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Research Article

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Lie symmetry analysis and conservation law for the equation arising from higher order Broer-Kaup equation

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Abstract: In this paper, Lie symmetry analysis is performed for the equation derived from $(2+1)$-dimensional higher order Broer-Kaup equation. Meanwhile, the optimal system and similarity reductions based on the Lie group method are obtained. Furthermore, the conservation law is studied via the Ibragimov's method.

Keywords: Lie symmetry analysis, optimal system, conservation laws

MSC: 35L65, 37K05, 70S10

1 Introduction

Nonlinear partial differential equations (PDEs) arising in many physical fields like the condense matter physics, plasma physics, fluid mechanics and optics and so on. In order to investigate the exact solution of PDEs, a fruitful techniques have been developed, such as traveling wave transformations, inverse scattering method [1], Darboux and Bäcklund transformations [2], Lie symmetry analysis [3–5]. Lie symmetry analysis is a very useful method to find the new solutions of PDEs, which was distribution by Sophus Lie (1842 – 1899). In addition on the base of symmetries, the integrability of the nonlinear PDEs, such as group classification, optimal system and conservation laws, can be considered. Lie groups, as a type of trandformation groups, can tranfer one solution to another one of a given PDE. In other words, if we get one solution of a PDE, we can obtain the other ones via the symmetry of the PDE. Based on this, we will investigate the Lie symmetry analysis of the given PDE.

Noether's theorem [6] establishes a connection between symmetries of differential equations and conservation laws. However, there are other methods to study the conservation laws, such as partial Noether's approach, multiplier approach and Ibragimov's method. As stated in [7], the former three methods are not applicable to the nonlinear PDEs that do not admit a Lagrangian. In order to overcome these difficulties, Ibragimov's method was proposed [8]. Especially state, on the contribution of Lie symmetry method, significant researches have been done on the integrability of the nonlinear PDEs, group classification, optimal system, reduced solutions and conservation laws, such as [9–14] and [15–21] published this year and last year.

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The (2 + 1)-dimensional higher order Broer-Kaup equation was considered in [22] and [23], whose expression is as follows:

$$\begin{aligned} &U_t + 4(U_{xx} + U^3 - 3UU_x + 3UW + 3P)_x = 0,\\ &V_t + 4(V_{xx} + VU^2 + UV_x + 3VW)_x = 0,\\ &W_y - V_x = 0,\\ &P_y - (UV)_x = 0. \end{aligned} \tag{1.1}$$

Li et al. and Mei et al. took the Bäcklund transformation of system (1.1) and obtained the relationship:

$$V = U_y, W = U_x, P = UU_x.$$

Such that (1.1) becomes a single differential equation:

$$U_t + 4(U_{xx} + U^3 + 3UU_x)_x = 0. \tag{1.2}$$

For (1.2), we consider its special case. That is, $U = U(x, t)$ is regarded as (1 + 1)-dimensional and replaced by u, then (1.2) becomes

$$u_t + 4(u_{xx} + u^3 + 3uu_x)_x = 0. \tag{1.3}$$

For convenience to cite later, we call (1.3) to be Li-Mei system, which is equivalent to

$$u_t + 4u_{xxx} + 12u^2u_x + 12u_x^2 + 12uu_{xx} = 0$$

The exact traveling wave solutions have been investigated in [24]. However, to the best of our knowledge, the Lie symmetry, optional system and conservation law of Li-Mei equation have not been researched, which is the original intention of this work.

This paper is organized as follows. In section 2, we perform Lie symmetry analysis of Li-Mei system. In section 3, the optimal system and similarity reductions are studied. Section 4 distributes to studying the conservation law in the method of Ibragimov's and construction the conserved vectors.

2 Lie symmetries of Li-Mei equation (1.3)

Lie symmetries analysis will be performed of Eq. (1.3) in this section. Consider a one-parameter Lie group of transformations:

$$\begin{aligned} &x \to x + \varepsilon\xi(x, t, u) + O(\varepsilon^2),\\ &t \to t + \varepsilon\tau(x, t, u) + O(\varepsilon^2),\\ &u \to u + \varepsilon\phi(x, t, u) + O(\varepsilon^2), \end{aligned} \tag{2.1}$$

With a small parameter $\varepsilon \ll 1$. The vector field associated with the above transformation group can assumed as:

$$V = \xi(x, t, u)\frac{\partial}{\partial x} + \tau(x, t, u)\frac{\partial}{\partial t} + \phi(x, t, u)\frac{\partial}{\partial u} \tag{2.2}$$

