpha\}, \quad \alpha > 0.$$

We also need the following characterization of a particular weak solution's of equation (1.1) with a scalar type spectral operator A in a complex Banach space being strongly Gevrey ultradifferentiable on a subinterval I of $[0, \infty)$.

Proposition 2.2 ([6, Proposition 3.2]).

Let A be a scalar type spectral operator in a complex Banach space $(X, \|\cdot\|)$, $0 \leq \beta < \infty$, and I be a subinterval of $[0, \infty)$. Then the restriction of a weak solution $y(\cdot)$ of equation (1.1) to I belongs to the Gevrey class $\mathcal{E}^{\{\beta\}}(I, X)$ ($\mathcal{E}^{(\beta)}(I, X)$) iff, for each $t \in I$,

$$y(t) \in \mathcal{E}^{\{\beta\}}(A) \text{ } (\mathcal{E}^{(\beta)}(A)), \text{ respectively),}$$

in which case, for every $n \in \mathbb{N}$,

$$y^{(n)}(t) = A^n y(t), \quad t \in I.$$

3 Gevrey ultradifferentiability of weak solutions

The case of the strong Gevrey ultradifferentiability of the weak solutions of equation (1.1) with a scalar type spectral operator in a complex Banach space on the *open semi-axis* $(0, \infty)$, similarly to the analogous setup

with a normal operator A in a complex Hilbert space [1], significantly differs from its counterpart over the closed semi-axis $[0, \infty)$ studied in [6].

First, let us consider the Roumieu type strong Gevrey ultradifferentiability of order $\beta \geq 1$.

Theorem 3.1. *Let A be a scalar type spectral operator in a complex Banach space $(X, \|\cdot\|)$ with spectral measure $E_A(\cdot)$ and $1 \leq \beta < \infty$. Every weak solution of equation (1.1) belongs to the β th-order Roumieu type Gevrey class $\mathcal{E}^{\{\beta\}}((0, \infty), X)$ iff there exist $b_+ > 0$ and $b_- > 0$ such that the set $\sigma(A) \setminus \mathcal{P}_{b_-, b_+}^\beta$, where*

$$\mathcal{P}_{b_-, b_+}^\beta := \left\{ \lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \leq -b_- |\operatorname{Im} \lambda|^{1/\beta} \text{ or } \operatorname{Re} \lambda \geq b_+ |\operatorname{Im} \lambda|^{1/\beta} \right\},$$

is bounded (see Figure 1).

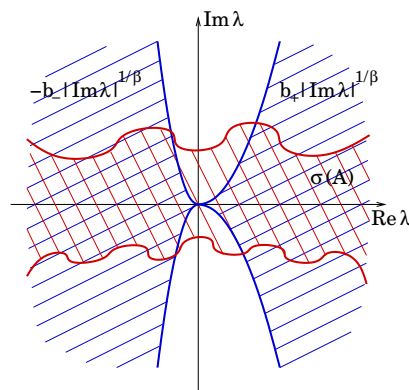


Figure 1: Gevrey ultradifferentiability of order $1 \leq \beta < \infty$.

Proof.

“If” Part. Suppose that there exist $b_+ > 0$ and $b_- > 0$ such that the set $\sigma(A) \setminus \mathcal{P}_{b_-, b_+}^\beta$ is bounded and let $y(\cdot)$ be an arbitrary weak solution of equation (1.1).

By Theorem 2.1,

$$y(t) = e^{tA} f, \quad t \geq 0, \quad \text{with some } f \in \bigcap_{t \geq 0} D(e^{tA}).$$

Our purpose is to show that $y(\cdot) \in \mathcal{E}^{\{\beta\}}((0, \infty), X)$, which, by Proposition 2.2 and (2.12), is attained by showing that, for each $t > 0$,

$$y(t) \in \mathcal{E}^{\{\beta\}}(A) = \bigcup_{s > 0} D(e^{s|A|^{1/\beta}}).$$

Let us proceed by proving that, for each $t > 0$, there exists an $s > 0$ such that

$$y(t) \in D(e^{s|A|^{1/\beta}})$$

via Proposition 2.1.

For an arbitrary $t > 0$, let us set

$$s := t(1 + b_-^\beta)^{-1/\beta} > 0. \quad (3.13)$$

Then, for any $g^* \in X^*$,

$$\begin{aligned} \int_{\sigma(A)} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) &= \int_{\sigma(A) \setminus \mathcal{P}_{b_-, b_+}^\beta} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\ &+ \int_{\{\lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid -1 < \operatorname{Re} \lambda < 1\}} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \end{aligned}$$

$$\begin{aligned}
& + \int_{\left\{ \lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \geq 1 \right\}} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} dv(f, g^*, \lambda) \\
& + \int_{\left\{ \lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \leq -1 \right\}} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} dv(f, g^*, \lambda) < \infty. \quad (3.14)
\end{aligned}$$

Indeed,

$$\int_{\sigma(A) \setminus \mathcal{P}_{b_-, b_+}^\beta} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} dv(f, g^*, \lambda) < \infty$$

and

$$\int_{\left\{ \lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid -1 < \operatorname{Re} \lambda < 1 \right\}} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} dv(f, g^*, \lambda) < \infty$$

due to the boundedness of the sets

$$\sigma(A) \setminus \mathcal{P}_{b_-, b_+}^\beta \text{ and } \left\{ \lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid -1 < \operatorname{Re} \lambda < 1 \right\},$$

the continuity of the integrated function on \mathbb{C} , and the finiteness of the measure $\nu(f, g^*, \cdot)$.

Further, for an arbitrary $t > 0$, $s > 0$ chosen as in (3.13), and any $g^* \in X^*$,

$$\begin{aligned}
& \int_{\left\{ \lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \geq 1 \right\}} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} dv(f, g^*, \lambda) \leq \int_{\left\{ \lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \geq 1 \right\}} e^{s[|\operatorname{Re} \lambda| + |\operatorname{Im} \lambda|]^{1/\beta}} e^{t \operatorname{Re} \lambda} dv(f, g^*, \lambda) \\
& \text{since, for } \lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \text{ with } \operatorname{Re} \lambda \geq 1, b_+^{-\beta} \operatorname{Re} \lambda^\beta \geq |\operatorname{Im} \lambda|; \\
& \leq \int_{\left\{ \lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \geq 1 \right\}} e^{s[\operatorname{Re} \lambda + b_+^{-\beta} \operatorname{Re} \lambda^\beta]^{1/\beta}} e^{t \operatorname{Re} \lambda} dv(f, g^*, \lambda) \\
& \text{since, in view of } \operatorname{Re} \lambda \geq 1 \text{ and } \beta \geq 1, \operatorname{Re} \lambda^\beta \geq \operatorname{Re} \lambda; \\
& \leq \int_{\left\{ \lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \geq 1 \right\}} e^{s(1+b_+^{-\beta})^{1/\beta} \operatorname{Re} \lambda} e^{t \operatorname{Re} \lambda} dv(f, g^*, \lambda) \\
& = \int_{\left\{ \lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \geq 1 \right\}} e^{[s(1+b_+^{-\beta})^{1/\beta} + t] \operatorname{Re} \lambda} dv(f, g^*, \lambda) \\
& \text{since } f \in \bigcap_{t \geq 0} D(e^{tA}), \text{ by Proposition 2.1;} \\
& < \infty. \quad (3.15)
\end{aligned}$$

Observe that, for the finiteness of the three preceding integrals, the special choice of $s > 0$ is superfluous.

Finally, for an arbitrary $t > 0$, $s > 0$ chosen as in (3.13), and any $g^* \in X^*$,

$$\begin{aligned}
& \int_{\left\{ \lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \leq -1 \right\}} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} dv(f, g^*, \lambda) \leq \int_{\left\{ \lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \leq -1 \right\}} e^{s[|\operatorname{Re} \lambda| + |\operatorname{Im} \lambda|]^{1/\beta}} e^{t \operatorname{Re} \lambda} dv(f, g^*, \lambda) \\
& \text{since, for } \lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \text{ with } \operatorname{Re} \lambda \leq -1, b_-^{-\beta} (-\operatorname{Re} \lambda)^\beta \geq |\operatorname{Im} \lambda|; \\
& \leq \int_{\left\{ \lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \leq -1 \right\}} e^{s[-\operatorname{Re} \lambda + b_-^{-\beta} (-\operatorname{Re} \lambda)^\beta]^{1/\beta}} e^{t \operatorname{Re} \lambda} dv(f, g^*, \lambda) \\
& \text{since, in view of } -\operatorname{Re} \lambda \geq 1 \text{ and } \beta \geq 1, (-\operatorname{Re} \lambda)^\beta \geq -\operatorname{Re} \lambda;
\end{aligned}$$

$$\begin{aligned}
&\leq \int_{\{\lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \leq -1\}} e^{s(1+b_-^{-\beta})^{1/\beta}(-\operatorname{Re} \lambda)} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\
&= \int_{\{\lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \leq -1\}} e^{\left[t-s(1+b_-^{-\beta})^{1/\beta}\right] \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\
&\text{since } s := t(1+b_-^{-\beta})^{-1/\beta} \text{ (see (3.13));} \\
&= \int_{\{\lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \leq -1\}} 1 d\nu(f, g^*, \lambda) \leq \int_{\sigma(A)} 1 d\nu(f, g^*, \lambda) = \nu(f, g^*, \sigma(A)) \\
&\text{by (2.6);} \\
&\leq 4M \|f\| \|g^*\| < \infty.
\end{aligned} \tag{3.16}$$

Also, for an arbitrary $t > 0$, $s > 0$ chosen as in (3.13), and any $n \in \mathbb{N}$,

$$\begin{aligned}
&\sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \mid e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n\}} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\
&\leq \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \setminus \mathcal{P}_{b_-, b_+}^\beta \mid e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n\}} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\
&+ \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid -1 < \operatorname{Re} \lambda < 1, e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n\}} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\
&+ \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \geq 1, e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n\}} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\
&+ \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \leq -1, e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n\}} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\
&\rightarrow 0, \quad n \rightarrow \infty.
\end{aligned} \tag{3.17}$$

Indeed, since, due to the boundedness of the sets

$$\sigma(A) \setminus \mathcal{P}_{b_-, b_+}^\beta \text{ and } \{\lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid -1 < \operatorname{Re} \lambda < 1\}$$

and the continuity of the integrated function on \mathbb{C} , the sets

$$\{\lambda \in \sigma(A) \setminus \mathcal{P}_{b_-, b_+}^\beta \mid e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n\}$$

and

$$\{\lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid -1 < \operatorname{Re} \lambda < 1, e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n\}$$

are empty for all sufficiently large $n \in \mathbb{N}$, we immediately infer that, for any $t > 0$ and $s > 0$ chosen as in (3.13),

$$\lim_{n \rightarrow \infty} \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \setminus \mathcal{P}_{b_-, b_+}^\beta \mid e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n\}} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) = 0$$

and

$$\lim_{n \rightarrow \infty} \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid -1 < \operatorname{Re} \lambda < 1, e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n\}} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) = 0.$$

Further, for an arbitrary $t > 0$, $s > 0$ chosen as in (3.13), and any $n \in \mathbb{N}$,

$$\begin{aligned}
 & \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\left\{ \lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \geq 1, e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n \right\}} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} dv(f, g^*, \lambda) \\
 & \text{as in (3.15);} \\
 & \leq \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\left\{ \lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \geq 1, e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n \right\}} e^{\left[s(1+b_+^{-\beta})^{1/\beta} + t \right] \operatorname{Re} \lambda} dv(f, g^*, \lambda) \\
 & \text{since } f \in \bigcap_{t \geq 0} D(e^{tA}), \text{ by (2.7);} \\
 & \leq \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} 4M \left\| E_A \left(\left\{ \lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \geq 1, e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n \right\} \right) e^{\left[s(1+b_+^{-\beta})^{1/\beta} + t \right] A} f \right\| \|g^*\| \\
 & \leq 4M \left\| E_A \left(\left\{ \lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \geq 1, e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n \right\} \right) e^{\left[s(1+b_+^{-\beta})^{1/\beta} + t \right] A} f \right\| \\
 & \text{by the strong continuity of the s.m.;} \\
 & \rightarrow 4M \left\| E_A(\emptyset) e^{\left[s(1+b_+^{-\beta})^{1/\beta} + t \right] A} f \right\| = 0, \quad n \rightarrow \infty.
 \end{aligned}$$

Finally, for an arbitrary $t > 0$, $s > 0$ chosen as in (3.13), and any $n \in \mathbb{N}$,

$$\begin{aligned}
 & \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\left\{ \lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \leq -1, e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n \right\}} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} dv(f, g^*, \lambda) \\
 & \text{as in (3.16);} \\
 & \leq \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\left\{ \lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \leq -1, e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n \right\}} e^{\left[t - s(1+b_-^{-\beta})^{1/\beta} \right] \operatorname{Re} \lambda} dv(f, g^*, \lambda) \\
 & \text{since } s := t(1 + b_-^{-\beta})^{-1/\beta} \text{ (see (3.13));} \\
 & = \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\left\{ \lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \leq -1, e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n \right\}} 1 dv(f, g^*, \lambda) \\
 & \text{by (2.7);} \\
 & \leq \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} 4M \left\| E_A \left(\left\{ \lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \leq -1, e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n \right\} \right) f \right\| \|g^*\| \\
 & \leq 4M \left\| E_A \left(\left\{ \lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \leq -1, e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n \right\} \right) f \right\| \\
 & \text{by the strong continuity of the s.m.;} \\
 & \rightarrow 4M \|E_A(\emptyset) f\| = 0, \quad n \rightarrow \infty.
 \end{aligned}$$

By Proposition 2.1 and the properties of the o.c. (see [15, Theorem XVIII.2.11 (f)]), (3.14) and (3.17) jointly imply that, for any $t > 0$,

$$f \in D(e^{s|A|^{1/\beta}} e^{tA})$$

with $s := t(1 + b_-^{-\beta})^{-1/\beta} > 0$, and hence, in view of (2.12), for each $t > 0$,

$$y(t) = e^{tA} f \in \bigcup_{s > 0} D(e^{s|A|^{1/\beta}}) = \mathcal{E}^{\{\beta\}}(A).$$

By Proposition 2.2, we infer that

$$y(\cdot) \in \mathcal{E}^{\{\beta\}}((0, \infty), X),$$

which completes the proof of the “if” part.

“Only if” part. Let us prove this part *by contrapositive* assuming that, for any $b_+ > 0$ and $b_- > 0$, the set $\sigma(A) \setminus \mathcal{P}_{b_-, b_+}^\beta$ is *unbounded*. In particular, this means that, for any $n \in \mathbb{N}$, unbounded is the set

$$\sigma(A) \setminus \mathcal{P}_{n^{-1}, n^{-2}}^\beta = \left\{ \lambda \in \sigma(A) \mid -n^{-1} |\operatorname{Im} \lambda|^{1/\beta} < \operatorname{Re} \lambda < n^{-2} |\operatorname{Im} \lambda|^{1/\beta} \right\}.$$

Hence, we can choose a sequence $\{\lambda_n\}_{n=1}^\infty$ of points in the complex plane as follows:

$$\begin{aligned} \lambda_n &\in \sigma(A), \quad n \in \mathbb{N}, \\ -n^{-1} |\operatorname{Im} \lambda_n|^{1/\beta} &< \operatorname{Re} \lambda_n < n^{-2} |\operatorname{Im} \lambda_n|^{1/\beta}, \quad n \in \mathbb{N}, \\ \lambda_0 &:= 0, \quad |\lambda_n| > \max[n, |\lambda_{n-1}|], \quad n \in \mathbb{N}. \end{aligned}$$

The latter implies, in particular, that the points λ_n , $n \in \mathbb{N}$, are *distinct* ($\lambda_i \neq \lambda_j$, $i \neq j$).

Since, for each $n \in \mathbb{N}$, the set

$$\left\{ \lambda \in \mathbb{C} \mid -n^{-1} |\operatorname{Im} \lambda|^{1/\beta} < \operatorname{Re} \lambda < n^{-2} |\operatorname{Im} \lambda|^{1/\beta}, \quad |\lambda| > \max[n, |\lambda_{n-1}|] \right\}$$

is *open* in \mathbb{C} , along with the point λ_n , it contains an *open disk*

$$\Delta_n := \{\lambda \in \mathbb{C} \mid |\lambda - \lambda_n| < \varepsilon_n\}$$

centered at λ_n of some radius $\varepsilon_n > 0$, i.e., for each $\lambda \in \Delta_n$,

$$-n^{-1} |\operatorname{Im} \lambda|^{1/\beta} < \operatorname{Re} \lambda < n^{-2} |\operatorname{Im} \lambda|^{1/\beta} \text{ and } |\lambda| > \max[n, |\lambda_{n-1}|]. \quad (3.18)$$

Furthermore, we can regard the radii of the disks to be small enough so that

$$\begin{aligned} 0 < \varepsilon_n < \frac{1}{n}, \quad n \in \mathbb{N}, \text{ and} \\ \Delta_i \cap \Delta_j &= \emptyset, \quad i \neq j \quad (\text{i.e., the disks are pairwise disjoint}). \end{aligned} \quad (3.19)$$

Whence, by the properties of the s.m.,

$$E_A(\Delta_i)E_A(\Delta_j) = 0, \quad i \neq j,$$

where 0 stands for the *zero operator* on X .

Observe also that the subspaces $E_A(\Delta_n)X$, $n \in \mathbb{N}$, are *nontrivial* since

$$\Delta_n \cap \sigma(A) \neq \emptyset, \quad n \in \mathbb{N},$$

with Δ_n being an *open set* in \mathbb{C} .

By choosing a unit vector $e_n \in E_A(\Delta_n)X$ for each $n \in \mathbb{N}$, we obtain a sequence $\{e_n\}_{n=1}^\infty$ in X such that

$$\|e_n\| = 1, \quad n \in \mathbb{N}, \text{ and } E_A(\Delta_i)e_j = \delta_{ij}e_j, \quad i, j \in \mathbb{N}, \quad (3.20)$$

where δ_{ij} is the *Kronecker delta*.

As is easily seen, (3.20) implies that the vectors e_n , $n \in \mathbb{N}$, are *linearly independent*.

Furthermore, there exists an $\varepsilon > 0$ such that

$$d_n := \operatorname{dist}(e_n, \operatorname{span}(\{e_i \mid i \in \mathbb{N}, i \neq n\})) \geq \varepsilon, \quad n \in \mathbb{N}. \quad (3.21)$$

Indeed, the opposite implies the existence of a subsequence $\{d_{n(k)}\}_{k=1}^\infty$ such that

$$d_{n(k)} \rightarrow 0, \quad k \rightarrow \infty.$$

Then, by selecting a vector

$$f_{n(k)} \in \operatorname{span}(\{e_i \mid i \in \mathbb{N}, i \neq n(k)\}), \quad k \in \mathbb{N},$$

such that

$$\|e_{n(k)} - f_{n(k)}\| < d_{n(k)} + 1/k, \quad k \in \mathbb{N},$$

we arrive at

$$\begin{aligned} 1 &= \|e_{n(k)}\| \text{ since, by (3.20), } E_A(\Delta_{n(k)})f_{n(k)} = 0; \\ &= \|E_A(\Delta_{n(k)})(e_{n(k)} - f_{n(k)})\| \leq \|E_A(\Delta_{n(k)})\| \|e_{n(k)} - f_{n(k)}\| \text{ by (2.5);} \\ &\leq M \|e_{n(k)} - f_{n(k)}\| \leq M [d_{n(k)} + 1/k] \rightarrow 0, \quad k \rightarrow \infty, \end{aligned}$$

which is a *contradiction* proving (3.21).

As follows from the *Hahn-Banach Theorem*, for any $n \in \mathbb{N}$, there is an $e_n^* \in X^*$ such that

$$\|e_n^*\| = 1, \quad n \in \mathbb{N}, \quad \text{and } \langle e_i, e_j^* \rangle = \delta_{ij} d_i, \quad i, j \in \mathbb{N}. \quad (3.22)$$

Let us consider separately the two possibilities concerning the sequence of the real parts $\{\operatorname{Re} \lambda_n\}_{n=1}^\infty$: its being *bounded* or *unbounded*.

First, suppose that the sequence $\{\operatorname{Re} \lambda_n\}_{n=1}^\infty$ is *bounded*, i.e., there exists an $\omega > 0$ such that

$$|\operatorname{Re} \lambda_n| \leq \omega, \quad n \in \mathbb{N}, \quad (3.23)$$

and consider the element

$$f := \sum_{k=1}^{\infty} k^{-2} e_k \in X,$$

which is well defined since $\{k^{-2}\}_{k=1}^\infty \in l_1$ (l_1 is the space of absolutely summable sequences) and $\|e_k\| = 1$, $k \in \mathbb{N}$ (see (3.20)).