Thus the third prolongation $\mathrm{pr}^{(3)}V$ is:

$$\mathrm{pr}^{(3)}V = V + \phi^x\frac{\partial}{\partial u_x} + \phi^t\frac{\partial}{\partial u_t} + \phi^{xx}\frac{\partial}{\partial u_{xx}} + \phi^{xxx}\frac{\partial}{\partial u_{xxx}}, \tag{2.3}$$

where only the terms involved in (1.3) appear in (2.3). In (2.3), ϕ^x, ϕ^t, ϕ^{xx} and ϕ^{xxx} are all undetermined functions, which are given by the following formulae.

$$\phi^x = D_x(\phi - \xi u_x - \tau u_t) + \xi u_{xx} + \tau u_{xt}, \tag{2.4}$$

$$\phi^t = D_t(\phi - \xi u_x - \tau u_t) + \xi u_{xt} + \tau u_{tt}, \tag{2.5}$$

$$\phi^{xx} = D_x^2(\phi - \xi u_x - \tau u_t) + \xi u_{xxx} + \tau u_{xxt}, \tag{2.6}$$

$$\phi^{xxx} = D_x^3(\phi - \xi u_x - \tau u_t) + \xi u_{xxxx} + \tau u_{xxxt}, \tag{2.7}$$

where D_x, D_t are denoted the total derivatives with respect to x and t, respectively.

The determining equation of Eq. (1.3) arises from the following invariance condition:

$$\mathrm{pr}^{(3)} V(\Delta)|_{\Delta=0} = 0. \tag{2.8}$$

where

$$\Delta = u_t + 4u_{xxx} + 12u^2 u_x + 12u_x^2 + 12uu_{xx}. \tag{2.9}$$

By (2.8), we have the following symmetry condition:

$$\phi^t + 4\phi^{xxx} + 24\phi uu_x + 12u^2\phi^x + 24\phi^x u_x + 12\phi u_{xx} + 12u\phi^{xx} = 0, \tag{2.10}$$

which $\xi(x, t, u)$, $\tau(x, t, u)$ and $\phi(x, t, u)$ must satisfy.

Substituting (2.4)-(2.7) into (2.10), replacing u_t by $-(4u_{xxx} + 12u^2u_x + 12u_x^2 + 12uu_{xx})$ whenever it appears, and comparing the coefficients of the various monomials in the first, second and third order partial derivatives, and solving the system, we obtain the expression of $\xi(x, t, u)$, $\tau(x, t, u)$ and $\phi(x, t, u)$.

$$\begin{aligned} \xi(x, t, u) &= c_1 x + c_2, \\ \tau(x, t, u) &= 3c_1 x + c_3, \\ \phi(x, t, u) &= -c_1 u, \end{aligned} \tag{2.11}$$

where c_1, c_2, c_3 are arbitrary constants.

Hence the infinitesimal generators of Eq. (1.3) can be listed as follows

$$V_1 = \frac{\partial}{\partial x},\ V_2 = \frac{\partial}{\partial t},\ V_3 = x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}. \tag{2.12}$$

By solving the following ordinary differential equations with initial condition:

$$\begin{aligned} \tfrac{dx^*}{d\varepsilon} &= \xi(x^*, t^*, u^*), x^*|_{\varepsilon=0} = x, \\ \tfrac{dt^*}{d\varepsilon} &= \tau(x^*, t^*, u^*), t^*|_{\varepsilon=0} = t, \\ \tfrac{du^*}{d\varepsilon} &= \phi(x^*, t^*, u^*), u^*|_{\varepsilon=0} = u. \end{aligned} \tag{2.13}$$

We therefore obtain the group transformation which is generated by the infinitesimal generators V_1, V_2, V_3, respectively:

$$G_1 : (x, t, u) \longmapsto (x + \varepsilon, t, u), \tag{2.14}$$

$$G_2 : (x, t, u) \longmapsto (x, t + \varepsilon, u), \tag{2.15}$$

$$G_3 : (x, t, u) \longmapsto (e^{\varepsilon}x, e^{3\varepsilon}t, e^{-\varepsilon}u). \tag{2.16}$$

Here G_1, G_2, G_3 are all one-dimensional Lie groups generated by their own generators $g_{i,\varepsilon}$, whose operation is manifested by (2.14),(2.15),(2.16), respectively.