In view of (3.20), by the properties of the s.m.,

$$E_A(\cup_{k=1}^\infty \Delta_k)f = f \text{ and } E_A(\Delta_k)f = k^{-2} e_k, \quad k \in \mathbb{N}. \quad (3.24)$$

For an arbitrary $t \geq 0$ and any $g^* \in X^*$,

$$\int_{\sigma(A)} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \quad \text{by (3.24);}$$

$$= \int_{\sigma(A)} e^{t \operatorname{Re} \lambda} d\nu(E_A(\cup_{k=1}^\infty \Delta_k)f, g^*, \lambda) \quad \text{by (2.9);}$$

$$= \sum_{k=1}^{\infty} \int_{\sigma(A) \cap \Delta_k} e^{t \operatorname{Re} \lambda} d\nu(E_A(\Delta_k)f, g^*, \lambda) \quad \text{by (3.24);}$$

$$= \sum_{k=1}^{\infty} k^{-2} \int_{\sigma(A) \cap \Delta_k} e^{t \operatorname{Re} \lambda} d\nu(e_k, g^*, \lambda)$$

since, for $\lambda \in \Delta_k$;

by (3.23) and (3.19), $\operatorname{Re} \lambda = \operatorname{Re} \lambda_k + (\operatorname{Re} \lambda - \operatorname{Re} \lambda_k) \leq \operatorname{Re} \lambda_k + |\lambda - \lambda_k| \leq \omega + \varepsilon_k \leq \omega + 1$;

$$\leq e^{t(\omega+1)} \sum_{k=1}^{\infty} k^{-2} \int_{\sigma(A) \cap \Delta_k} 1 d\nu(e_k, g^*, \lambda) = e^{t(\omega+1)} \sum_{k=1}^{\infty} k^{-2} \nu(e_k, g^*, \Delta_k)$$

by (2.6);

$$\leq e^{t(\omega+1)} \sum_{k=1}^{\infty} k^{-2} 4M \|e_k\| \|g^*\| = 4M e^{t(\omega+1)} \|g^*\| \sum_{k=1}^{\infty} k^{-2} < \infty. \quad (3.25)$$

Similarly, for an arbitrary $t \geq 0$ and any $n \in \mathbb{N}$,

$$\begin{aligned}
& \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\}} e^{t \operatorname{Re} \lambda} dv(f, g^*, \lambda) \\
& \leq \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} e^{t(\omega+1)} \sum_{k=1}^{\infty} k^{-2} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\} \cap \Delta_k} 1 dv(e_k, g^*, \lambda) \\
& \quad \text{by (3.24);} \\
& = e^{t(\omega+1)} \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \sum_{k=1}^{\infty} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\} \cap \Delta_k} 1 dv(E_A(\Delta_k)f, g^*, \lambda) \\
& \quad \text{by (2.9);} \\
& = e^{t(\omega+1)} \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\}} 1 dv(E_A(\cup_{k=1}^{\infty} \Delta_k)f, g^*, \lambda) \\
& \quad \text{by (3.24);} \\
& = e^{t(\omega+1)} \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\}} 1 dv(f, g^*, \lambda) \\
& \quad \text{by (2.7);} \\
& \leq e^{t(\omega+1)} \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} 4M \left\| E_A \left(\left\{ \lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n \right\} \right) f \right\| \|g^*\| \\
& \leq 4Me^{t(\omega+1)} \left\| E_A \left(\left\{ \lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n \right\} \right) f \right\| \\
& \quad \text{by the strong continuity of the s.m.;} \\
& \rightarrow 4Me^{t(\omega+1)} \|E_A(\emptyset)f\| = 0, \quad n \rightarrow \infty. \tag{3.26}
\end{aligned}$$

By Proposition 2.1, (3.25) and (3.26) jointly imply that

$$f \in \bigcap_{t \geq 0} D(e^{tA}),$$

and hence, by Theorem 2.1,

$$y(t) := e^{tA}f, \quad t \geq 0,$$

is a weak solution of equation (1.1).

Let

$$h^* := \sum_{k=1}^{\infty} k^{-2} e_k^* \in X^*, \tag{3.27}$$

the functional being well defined since $\{k^{-2}\}_{k=1}^{\infty} \in l_1$ and $\|e_k^*\| = 1$, $k \in \mathbb{N}$ (see (3.22)).

In view of (3.22) and (3.21), we have:

$$\langle e_k, h^* \rangle = \langle e_k, k^{-2} e_k^* \rangle = d_k k^{-2} \geq \varepsilon k^{-2}, \quad k \in \mathbb{N}. \tag{3.28}$$

For any $s > 0$,

$$\begin{aligned}
& \int_{\sigma(A)} e^{s|\lambda|^{1/\beta}} e^{\operatorname{Re} \lambda} dv(f, h^*, \lambda) \quad \text{by (2.9) as in (3.25);} \\
& = \sum_{k=1}^{\infty} k^{-2} \int_{\sigma(A) \cap \Delta_k} e^{s|\lambda|^{1/\beta}} e^{\operatorname{Re} \lambda} dv(e_k, h^*, \lambda) \\
& \quad \text{since, for } \lambda \in \Delta_k, \text{ by (3.18), (3.23), and (3.19), } |\lambda| \geq k \text{ and} \\
& \quad \operatorname{Re} \lambda = \operatorname{Re} \lambda_k - (\operatorname{Re} \lambda_k - \operatorname{Re} \lambda) \geq \operatorname{Re} \lambda_k - |\operatorname{Re} \lambda_k - \operatorname{Re} \lambda| \geq -\omega - \varepsilon_k \geq -\omega - 1; \\
& \geq \sum_{k=1}^{\infty} k^{-2} e^{s k^{1/\beta}} e^{-(\omega+1)} v(e_k, h^*, \Delta_k) \geq \sum_{k=1}^{\infty} e^{-(\omega+1)} k^{-2} e^{s k^{1/\beta}} |\langle E_A(\Delta_k) e_k, h^* \rangle|
\end{aligned}$$

by (3.20) and (3.28);

$$\geq \sum_{k=1}^{\infty} \varepsilon e^{-(\omega+1)} k^{-4} e^{s k^{1/\beta}} = \infty. \quad (3.29)$$

By Proposition 2.1 and the properties of the o.c. (see [15, Theorem XVIII.2.11 (f)]), (3.29) implies that, for any $s > 0$,

$$f \notin D(e^{s|A|^{1/\beta}} e^A),$$

and hence, in view of (2.12),

$$y(1) = e^A f \notin \bigcup_{s>0} D(e^{s|A|^{1/\beta}}) = \mathcal{E}^{\{\beta\}}(A).$$

By Proposition 2.2, we infer that the weak solution $y(t) = e^{tA} f$, $t \geq 0$, of equation (1.1) does not belong to the Roumieu type Gevrey class $\mathcal{E}^{\{\beta\}}((0, \infty), X)$, which completes our consideration of the case of the sequence's $\{\operatorname{Re} \lambda_n\}_{n=1}^{\infty}$ being *bounded*.

Now, suppose that the sequence $\{\operatorname{Re} \lambda_n\}_{n=1}^{\infty}$ is *unbounded*.

Therefore, there is a subsequence $\{\operatorname{Re} \lambda_{n(k)}\}_{k=1}^{\infty}$ such that

$$\operatorname{Re} \lambda_{n(k)} \rightarrow \infty \text{ or } \operatorname{Re} \lambda_{n(k)} \rightarrow -\infty, \quad k \rightarrow \infty.$$

Let us consider separately each of the two cases.

First, suppose that

$$\operatorname{Re} \lambda_{n(k)} \rightarrow \infty, \quad k \rightarrow \infty.$$

Then, without loss of generality, we can regard that

$$\operatorname{Re} \lambda_{n(k)} \geq k, \quad k \in \mathbb{N}. \quad (3.30)$$

Consider the elements

$$f := \sum_{k=1}^{\infty} e^{-n(k) \operatorname{Re} \lambda_{n(k)}} e_{n(k)} \in X \text{ and } h := \sum_{k=1}^{\infty} e^{-\frac{n(k)}{2} \operatorname{Re} \lambda_{n(k)}} e_{n(k)} \in X,$$

well defined since, by (3.30),

$$\left\{ e^{-n(k) \operatorname{Re} \lambda_{n(k)}} \right\}_{k=1}^{\infty}, \left\{ e^{-\frac{n(k)}{2} \operatorname{Re} \lambda_{n(k)}} \right\}_{k=1}^{\infty} \in l_1$$

and $\|e_{n(k)}\| = 1$, $k \in \mathbb{N}$ (see (3.20)).

By (3.20),

$$E_A(\cup_{k=1}^{\infty} \Delta_{n(k)})f = f \text{ and } E_A(\Delta_{n(k)})f = e^{-n(k) \operatorname{Re} \lambda_{n(k)}} e_{n(k)}, \quad k \in \mathbb{N}, \quad (3.31)$$

and

$$E_A(\cup_{k=1}^{\infty} \Delta_{n(k)})h = h \text{ and } E_A(\Delta_{n(k)})h = e^{-\frac{n(k)}{2} \operatorname{Re} \lambda_{n(k)}} e_{n(k)}, \quad k \in \mathbb{N}. \quad (3.32)$$

For an arbitrary $t \geq 0$ and any $g^* \in X^*$,

$$\int_{\sigma(A)} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \quad \text{by (2.9) as in (3.25);}$$

$$= \sum_{k=1}^{\infty} e^{-n(k) \operatorname{Re} \lambda_{n(k)}} \int_{\sigma(A) \cap \Delta_{n(k)}} e^{t \operatorname{Re} \lambda} d\nu(e_{n(k)}, g^*, \lambda)$$

$$\begin{aligned} \text{since, for } \lambda \in \Delta_{n(k)}, \text{ by (3.19), } \operatorname{Re} \lambda &= \operatorname{Re} \lambda_{n(k)} + (\operatorname{Re} \lambda - \operatorname{Re} \lambda_{n(k)}) \\ &\leq \operatorname{Re} \lambda_{n(k)} + |\lambda - \lambda_{n(k)}| \leq \operatorname{Re} \lambda_{n(k)} + 1; \end{aligned}$$

$$\leq \sum_{k=1}^{\infty} e^{-n(k) \operatorname{Re} \lambda_{n(k)}} e^{t(\operatorname{Re} \lambda_{n(k)}+1)} \int_{\sigma(A) \cap \Delta_{n(k)}} 1 d\nu(e_{n(k)}, g^*, \lambda)$$

$$\begin{aligned}
&= e^t \sum_{k=1}^{\infty} e^{-[n(k)-t] \operatorname{Re} \lambda_{n(k)}} \nu(e_{n(k)}, g^*, \Delta_{n(k)}) && \text{by (2.6);} \\
&\leq e^t \sum_{k=1}^{\infty} e^{-[n(k)-t] \operatorname{Re} \lambda_{n(k)}} 4M \|e_{n(k)}\| \|g^*\| = 4Me^t \|g^*\| \sum_{k=1}^{\infty} e^{-[n(k)-t] \operatorname{Re} \lambda_{n(k)}} \\
&< \infty. && (3.33)
\end{aligned}$$

Indeed, for all $k \in \mathbb{N}$ sufficiently large so that

$$n(k) \geq t + 1,$$

in view of (3.30),

$$e^{-[n(k)-t] \operatorname{Re} \lambda_{n(k)}} \leq e^{-k}.$$

Similarly, for an arbitrary $t \geq 0$ and any $n \in \mathbb{N}$,

$$\begin{aligned}
&\sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\
&\leq \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} e^t \sum_{k=1}^{\infty} e^{-[n(k)-t] \operatorname{Re} \lambda_{n(k)}} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\} \cap \Delta_{n(k)}} 1 d\nu(e_{n(k)}, g^*, \lambda) \\
&= e^t \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \sum_{k=1}^{\infty} e^{-\left[\frac{n(k)}{2}-t\right] \operatorname{Re} \lambda_{n(k)}} e^{-\frac{n(k)}{2} \operatorname{Re} \lambda_{n(k)}} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\} \cap \Delta_{n(k)}} 1 d\nu(e_{n(k)}, g^*, \lambda)
\end{aligned}$$

since, by (3.30), there is an $L > 0$ such that $e^{-\left[\frac{n(k)}{2}-t\right] \operatorname{Re} \lambda_{n(k)}} \leq L$, $k \in \mathbb{N}$;

$$\begin{aligned}
&\leq Le^t \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \sum_{k=1}^{\infty} e^{-\frac{n(k)}{2} \operatorname{Re} \lambda_{n(k)}} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\} \cap \Delta_{n(k)}} 1 d\nu(e_{n(k)}, g^*, \lambda) \\
& && \text{by (3.32);}
\end{aligned}$$

$$\begin{aligned}
&= Le^t \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \sum_{k=1}^{\infty} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\} \cap \Delta_{n(k)}} 1 d\nu(E_A(\Delta_{n(k)})h, g^*, \lambda) \\
& && \text{by (2.9);}
\end{aligned}$$

$$\begin{aligned}
&= Le^t \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\}} 1 d\nu(E_A(\cup_{k=1}^{\infty} \Delta_{n(k)})h, g^*, \lambda) \\
& && \text{by (3.32);}
\end{aligned}$$

$$\begin{aligned}
&= Le^t \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\}} 1 d\nu(h, g^*, \lambda) \\
& && \text{by (2.7);}
\end{aligned}$$

$$\begin{aligned}
&\leq Le^t \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} 4M \left\| E_A \left(\left\{ \lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n \right\} \right) h \right\| \|g^*\| \\
&\leq 4LMe^t \|E_A(\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\})h\|
\end{aligned}$$

by the strong continuity of the s.m.;

$$\rightarrow 4LMe^t \|E_A(\emptyset)h\| = 0, \quad n \rightarrow \infty. \quad (3.34)$$

By Proposition 2.1, (3.33) and (3.34) jointly imply that

$$f \in \bigcap_{t \geq 0} D(e^{tA}),$$

and hence, by Theorem 2.1,

$$y(t) := e^{tA}f, \quad t \geq 0,$$

is a weak solution of equation (1.1).

Since, for any $\lambda \in \Delta_{n(k)}$, $k \in \mathbb{N}$, by (3.19), (3.30),

$$\begin{aligned} \operatorname{Re} \lambda &= \operatorname{Re} \lambda_{n(k)} - (\operatorname{Re} \lambda_{n(k)} - \operatorname{Re} \lambda) \geq \operatorname{Re} \lambda_{n(k)} - |\operatorname{Re} \lambda_{n(k)} - \operatorname{Re} \lambda| \\ &\geq \operatorname{Re} \lambda_{n(k)} - \varepsilon_{n(k)} \geq \operatorname{Re} \lambda_{n(k)} - 1/n(k) \geq k - 1 \geq 0 \end{aligned}$$

and, by (3.18),

$$\operatorname{Re} \lambda < n(k)^{-2} |\operatorname{Im} \lambda|^{1/\beta},$$

we infer that, for any $\lambda \in \Delta_{n(k)}$, $k \in \mathbb{N}$,

$$|\lambda| \geq |\operatorname{Im} \lambda| \geq \left[n(k)^2 \operatorname{Re} \lambda \right]^\beta \geq \left[n(k)^2 (\operatorname{Re} \lambda_{n(k)} - 1/n(k)) \right]^\beta.$$

Using this estimate, for an arbitrary $s > 0$ and the functional $h^* \in X^*$ defined by (3.27), we have:

$$\begin{aligned} \int_{\sigma(A)} e^{s|\lambda|^{1/\beta}} dv(f, h^*, \lambda) & \quad \text{by (2.9) as in (3.25);} \\ &= \sum_{k=1}^{\infty} e^{-n(k) \operatorname{Re} \lambda_{n(k)}} \int_{\sigma(A) \cap \Delta_{n(k)}} e^{s|\lambda|^{1/\beta}} dv(e_{n(k)}, h^*, \lambda) \\ &\geq \sum_{k=1}^{\infty} e^{-n(k) \operatorname{Re} \lambda_{n(k)}} e^{sn(k)^2 (\operatorname{Re} \lambda_{n(k)} - 1/n(k))} v(e_{n(k)}, h^*, \Delta_{n(k)}) \\ &\geq \sum_{k=1}^{\infty} e^{-n(k) \operatorname{Re} \lambda_{n(k)}} e^{sn(k)^2 (\operatorname{Re} \lambda_{n(k)} - 1/n(k))} |\langle E_A(\Delta_{n(k)}) e_{n(k)}, h^* \rangle| \\ & \quad \text{by (3.20) and (3.28);} \\ &\geq \sum_{k=1}^{\infty} \varepsilon e^{(sn(k)-1)n(k) \operatorname{Re} \lambda_{n(k)} - sn(k)} n(k)^{-2} = \infty. \end{aligned} \quad (3.35)$$

Indeed, for all $k \in \mathbb{N}$ sufficiently large so that

$$sn(k) \geq s + 2,$$

in view of (3.30),

$$e^{(sn(k)-1)n(k) \operatorname{Re} \lambda_{n(k)} - sn(k)} n(k)^{-2} \geq e^{(s+1)n(k) - sn(k)} n(k)^{-2} = e^{n(k)} n(k)^{-2} \rightarrow \infty, \quad k \rightarrow \infty.$$

By Proposition 2.1 and the properties of the o.c. (see [15, Theorem XVIII.2.11 (f)]), (3.35) implies that, for any $s > 0$,

$$f \notin D(e^{s|A|^{1/\beta}} e^A),$$

which, in view of (2.12), further implies that

$$y(1) = e^A f \notin \bigcup_{s>0} D(e^{s|A|^{1/\beta}}) = \mathcal{E}^{\{\beta\}}(A).$$

Whence, by Proposition 2.2, we infer that the weak solution $y(t) = e^{tA}f$, $t \geq 0$, of equation (1.1) does not belong to the Roumieu type Gevrey class $\mathcal{E}^{\{\beta\}}((0, \infty), X)$.

Now, suppose that

$$\operatorname{Re} \lambda_{n(k)} \rightarrow -\infty, \quad k \rightarrow \infty$$

Then, without loss of generality, we can regard that

$$\operatorname{Re} \lambda_{n(k)} \leq -k, \quad k \in \mathbb{N}. \quad (3.36)$$

Consider the element

$$f := \sum_{k=1}^{\infty} k^{-2} e_{n(k)} \in X,$$

which is well defined since $\{k^{-2}\}_{k=1}^{\infty} \in l_1$ and $\|e_{n(k)}\| = 1, k \in \mathbb{N}$ (see (3.20)).

By (3.20),

$$E_A(\cup_{k=1}^{\infty} \Delta_{n(k)})f = f \text{ and } E_A(\Delta_{n(k)})f = k^{-2} e_{n(k)}, \quad k \in \mathbb{N}. \quad (3.37)$$

For an arbitrary $t \geq 0$ and any $g^* \in X^*$,

$$\int_{\sigma(A)} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \quad \text{by (2.9) as in (3.25);}$$

$$= \sum_{k=1}^{\infty} k^{-2} \int_{\sigma(A) \cap \Delta_{n(k)}} e^{t \operatorname{Re} \lambda} d\nu(e_{n(k)}, g^*, \lambda)$$

since, for $\lambda \in \Delta_{n(k)}$, by (3.36) and (3.19),

$$\operatorname{Re} \lambda = \operatorname{Re} \lambda_{n(k)} + (\operatorname{Re} \lambda - \operatorname{Re} \lambda_{n(k)}) \leq \operatorname{Re} \lambda_{n(k)} + |\lambda - \lambda_{n(k)}| \leq -k + 1 \leq 0;$$

$$\leq \sum_{k=1}^{\infty} k^{-2} \int_{\sigma(A) \cap \Delta_{n(k)}} 1 d\nu(e_{n(k)}, g^*, \lambda) = \sum_{k=1}^{\infty} k^{-2} \nu(e_{n(k)}, g^*, \Delta_{n(k)}) \quad \text{by (2.6);}$$

$$\leq \sum_{k=1}^{\infty} k^{-2} 4M \|e_{n(k)}\| \|g^*\| = 4M \|g^*\| \sum_{k=1}^{\infty} k^{-2} < \infty. \quad (3.38)$$

Similarly, for an arbitrary $t \geq 0$ and any $g^* \in X^*$,

$$\begin{aligned} & \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\ & \leq \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \sum_{k=1}^{\infty} k^{-2} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\} \cap \Delta_{n(k)}} 1 d\nu(e_{n(k)}, g^*, \lambda) \end{aligned} \quad \text{by (3.37);}$$

$$\begin{aligned} & \leq \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \sum_{k=1}^{\infty} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\} \cap \Delta_{n(k)}} 1 d\nu(E_A(\Delta_{n(k)})f, g^*, \lambda) \\ & \quad \text{by (2.9);} \end{aligned}$$

$$= \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\}} 1 d\nu(E_A(\cup_{k=1}^{\infty} \Delta_{n(k)})f, g^*, \lambda) \quad \text{by (3.37);}$$

$$= \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\}} 1 d\nu(f, g^*, \lambda) \quad \text{by (2.7);}$$

$$\begin{aligned} & \leq \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} 4M \left\| E_A \left(\left\{ \lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n \right\} \right) f \right\| \|g^*\| \\ & \leq 4M \|E_A(\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\})f\| \end{aligned}$$

by the strong continuity of the s.m.;

$$\rightarrow 4M \|E_A(\emptyset)f\| = 0, \quad n \rightarrow \infty. \quad (3.39)$$

By Proposition 2.1, (3.38) and (3.39) jointly imply that

$$f \in \bigcap_{t \geq 0} D(e^{tA}),$$

and hence, by Theorem 2.1,

$$y(t) := e^{tA}f, \quad t \geq 0,$$

is a weak solution of equation (1.1).