It is trivial that V_1, V_2, V_3 form a 3-dimensional Lie algebra L with the following Lie bracket:

$$[V_1, V_2] = 0, [V_1, V_3] = V_1, [V_2, V_3] = 3V_2. \tag{2.17}$$

Remark 1. *In (2.14)-(2.16), an arbitrary element in $G_i (i = 1, 2, 3)$ can transfer one solution of Eq. (1.3) to another one, so do the products of the elements from G_1, G_2 and G_3.*

Remark 2. *The Lie group $G_1 \times G_2$ is a normal Lie subgroup of $G_1G_2G_3$. The Lie algebra generated by V_1 and V_2 is an ideal of L.*

Theorem 1. *The vector fields V_1, V_2 and V_3 supply a representation of the Lie algebra*

$$\mathfrak{g} = \mathrm{span}\{x_1, x_2, x_3\},$$

where the Lie bracket is

$$[x_1, x_2] = 0, [x_1, x_3] = x_1, [x_2, x_3] = 3x_2. \tag{2.18}$$

The definition of representations of Lie algebras see [25].

Proof. It is suffice if we take the representation space to be the set of all the analytic functions and the linear mapping $\rho : x_i \mapsto V_i$ for $i = 1, 2, 3$. □

Remark 3. *The vector fields V_1 and V_2 have trivial prolongation. However, the prolongation of V_3 can be computed:*

$$\mathrm{pr}^{(3)}V_3 = x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} - 4u_t\frac{\partial}{\partial u_t} - 2u_x\frac{\partial}{\partial u_x} - 3u_{xx}\frac{\partial}{\partial u_{xx}} - 4u_{xxx}\frac{\partial}{\partial u_{xxx}}. \tag{2.19}$$

It is easy to check $\mathrm{pr}^{(3)}V_3(\Delta) = -4 \cdot \Delta$, which is called the symmetry invariance of differential equation (1.3).

We are now to take an example to illustrate the applications of Lie symmetry analysis. We take $u_t = u_{xx}$ as an example rather than Eq. (1.3) since it is difficult to find the analytical solution. The vector fields of this equation is $V_1 = \partial_x$, $V_2 = \partial_t$, $V_3 = \partial_u$, $V_4 = x/2\partial_x + t\partial_t + u\partial_u$. It is not difficult to find a special solution $u(x, t) = e^t(e^x + e^{-x})$. Under the operation of Lie group generated by V_1-V_4, we can check that

$$\begin{aligned} u^{(1)} &= e^t(e^{x-\varepsilon} + e^{-x-\varepsilon}) \\ u^{(2)} &= e^{t-\varepsilon}(e^x + e^{-x}) \\ u^{(3)} &= e^{\varepsilon}e^t(e^x + e^{-x}) \\ u^{(3)} &= e^{\varepsilon}e^{e^{\varepsilon}t}(e^{e^{\frac{\varepsilon}{2}}x} + e^{-e^{\frac{\varepsilon}{2}}x}) \end{aligned}$$

are all the solutions of $u_t = u_{xx}$.

3 Optimal system of one-dimensional subalgebras

The more technical matters arose in order to classify the subalgebra of Lie algebra generated by Lie point symmetries, for instance [26] and [3]. A concise method to get the optimal system was presented by Ibragimov in 2010 [27]. In this section we shall construct an optimal system of one-dimensional subalgebra.

Theorem 2. *The following operators provide two optimal systems of one-dimensional subalgebras of the Lie algebra spanned by V_1, V_2, V_3 of Eq. (1.3):*

$$I: \ \{V_1, \nu V_1 + V_2, V_3\}, \tag{3.1}$$

and

$$II: \ \{V_2, V_1 + \mu V_2, V_3\}, \tag{3.2}$$

where both ν and μ are arbitrary constants.

Proof. Suppose W and V are two vector field and

$$\frac{dW}{d\varepsilon} = adV|_W, \quad W(0) = w_0.$$

By solving this ODE we have

$$W(\varepsilon) = \mathrm{Ad}(\exp(\varepsilon V))W_0, \tag{3.3}$$

by summing the Lie series[3]

$$\begin{aligned}\mathrm{Ad}(\exp(\varepsilon V))W_0 &= \sum_{n=0}^{\infty}\frac{\varepsilon^n}{n!}(adV)^n(W_0)\\ &= W_0 - \varepsilon[V, W_0] + \frac{\varepsilon^2}{2!}[V,[V,W_0]] - \cdots .\end{aligned} \tag{3.4}$$

In view of (3.4), we obtain

$$\begin{aligned}&\mathrm{Ad}(\exp(\varepsilon V_i))V_i = V_i,\ \ i = 1,2,3;\\ &\mathrm{Ad}(\exp(\varepsilon V_1))V_2 = V_2,\ \ \mathrm{Ad}(\exp(\varepsilon V_1))V_3 = V_3 - \varepsilon V_1;\\ &\mathrm{Ad}(\exp(\varepsilon V_2))V_1 = V_1,\ \ \mathrm{Ad}(\exp(\varepsilon V_2))V_3 = V_3 - 3\varepsilon V_2;\\ &\mathrm{Ad}(\exp(\varepsilon V_3))V_1 = e^{\varepsilon}V_1,\ \ \mathrm{Ad}(\exp(\varepsilon V_3))V_2 = e^{3\varepsilon}V_2.\end{aligned} \tag{3.5}$$

For an arbitrary nonzero vector

$$V = a_1V_1 + a_2V_2 + a_3V_3,$$

our task is to simplify as many of the coefficients a_i as possible through the applications of adjoint maps to V.