Let

$$h^* := \sum_{k=1}^{\infty} k^{-2} e_{n(k)}^* \in X^*, \quad (3.40)$$

the functional being well defined since $\{k^{-2}\}_{k=1}^{\infty} \in l_1$ and $\|e_{n(k)}^*\| = 1, k \in \mathbb{N}$ (see (3.22)).

In view of (3.22) and (3.21), we have:

$$\langle e_{n(k)}, h^* \rangle = \langle e_{n(k)}, k^{-2} e_{n(k)}^* \rangle = d_{n(k)} k^{-2} \geq \varepsilon k^{-2}, \quad k \in \mathbb{N}. \quad (3.41)$$

Since, for any $\lambda \in \Delta_{n(k)}, k \in \mathbb{N}$, by (3.36) and (3.19),

$$\begin{aligned} \operatorname{Re} \lambda &= \operatorname{Re} \lambda_{n(k)} + (\operatorname{Re} \lambda - \operatorname{Re} \lambda_{n(k)}) \leq \operatorname{Re} \lambda_{n(k)} + |\operatorname{Re} \lambda - \operatorname{Re} \lambda_{n(k)}| \\ &\leq \operatorname{Re} \lambda_{n(k)} + \varepsilon_{n(k)} \leq -k + 1 \leq 0 \end{aligned} \quad (3.42)$$

and, by (3.18),

$$-n(k)^{-1} |\operatorname{Im} \lambda|^{1/\beta} < \operatorname{Re} \lambda,$$

we infer that, for any $\lambda \in \Delta_{n(k)}, k \in \mathbb{N}$,

$$|\lambda| \geq |\operatorname{Im} \lambda| \geq [n(k)(-\operatorname{Re} \lambda)]^\beta.$$

Using this estimate, for an arbitrary $s > 0$ and the functional $h^* \in X^*$ defined by (3.59), we have:

$$\begin{aligned} \int_{\sigma(A)} e^{s|\lambda|^{1/\beta}} e^{\operatorname{Re} \lambda} d\nu(f, h^*, \lambda) & \quad \text{by (2.9) as in (3.25);} \\ &= \sum_{k=1}^{\infty} k^{-2} \int_{\sigma(A) \cap \Delta_{n(k)}} e^{s|\lambda|^{1/\beta}} e^{\operatorname{Re} \lambda} d\nu(e_{n(k)}, h^*, \lambda) \\ &\geq \sum_{k=1}^{\infty} k^{-2} \int_{\sigma(A) \cap \Delta_{n(k)}} e^{[sn(k)-1](-\operatorname{Re} \lambda)} d\nu(e_{n(k)}, h^*, \lambda) = \infty. \end{aligned} \quad (3.43)$$

Indeed, for all $k \in \mathbb{N}$ sufficiently large so that

$$sn(k) \geq 2,$$

we have:

$$\begin{aligned} k^{-2} \int_{\sigma(A) \cap \Delta_{n(k)}} e^{[sn(k)-1](-\operatorname{Re} \lambda)} d\nu(e_{n(k)}, h^*, \lambda) &\geq k^{-2} \int_{\sigma(A) \cap \Delta_{n(k)}} e^{-\operatorname{Re} \lambda} d\nu(e_{n(k)}, h^*, \lambda) \\ & \quad \text{by (3.42);} \\ &\geq k^{-2} e^{k-1} \int_{\sigma(A) \cap \Delta_{n(k)}} 1 d\nu(e_{n(k)}, h^*, \lambda) = k^{-2} e^{k-1} \nu(e_{n(k)}, h^*, \Delta_{n(k)}) \\ &\geq k^{-2} e^{k-1} |\langle E_A(\Delta_{n(k)}) e_{n(k)}, h^* \rangle| \quad \text{by (3.20) and (3.41);} \\ &\geq \varepsilon k^{-4} e^{k-1} \rightarrow \infty, \quad k \rightarrow \infty. \end{aligned}$$

By Proposition 2.1 and the properties of the o.c. (see [15, Theorem XVIII.2.11 (f)]), (3.43) implies that, for any $s > 0$,

$$f \notin D(e^{s|A|^{1/\beta}} e^A),$$

which, in view of (2.12), further implies that

$$y(1) = e^A f \notin \bigcup_{s>0} D(e^{s|A|^{1/\beta}}) = \mathcal{E}^{\{\beta\}}(A).$$

Whence, by Proposition 2.2, we infer that the weak solution $y(t) = e^{tA}f$, $t \geq 0$, of equation (1.1) does not belong to the Roumieu type Gevrey class $\mathcal{E}^{\{\beta\}}((0, \infty), X)$, which completes our consideration of the case of the sequence's $\{\operatorname{Re} \lambda_n\}_{n=1}^\infty$ being *unbounded*.

With every possibility concerning $\{\operatorname{Re} \lambda_n\}_{n=1}^\infty$ considered, the proof by contrapositive of the “only if” part is complete and so is the proof of the entire statement. \square

For $\beta = 1$, we obtain the following important particular case.

Corollary 3.1 (Characterization of the analyticity of weak solutions on $(0, \infty)$). *Let A be a scalar type spectral operator in a complex Banach space $(X, \|\cdot\|)$. Every weak solution of the equation (1.1) is analytic on $(0, \infty)$ iff there exist $b_+ > 0$ and $b_- > 0$ such that the set $\sigma(A) \setminus \mathcal{P}_{b_-, b_+}^1$, where*

$$\mathcal{P}_{b_-, b_+}^1 := \{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \leq -b_- |\operatorname{Im} \lambda| \text{ or } \operatorname{Re} \lambda \geq b_+ |\operatorname{Im} \lambda|\},$$

is bounded (see Figure 2).

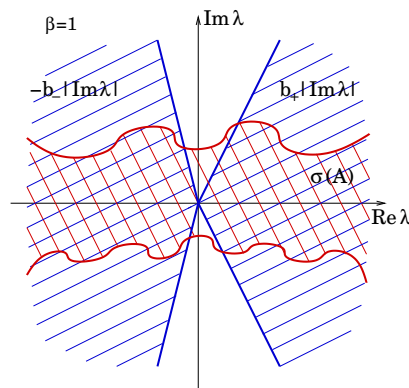


Figure 2: The case of $\beta = 1$.

Remark 3.1. Thus, we have obtained a generalization of [1, Theorem 4.2], the counterpart for a normal operator A in a complex Hilbert space, and of [2, Theorem 5.1] (cf. [3]), a characterization of the generation of a Roumieu type Gevrey ultradifferentiable C_0 -semigroup by a scalar type spectral operator A .

Now, let us treat the Beurling type strong Gevrey ultradifferentiability of order $\beta > 1$. Observe that the case of *entireness* ($\beta = 1$) is included in [6, Theorem 4.1] (see also [6, Corollary 4.1]).

Theorem 3.2. *Let A be a scalar type spectral operator in a complex Banach space $(X, \|\cdot\|)$ with spectral measure $E_A(\cdot)$ and $1 < \beta < \infty$. Every weak solution of equation (1.1) belongs to the β th-order Beurling type Gevrey class $\mathcal{E}^{(\beta)}((0, \infty), X)$ iff there exists a $b_+ > 0$ such that, for any $b_- > 0$, the set $\sigma(A) \setminus \mathcal{P}_{b_-, b_+}^\beta$, where*

$$\mathcal{P}_{b_-, b_+}^\beta := \left\{ \lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \leq -b_- |\operatorname{Im} \lambda|^{1/\beta} \text{ or } \operatorname{Re} \lambda \geq b_+ |\operatorname{Im} \lambda|^{1/\beta} \right\},$$

is bounded (see Figure 1).

Proof.

“If” Part. Suppose that there exists a $b_+ > 0$ such that, for any $b_- > 0$, the set $\sigma(A) \setminus \mathcal{P}_{b_-, b_+}^\beta$ is bounded and let $y(\cdot)$ be an arbitrary weak solution of equation (1.1).

By Theorem 2.1,

$$y(t) = e^{tA}f, \quad t \geq 0, \quad \text{with some } f \in \bigcap_{t \geq 0} D(e^{tA}).$$

Our purpose is to show that $y(\cdot) \in \mathcal{E}^{(\beta)}((0, \infty), X)$, which, by Proposition 2.2 and (2.12), is attained by showing that, for each $t > 0$,

$$y(t) \in \mathcal{E}^{(\beta)}(A) = \bigcap_{s > 0} D(e^{s|A|^{1/\beta}}).$$

Let us proceed by proving that, for any $t > 0$ and $s > 0$,

$$y(t) \in D(e^{s|A|^{1/\beta}})$$

via Proposition 2.1.

Since $\beta > 1$, for any $b_- > 0$, there exists a $c(b_-) > 0$ such that

$$x \leq b_-^{-\beta} x^\beta, \quad x \geq c(b_-). \quad (3.44)$$

Fixing arbitrary $t > 0$ and $s > 0$, since $b_- > 0$ is random, we can set

$$b_- := 2^{1/\beta} s t^{-1} > 0, \quad (3.45)$$

such a peculiar choice explaining itself in the process.

For arbitrary $t > 0$ and $s > 0$, $b_- > 0$ chosen as in (3.45), and any $g^* \in X^*$,

$$\begin{aligned} \int_{\sigma(A)} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) &= \int_{\sigma(A) \setminus \mathcal{P}_{b_-, b_+}^\beta} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\ &+ \int_{\left\{ \lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid -c(b_-) < \operatorname{Re} \lambda < 1 \right\}} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\ &+ \int_{\left\{ \lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \geq 1 \right\}} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\ &+ \int_{\left\{ \lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \leq -c(b_-) \right\}} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) < \infty. \end{aligned} \quad (3.46)$$

Indeed,

$$\int_{\sigma(A) \setminus \mathcal{P}_{b_-, b_+}^\beta} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) < \infty$$

and

$$\int_{\left\{ \lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid -c(b_-) < \operatorname{Re} \lambda < 1 \right\}} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) < \infty$$

due to the boundedness of the sets

$$\sigma(A) \setminus \mathcal{P}_{b_-, b_+}^\beta \quad \text{and} \quad \left\{ \lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid -c(b_-) < \operatorname{Re} \lambda < 1 \right\},$$

the continuity of the integrated function on \mathbb{C} , and the finiteness of the measure $\nu(f, g^*, \cdot)$.

Further, for arbitrary $t > 0$, $s > 0$, $b_- > 0$ chosen as in (3.45), and any $g^* \in X^*$,

Also, for arbitrary $t > 0$ and $s > 0$, $b_- > 0$ chosen as in (3.45), and any $g^* \in X^*$,

$$\begin{aligned}
 & \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\left\{\lambda \in \sigma(A) \mid e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n\right\}} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\
 & \leq \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\left\{\lambda \in \sigma(A) \setminus \mathcal{P}_{b_-, b_+}^\beta \mid e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n\right\}} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\
 & + \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\left\{\lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid -c(b_-) < \operatorname{Re} \lambda < 1, e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n\right\}} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\
 & + \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\left\{\lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \geq 1, e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n\right\}} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\
 & + \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\left\{\lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \leq -c(b_-), e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n\right\}} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\
 & \rightarrow 0, \quad n \rightarrow \infty.
 \end{aligned} \tag{3.49}$$

Indeed, since, due to the boundedness of the sets

$$\sigma(A) \setminus \mathcal{P}_{b_-, b_+}^\beta \text{ and } \left\{\lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid -c(b_-) < \operatorname{Re} \lambda < 1\right\}$$

and the continuity of the integrated function on \mathbb{C} , the sets

$$\left\{\lambda \in \sigma(A) \setminus \mathcal{P}_{b_-, b_+}^\beta \mid e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n\right\}$$

and

$$\left\{\lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid -c(b_-) < \operatorname{Re} \lambda < 1, e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n\right\}$$

are empty for all sufficiently large $n \in \mathbb{N}$, we immediately infer that, for any $t > 0$, $s > 0$ and $b_- > 0$ chosen as in (3.45),

$$\lim_{n \rightarrow \infty} \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\left\{\lambda \in \sigma(A) \setminus \mathcal{P}_{b_-, b_+}^\beta \mid e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n\right\}} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) = 0$$

and

$$\lim_{n \rightarrow \infty} \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\left\{\lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid -c(b_-) < \operatorname{Re} \lambda < 1, e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n\right\}} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) = 0.$$

Further, for arbitrary $t > 0$, $s > 0$, $b_- > 0$ chosen as in (3.45), and any $g^* \in X^*$,

$$\begin{aligned}
 & \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\left\{\lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \geq 1, e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n\right\}} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \\
 & \leq \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\left\{\lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \geq 1, e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n\right\}} e^{\left[s(1+b_+^{-\beta})^{1/\beta} + t\right] \operatorname{Re} \lambda} d\nu(f, g^*, \lambda)
 \end{aligned} \tag{3.47};$$

since $f \in \bigcap_{t \geq 0} D(e^{tA})$, by (2.7);

$$\leq \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} 4M \left\| E_A \left(\left\{\lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \geq 1, e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n\right\} \right) e^{\left[s(1+b_+^{-\beta})^{1/\beta} + t\right] A} f \right\| \|g^*\|$$

$$\leq 4M \left\| E_A \left(\left\{ \lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \geq 1, e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n \right\} \right) e^{\left[s(1+b_+^{-\beta})^{1/\beta} + t \right] A} f \right\|$$

by the strong continuity of the *s.m.*;

$$\rightarrow 4M \left\| E_A(\emptyset) e^{\left[s(1+b_+^{-\beta})^{1/\beta} + t \right] A} f \right\| = 0, \quad n \rightarrow \infty.$$

Finally, for arbitrary $t > 0$ and $s > 0$, $b_- > 0$ chosen as in (3.45), and any $g^* \in X^*$,

$$\sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\left\{ \lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \leq -c(b_-), e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n \right\}} e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} dv(f, g^*, \lambda)$$

as in (3.48);

$$\leq \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\left\{ \lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \leq -c(b_-), e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n \right\}} e^{\left[t - s2^{1/\beta} b_-^{-1} \right] \operatorname{Re} \lambda} dv(f, g^*, \lambda)$$

by the choice of $b_- > 0$ (see (3.45));

$$= \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\left\{ \lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \leq -c(b_-), e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n \right\}} 1 dv(f, g^*, \lambda)$$

by (2.7);

$$\leq \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} 4M \left\| E_A \left(\left\{ \lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \leq -c(b_-), e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n \right\} \right) f \right\| \|g^*\|$$

$$\leq 4M \left\| E_A \left(\left\{ \lambda \in \sigma(A) \cap \mathcal{P}_{b_-, b_+}^\beta \mid \operatorname{Re} \lambda \leq -c(b_-), e^{s|\lambda|^{1/\beta}} e^{t \operatorname{Re} \lambda} > n \right\} \right) f \right\|$$

by the strong continuity of the *s.m.*;

$$\rightarrow 4M \|E_A(\emptyset) f\| = 0, \quad n \rightarrow \infty.$$

By Proposition 2.1 and the properties of the o.c. (see [15, Theorem XVIII.2.11 (f)]), (3.46) and (3.49) jointly imply that, for any $t > 0$ and $s > 0$,

$$f \in D(e^{s|A|^{1/\beta}} e^{tA}),$$

which, in view of (2.12), further implies that, for each $t > 0$,

$$y(t) = e^{tA} f \in \bigcap_{s>0} D(e^{s|A|^{1/\beta}}) = \mathcal{E}^{(\beta)}(A).$$

Whence, by Proposition 2.2, we infer that

$$y(\cdot) \in \mathcal{E}^{(\beta)}((0, \infty), X),$$

which completes the proof of the “if” part.

“Only if” part. Let us prove this part *by contrapositive* assuming that, for any $b_+ > 0$, there exists a $b_- > 0$ such that the set $\sigma(A) \setminus \mathcal{P}_{b_-, b_+}^\beta$ is *unbounded*.

Let us show that, under the circumstances, we can equivalently set the following seemingly stronger hypothesis: there exists a $b_- > 0$ such that, for any $b_+ > 0$, the set $\sigma(A) \setminus \mathcal{P}_{b_-, b_+}^\beta$ is unbounded.

Indeed, under the premise, there are two possibilities:

1. For some $b_- > 0$, the set

$$\left\{ \lambda \in \sigma(A) \mid -b_- |\operatorname{Im} \lambda|^{1/\beta} < \operatorname{Re} \lambda \leq 0 \right\}$$

is *unbounded*.

2. For any $b_- > 0$, the set

$$\left\{ \lambda \in \sigma(A) \mid -b_- |\operatorname{Im} \lambda|^{1/\beta} < \operatorname{Re} \lambda \leq 0 \right\}$$

is *bounded*.

In the first case, as is easily seen, the set $\sigma(A) \setminus \mathcal{P}_{b_-, b_+}^\beta$ is also unbounded for some $b_- > 0$ and any $b_+ > 0$.

In the second case, by the premise, we infer that, for any $b_+ > 0$, unbounded is the set

$$\left\{ \lambda \in \sigma(A) \mid 0 < \operatorname{Re} \lambda < b_+ |\operatorname{Im} \lambda|^{1/\beta} \right\},$$

which makes the set $\sigma(A) \setminus \mathcal{P}_{b_-, b_+}^\beta$ unbounded for any $b_- > 0$ and $b_+ > 0$.

The foregoing equivalent version of the premise implies, in particular, that, for some $b_- > 0$ and any $n \in \mathbb{N}$, unbounded is the set

$$\sigma(A) \setminus \mathcal{P}_{b_-, n^{-2}}^\beta = \left\{ \lambda \in \sigma(A) \mid -b_- |\operatorname{Im} \lambda|^{1/\beta} < \operatorname{Re} \lambda < n^{-2} |\operatorname{Im} \lambda|^{1/\beta} \right\}.$$

Hence, we can choose a sequence $\{\lambda_n\}_{n=1}^\infty$ of points in the complex plane as follows:

$$\begin{aligned} \lambda_n &\in \sigma(A), \quad n \in \mathbb{N}, \\ -b_- |\operatorname{Im} \lambda_n|^{1/\beta} &< \operatorname{Re} \lambda_n < n^{-2} |\operatorname{Im} \lambda_n|^{1/\beta}, \quad n \in \mathbb{N}, \\ \lambda_0 &:= 0, \quad |\lambda_n| > \max[n, |\lambda_{n-1}|], \quad n \in \mathbb{N}. \end{aligned}$$

The latter implies, in particular, that the points λ_n , $n \in \mathbb{N}$, are *distinct* ($\lambda_i \neq \lambda_j$, $i \neq j$).

Since, for each $n \in \mathbb{N}$, the set

$$\left\{ \lambda \in \mathbb{C} \mid -b_- |\operatorname{Im} \lambda|^{1/\beta} < \operatorname{Re} \lambda < n^{-2} |\operatorname{Im} \lambda|^{1/\beta}, \quad |\lambda| > \max[n, |\lambda_{n-1}|] \right\}$$

is *open* in \mathbb{C} , along with the point λ_n , it contains an *open disk*

$$\Delta_n := \{ \lambda \in \mathbb{C} \mid |\lambda - \lambda_n| < \varepsilon_n \}$$

centered at λ_n of some radius $\varepsilon_n > 0$, i.e., for each $\lambda \in \Delta_n$,

$$-b_- |\operatorname{Im} \lambda|^{1/\beta} < \operatorname{Re} \lambda < n^{-2} |\operatorname{Im} \lambda|^{1/\beta} \text{ and } |\lambda| > \max[n, |\lambda_{n-1}|]. \quad (3.50)$$

Furthermore, under the circumstances, we can regard the radii of the disks to be small enough so that

$$\begin{aligned} 0 < \varepsilon_n &< \frac{1}{n}, \quad n \in \mathbb{N}, \text{ and} \\ \Delta_i \cap \Delta_j &= \emptyset, \quad i \neq j \quad (\text{i.e., the disks are pairwise disjoint}). \end{aligned} \quad (3.51)$$

Whence, by the properties of the *s.m.*,

$$E_A(\Delta_i)E_A(\Delta_j) = 0, \quad i \neq j,$$

where 0 stands for the *zero operator* on X .