Case 1. $a_3 \neq 0$. Scaling V if necessary, we can assume that $a_3 = 1$. By making use of (3.5) and acting on such a V by $\mathrm{Ad}(\exp(\varepsilon\frac{a_2}{3}V_2))$, we can make the coefficient of V_2 vanish:

$$V' = \mathrm{Ad}(\exp(\varepsilon a_2V_2))V = a_1V_1 + V_3.$$

Next we act on V' by $\mathrm{Ad}(\exp(\varepsilon a_1V_1))$, to cancel the coefficient of V_1. Hence V is equivalent to V_3 under the adjoint representation.

Case 2. $a_3 = 0$.

Subcase 1. $a_2 \neq 0$, $a_1 \neq 0$. Without losing generality, we can assume that $a_2 = 1$. One can easily figure out that the adjoint representation induced by any combinations of V_1, V_2, V_3 shall make $a_1V_1 + V_2$ invariant. In other words, any one-dimensional subalgebra generated by V is equivalent to the subalgebra generated by $a_1V_1 + V_2$.

Subcase 2. $a_2 = 0$, $a_1 \neq 0$. Similarly to the discussion of Subcase 1, we can conclude that V is equivalent to V_1 under the adjoint representation.

The other optimal system can be obtained similarly. □

4 Similarity reductions and exact solutions for Eq. (1.3)

In the preceding section, we got the optimal system of Eq. (1.3). We are now in the position to deal with the symmetry reduction and exact solutions via constructing similarity variables.

(1) For the generator V_1, we assume $\zeta = t$, $u = f(\zeta)$ and the we obtain the trivial solution $f = c$, where c is an arbitrary constant.

(2) For the linear combination $\nu V_1 + V_2$, we have

$$u = f(\zeta), \tag{4.1}$$

where $\zeta = x - \nu t$, which is a traveling wave transformation. By substituting (4.1) into Eq. (1.3), we reduce this equation to the following ODE

$$4f''' + 12ff'' + 12f'^2 + 12f^2f' - \nu f' = 0, \tag{4.2}$$

where $f' = \frac{\mathrm{d}f}{\mathrm{d}\zeta}$, $\nu \neq 0$.

The traveling wave solutions were obtained in [24].

(3) For the generator V_3, we have

$$u = t^{-\frac{1}{3}}f(\zeta), \tag{4.3}$$

where $\zeta = xt^{-1/3}$. Substituting (4.3) into Eq. (1.3), we reduce it to the following ODE

$$4f^{'''} + 12ff^{''} + 12f^{'2} + 12f^2f^{'} - \frac{1}{3}\zeta f^{'} - \frac{1}{3}f = 0, \tag{4.4}$$

where $f^{'} = \frac{\mathrm{d}f}{\mathrm{d}\zeta}$.

For optimal system II, we only discuss the similarity reductions of V_2 and $V_1 + \mu V_2$.

(4) For the generator V_2, we have

$$u = f(\zeta), \tag{4.5}$$

where $\zeta = x$. By substituting (4.5) into Eq. (1.3), we reduce this equation to the following ODE

$$4f^{'''} + 12ff^{''} + 12f^{'2} + 12f^2f^{'} = 0, \tag{4.6}$$

where $f^{'} = \frac{\mathrm{d}f}{\mathrm{d}\zeta}$.

(5) For the linear combination $V_1 + \mu V_2$, we have

$$u = f(\zeta), \tag{4.7}$$

where $\zeta = \mu x - t$, which follows that this ODE

$$4\mu^3 f^{'''} + 12\mu^2 ff^{''} + 12\mu^2 f^{'2} + 12\mu f^2 f^{'} - f' = 0, \tag{4.8}$$

where $f^{'} = \frac{\mathrm{d}f}{\mathrm{d}\zeta}$ and $\mu \neq 0$.