Observe also that the subspaces $E_A(\Delta_n)X$, $n \in \mathbb{N}$, are *nontrivial* since

$$\Delta_n \cap \sigma(A) \neq \emptyset, \quad n \in \mathbb{N},$$

with Δ_n being an *open set* in \mathbb{C} .

By choosing a unit vector $e_n \in E_A(\Delta_n)X$ for each $n \in \mathbb{N}$, we obtain a sequence $\{e_n\}_{n=1}^\infty$ such that

$$\|e_n\| = 1, \quad n \in \mathbb{N}, \text{ and } E_A(\Delta_i)e_j = \delta_{ij}e_j, \quad i, j \in \mathbb{N}, \quad (3.52)$$

where δ_{ij} is the *Kronecker delta*.

As is easily seen, (3.52) implies that the vectors e_n , $n \in \mathbb{N}$, are *linearly independent*.

Furthermore, there exists an $\varepsilon > 0$ such that

$$d_n := \text{dist}(e_n, \text{span}(\{e_i \mid i \in \mathbb{N}, i \neq n\})) \geq \varepsilon, \quad n \in \mathbb{N}. \quad (3.53)$$

Indeed, the opposite implies the existence of a subsequence $\{d_{n(k)}\}_{k=1}^\infty$ such that

$$d_{n(k)} \rightarrow 0, \quad k \rightarrow \infty.$$

Then, by selecting a vector

$$f_{n(k)} \in \text{span}(\{e_i \mid i \in \mathbb{N}, i \neq n(k)\}), \quad k \in \mathbb{N},$$

such that

$$\|e_{n(k)} - f_{n(k)}\| < d_{n(k)} + 1/k, \quad k \in \mathbb{N},$$

we arrive at

$$\begin{aligned} 1 &= \|e_{n(k)}\| && \text{since, by (3.52), } E_A(\Delta_{n(k)})f_{n(k)} = 0; \\ &= \|E_A(\Delta_{n(k)})(e_{n(k)} - f_{n(k)})\| \leq \|E_A(\Delta_{n(k)})\| \|e_{n(k)} - f_{n(k)}\| && \text{by (2.5);} \\ &\leq M \|e_{n(k)} - f_{n(k)}\| \leq M [d_{n(k)} + 1/k] \rightarrow 0, \quad k \rightarrow \infty, \end{aligned}$$

which is a *contradiction* proving (3.53).

As follows from the *Hahn-Banach Theorem*, for any $n \in \mathbb{N}$, there is an $e_n^* \in X^*$ such that

$$\|e_n^*\| = 1, \quad n \in \mathbb{N}, \quad \text{and } \langle e_i, e_j^* \rangle = \delta_{ij} d_i, \quad i, j \in \mathbb{N}. \quad (3.54)$$

Let us consider separately the two possibilities concerning the sequence of the real parts $\{\text{Re } \lambda_n\}_{n=1}^\infty$: its being *bounded* or *unbounded*.

The case of the sequence's $\{\text{Re } \lambda_n\}_{n=1}^\infty$ being *bounded* is considered in absolutely the same manner as the corresponding case in the proof of the "*only if*" part of Theorem 3.1 and furnishes a weak solution $y(\cdot)$ of equation (1.1) such that

$$y(1) \notin \mathcal{E}^{\{\beta\}}(A).$$

Hence, by Proposition 2.2, $y(\cdot)$ does not belong to the Roumieu type Gevrey class $\mathcal{E}^{\{\beta\}}((0, \infty), X)$, and the more so, the narrower Beurling type Gevrey class $\mathcal{E}^{(\beta)}((0, \infty), X)$.

Now, suppose that the sequence $\{\text{Re } \lambda_n\}_{n=1}^\infty$ is *unbounded*.

Therefore, there is a subsequence $\{\text{Re } \lambda_{n(k)}\}_{k=1}^\infty$ such that

$$\text{Re } \lambda_{n(k)} \rightarrow \infty \text{ or } \text{Re } \lambda_{n(k)} \rightarrow -\infty, \quad k \rightarrow \infty.$$

Let us consider separately each of the two cases.

The case of

$$\text{Re } \lambda_{n(k)} \rightarrow \infty, \quad k \rightarrow \infty$$

is also considered in the same manner as the corresponding case in the proof of the "*only if*" part of Theorem 3.1, and again furnishes a weak solution $y(\cdot)$ of equation (1.1) such that

$$y(1) \notin \mathcal{E}^{\{\beta\}}(A).$$

Hence, by Proposition 2.2, $y(\cdot)$ does not belong to the Roumieu type Gevrey class $\mathcal{E}^{\{\beta\}}((0, \infty), X)$, let alone, the narrower Beurling type Gevrey class $\mathcal{E}^{(\beta)}((0, \infty), X)$.

Suppose that

$$\text{Re } \lambda_{n(k)} \rightarrow -\infty, \quad k \rightarrow \infty.$$

Then, without loss of generality, we can regard that

$$\text{Re } \lambda_{n(k)} \leq -k, \quad k \in \mathbb{N}. \quad (3.55)$$

Consider the element

$$f := \sum_{k=1}^{\infty} k^{-2} e_{n(k)} \in X,$$

which is well defined since $\{k^{-2}\}_{k=1}^{\infty} \in l_1$ and $\|e_{n(k)}\| = 1$, $k \in \mathbb{N}$ (see (3.52)).

By (3.52),

$$E_A(\cup_{k=1}^{\infty} \Delta_{n(k)})f = f \text{ and } E_A(\Delta_{n(k)})f = k^{-2} e_{n(k)}, \quad k \in \mathbb{N}. \quad (3.56)$$

For arbitrary $t \geq 0$ and any $g^* \in X^*$,

$$\begin{aligned} & \int_{\sigma(A)} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \quad \text{by (2.9) as in (3.25);} \\ &= \sum_{k=1}^{\infty} k^{-2} \int_{\sigma(A) \cap \Delta_{n(k)}} e^{t \operatorname{Re} \lambda} d\nu(e_{n(k)}, g^*, \lambda) \\ & \quad \text{since, for } \lambda \in \Delta_{n(k)}, \text{ by (3.55) and (3.51),} \\ & \quad \operatorname{Re} \lambda = \operatorname{Re} \lambda_{n(k)} + (\operatorname{Re} \lambda - \operatorname{Re} \lambda_{n(k)}) \leq \operatorname{Re} \lambda_{n(k)} + |\lambda - \lambda_{n(k)}| \leq -k + 1 \leq 0; \\ & \leq \sum_{k=1}^{\infty} k^{-2} \int_{\sigma(A) \cap \Delta_{n(k)}} 1 d\nu(e_{n(k)}, g^*, \lambda) = \sum_{k=1}^{\infty} k^{-2} \nu(e_{n(k)}, g^*, \Delta_{n(k)}) \quad \text{by (2.6);} \\ & \leq \sum_{k=1}^{\infty} k^{-2} 4M \|e_{n(k)}\| \|g^*\| = 4M \|g^*\| \sum_{k=1}^{\infty} k^{-2} < \infty. \end{aligned} \quad (3.57)$$

Similarly, for arbitrary $t \geq 0$ and any $g^* \in X^*$,

$$\begin{aligned} & \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\}} e^{t \operatorname{Re} \lambda} d\nu(f, g^*, \lambda) \quad \text{as in (3.57);} \\ & \leq \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \sum_{k=1}^{\infty} k^{-2} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\} \cap \Delta_{n(k)}} 1 d\nu(e_{n(k)}, g^*, \lambda) \\ & \quad \text{by (3.56);} \\ & = \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \sum_{k=1}^{\infty} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\} \cap \Delta_{n(k)}} 1 d\nu(E_A(\Delta_{n(k)})f, g^*, \lambda) \\ & \quad \text{by (2.9);} \\ & = \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\}} 1 d\nu(E_A(\cup_{k=1}^{\infty} \Delta_{n(k)})f, g^*, \lambda) \quad \text{by (3.56);} \\ & = \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} \int_{\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\}} 1 d\nu(f, g^*, \lambda) \quad \text{by (2.7);} \\ & \leq \sup_{\{g^* \in X^* \mid \|g^*\|=1\}} 4M \left\| E_A \left(\left\{ \lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n \right\} \right) f \right\| \|g^*\| \\ & \leq 4M \|E_A(\{\lambda \in \sigma(A) \mid e^{t \operatorname{Re} \lambda} > n\})f\| \\ & \quad \text{by the strong continuity of the s.m.;} \\ & \rightarrow 4M \|E_A(\emptyset)f\| = 0, \quad n \rightarrow \infty. \end{aligned} \quad (3.58)$$

By Proposition 2.1, (3.57) and (3.58) jointly imply that

$$f \in \bigcap_{t \geq 0} D(e^{tA}),$$

and hence, by Theorem 2.1,

$$y(t) := e^{tA}f, \quad t \geq 0,$$

is a weak solution of equation (1.1).

Let

$$h^* := \sum_{k=1}^{\infty} k^{-2} e_{n(k)}^* \in X^*, \quad (3.59)$$

the functional being well defined since $\{k^{-2}\}_{k=1}^{\infty} \in l_1$ and $\|e_{n(k)}^*\| = 1$, $k \in \mathbb{N}$ (see (3.54)).

In view of (3.54) and (3.53), we have:

$$\langle e_{n(k)}, h^* \rangle = \langle e_{n(k)}, k^{-2} e_{n(k)}^* \rangle = d_{n(k)} k^{-2} \geq \varepsilon k^{-2}, \quad k \in \mathbb{N}. \quad (3.60)$$

Since, for any $\lambda \in \Delta_{n(k)}$, $k \in \mathbb{N}$, by (3.55) and (3.51),

$$\begin{aligned} \operatorname{Re} \lambda &= \operatorname{Re} \lambda_{n(k)} + (\operatorname{Re} \lambda - \operatorname{Re} \lambda_{n(k)}) \leq \operatorname{Re} \lambda_{n(k)} + |\operatorname{Re} \lambda - \operatorname{Re} \lambda_{n(k)}| \\ &\leq \operatorname{Re} \lambda_{n(k)} + \varepsilon_{n(k)} \leq -k + 1 \leq 0 \end{aligned} \quad (3.61)$$

and, by (3.50),

$$-b_- |\operatorname{Im} \lambda|^{1/\beta} < \operatorname{Re} \lambda,$$

we infer that, for any $\lambda \in \Delta_{n(k)}$, $k \in \mathbb{N}$,

$$|\lambda| \geq |\operatorname{Im} \lambda| \geq \left[b_-^{-1} (-\operatorname{Re} \lambda) \right]^\beta.$$

Using this estimate, for

$$s := 2b_- > 0 \quad (3.62)$$

and the functional $h^* \in X^*$ defined by (3.59), we have:

$$\begin{aligned} &\int_{\sigma(A)} e^{s|\lambda|^{1/\beta}} e^{\operatorname{Re} \lambda} d\nu(f, h^*, \lambda) && \text{by (2.9) as in (3.25);} \\ &= \sum_{k=1}^{\infty} k^{-2} \int_{\sigma(A) \cap \Delta_{n(k)}} e^{s|\lambda|^{1/\beta}} e^{\operatorname{Re} \lambda} d\nu(e_{n(k)}, h^*, \lambda) \\ &\geq \sum_{k=1}^{\infty} k^{-2} \int_{\Delta_{n(k)}} e^{[sb_-^{-1}-1](-\operatorname{Re} \lambda)} d\nu(e_{n(k)}, h^*, \lambda) && \text{since } s := 2b_- > 0 \text{ (see (3.62));} \\ &= \sum_{k=1}^{\infty} k^{-2} \int_{\sigma(A) \cap \Delta_{n(k)}} e^{-\operatorname{Re} \lambda} d\nu(e_{n(k)}, h^*, \lambda) && \text{by (3.61);} \\ &\geq \sum_{k=1}^{\infty} k^{-2} e^{k-1} \int_{\sigma(A) \cap \Delta_{n(k)}} 1 d\nu(e_{n(k)}, h^*, \lambda) = \sum_{k=1}^{\infty} k^{-2} e^{k-1} \nu(e_{n(k)}, h^*, \Delta_{n(k)}) \\ &\geq \sum_{k=1}^{\infty} k^{-2} e^{k-1} |\langle E_A(\Delta_{n(k)}) e_{n(k)}, h^* \rangle| && \text{by (3.52) and (3.60);} \\ &\geq \sum_{k=1}^{\infty} \varepsilon k^{-4} e^{k-1} = \infty. && (3.63) \end{aligned}$$

By Proposition 2.1 and the properties of the o.c. (see [15, Theorem XVIII.2.11 (f)]), (3.63) implies that

$$f \notin D(e^{s|A|^{1/\beta}} e^A)$$

with $s = 2b_- > 0$, which, in view of (2.12), further implies that

$$y(1) = e^A f \notin \bigcap_{s>0} D(e^{s|A|^{1/\beta}}) = \mathcal{E}^{(\beta)}(A).$$

Whence, by Proposition 2.2, we infer that the weak solution $y(t) = e^{tA}f$, $t \geq 0$, of equation (1.1) does not belong to the Beurling type Gevrey class $\mathcal{E}^{(\beta)}((0, \infty), X)$, which completes our consideration of the case of the sequence's $\{\operatorname{Re} \lambda_n\}_{n=1}^\infty$ being *unbounded*.

With every possibility concerning $\{\operatorname{Re} \lambda_n\}_{n=1}^\infty$ considered, the proof by contrapositive of the “only if” part is complete and so is the proof of the entire statement. \square

Remark 3.2. Thus, we have obtained a generalization of [1, Theorem 4.3], the counterpart for a normal operator A in a complex Hilbert space, and of [4, Corollary 4.1], a characterization of the generation of a Beurling type Gevrey ultradifferentiable C_0 -semigroup by a scalar type spectral operator A .

4 Inherent Smoothness Improvement Effect

Now, let us see that there is more to be said about the important particular case of *analyticity* ($\beta = 1$) in Theorem 3.1 (see Corollary 3.1).

Proposition 4.1. *Let A be a scalar type spectral operator in a complex Banach space $(X, \|\cdot\|)$. If every weak solution of equation (1.1) is analytically continuable into a complex neighborhood of $(0, \infty)$ (each one into its own), then all of them are analytically continuable into the open sector*

$$\Sigma_\theta := \{\lambda \in \mathbb{C} \mid |\arg \lambda| < \theta\} \setminus \{0\}$$

with

$$\theta := \sup \{0 < \varphi < \pi/2 \mid \{\lambda \in \sigma(A) \mid \operatorname{Re} \lambda < 0, |\arg \lambda| \leq \pi/2 + \varphi\} \text{ is bounded}\},$$

where $-\pi < \arg \lambda \leq \pi$ is the principal value of the argument of λ ($\arg 0 := 0$).

Proof. By Corollary 3.1, the analyticity of all weak solutions of equation (1.1) on $(0, \infty)$ is equivalent to the existence of $b_+ > 0$ and $b_- > 0$ such that the set

$$\sigma(A) \setminus \{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \leq -b_- |\operatorname{Im} \lambda| \text{ or } \operatorname{Re} \lambda \geq b_+ |\operatorname{Im} \lambda|\}$$

is bounded (see Figure 2).

As is easily seen, this implies, in particular, that the set

$$\Phi := \{0 < \varphi < \pi/2 \mid \{\lambda \in \sigma(A) \mid \operatorname{Re} \lambda < 0, |\arg \lambda| \leq \pi/2 + \varphi\} \text{ is bounded}\} \neq \emptyset.$$

For any $\varphi \in \Phi$,

$$A = A_\varphi^- + A_\varphi^+,$$

where the scalar type spectral operators A_φ^- and A_φ^+ are defined as follows:

$$\begin{aligned} A_\varphi^- &:= AE_A(\{\lambda \in \sigma(A) \mid |\arg \lambda| \geq \pi/2 + \varphi\}), \\ A_\varphi^+ &:= AE_A(\{\lambda \in \sigma(A) \mid |\arg \lambda| < \pi/2 + \varphi\}) \end{aligned}$$

(see [15, Theorem XVIII.2.11 (f)]).

By the properties of the o.c. (see [15, Theorem XVIII.2.11 (h), (c)]), for any $\varphi \in \Phi$,

$$\begin{aligned} \sigma(A_\varphi^-) &\subseteq \{\lambda \in \sigma(A) \mid |\arg \lambda| \geq \pi/2 + \varphi\} \cup \{0\}, \\ \sigma(A_\varphi^+) &\subseteq \{\lambda \in \sigma(A) \mid |\arg \lambda| \leq \pi/2 + \varphi\}. \end{aligned}$$

Hence, by [12, Proposition 4.1] (cf. also [2]), for any $\varphi \in \Phi$, the operator A_φ^- generates the C_0 -semigroup $\{e^{tA_\varphi^-}\}_{t \geq 0}$ of the operator exponentials (see Preliminaries) *analytic* in the open sector

$$\Sigma_\varphi := \{\lambda \in \mathbb{C} \mid \arg \lambda < \varphi\} \setminus \{0\}$$

(see also [9]).

As follows from the premise, for any $\varphi \in \Phi$, the set

$$\sigma(A_\varphi^+) \setminus \{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda \geq b_+ \mid \operatorname{Im} \lambda\},$$

is *bounded*, which, by [6, Corollary 4.1], implies that all weak solutions of the equation

$$y'(t) = A_\varphi^+ y(t), \quad t \geq 0,$$

i.e., by Theorem 2.1, all vector functions of the form

$$y(t) = e^{tA_\varphi^+} f, \quad t \geq 0, f \in \bigcap_{t \geq 0} D(e^{tA_\varphi^+})$$

are *entire*.

By the properties of the o.c. (see [15, Theorem XVIII.2.11]),

$$e^{tA} = e^{tA_\varphi^-} + e^{tA_\varphi^+} - I, \quad t \geq 0.$$

In view of the fact that

$$D(e^{tA_\varphi^-}) = X, \quad t \geq 0,$$

for each

$$f \in \bigcap_{t \geq 0} D(e^{tA}) = \bigcap_{t \geq 0} D(e^{tA_\varphi^+}),$$

the vector function

$$y_+(t) := [e^{tA_\varphi^+} - I] f, \quad t \geq 0,$$

is entire, whereas the vector function

$$y_-(t) := e^{tA_\varphi^-} f, \quad t \geq 0,$$

is analytically continuable into the open sector Σ_φ , which makes the vector function

$$y(t) := e^{tA} f = y_-(t) + y_+(t), \quad t \geq 0,$$

to be analytically continuable into the open sector Σ_φ .

Considering that

$$\varphi \in \Phi \text{ and } f \in \bigcap_{t \geq 0} D(e^{tA})$$

are arbitrary, by Theorem 2.1, we infer that every weak solution of equation (1.1) is analytically continuable into the open sector

$$\Sigma_\theta := \{\lambda \in \mathbb{C} \mid |\arg \lambda| < \theta\} \setminus \{0\}$$

with $\theta := \sup \Phi$. □

Remarks 4.1.

- Thus, we have obtained a generalization of [1, Proposition 5.2], the counterpart for a normal operator A in a complex Hilbert space.
- It is noteworthy that Corollary 3.1 (i.e., Theorem 3.1 with $\beta = 1$) and Proposition 4.1 with $\theta = \pi/2$ apply to equation (1.1) with a *self-adjoint operator* in a complex Hilbert space, which implies that, for such an equation, all weak solutions are analytically continuable into the open right half-plane

$$\{\lambda \in \mathbb{C} \mid \operatorname{Re} \lambda > 0\}$$

(see [1, Corollary 5.1] and, for symmetric operators, [1, Theorem 6.1]).

5 Concluding remark

Due to the *scalar type spectrality* of the operator A , Theorems 3.1 and 3.2 are stated exclusively in terms of the location of its *spectrum* in the complex plane, similarly to the celebrated *Lyapunov stability theorem* [40] (cf. [9, Ch. I, Theorem 2.10]), and thus, are intrinsically qualitative statements (cf. [5, 6, 41]).

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