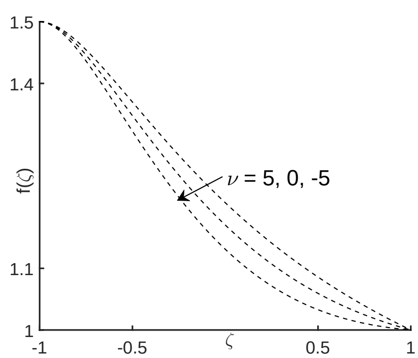

Figure 1: The graph of $f(\zeta)$ given by Eq. (4.2) as ν takes $-5, 0, 5$.

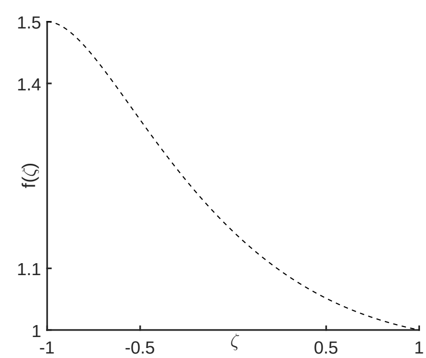

Figure 2: The graph of $f(\zeta)$ given by Eq. (4.4).

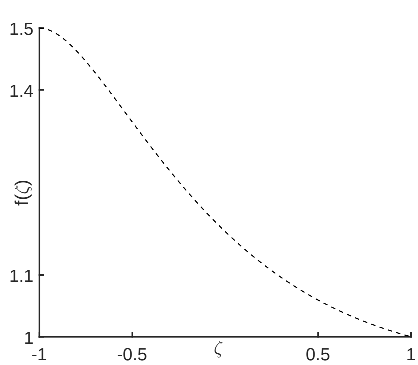

Figure 3: The graph of $f(\zeta)$ given by Eq. (4.6).

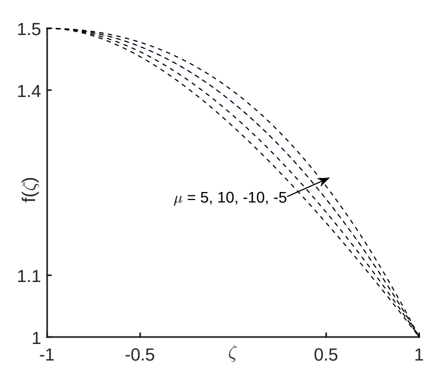

Figure 4: The graph of $f(\zeta)$ given by Eq. (4.8) for $\mu = -10, -5, 5, 10$.

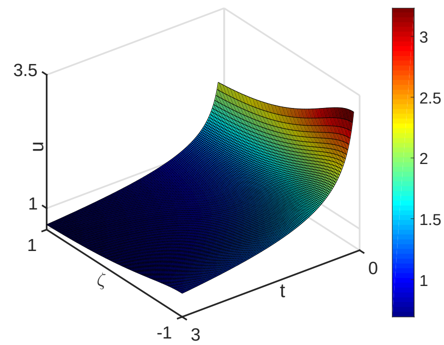

Figure 5: The graph of $u(x, t)$ given by Eq. (4.3) and Eq. (4.4).

In the above, we sketch the graphs of $f(\zeta)$ in Eqs. (4.2), (4.4), (4.6), (4.8) and 3D-plot of $u(x, t)$ in Eqs. (4.3), (4.4) under the initial conditions $f(0) = \frac{3}{2}$, $f(1) = 1$, $f'(0) = 0$.

5 Nonlinear self-adjointness and conservation law

First of all we show that Li-Mei equation is nonlinearly self-adjoint.

For a given PDEs

$$R^{\beta}(x, u, u_{(1)}, \cdots, u_{(k)}) = 0, \tag{5.1}$$

define the Euler-Lagrange operator

$$\frac{\delta}{\delta u^{\alpha}} \equiv \frac{\partial}{\partial u^{\alpha}} + \sum_{j=1}^{\infty} (-1)^j D_{i_1} \cdots D_{i_j} \frac{\partial}{\partial u^{\alpha}_{i_1 \cdots i_j}}, \alpha = 1, 2, \cdots, m, \tag{5.2}$$

and the formal Lagrangian

$$\mathcal{L} = \sum_{\beta=1}^{m} v^{\beta} R^{\beta}(x, u, u_{(1)}, \cdots, u_{(k)}). \tag{5.3}$$

The adjoint equations

$$(R^{\alpha})^{*}(x, u, u_{(1)}, \cdots, u_{(k)}) = \frac{\delta \mathcal{L}}{\delta u^{\alpha}} = 0, \ \ \alpha = 1, 2, \cdots, m, \ v = v(x). \tag{5.4}$$

Definition 1. *The system (5.1) is said to be nonlinearly self-adjoint if the adjoint system (5.4) is satisfied for all solutions u of system (5.1) upon a substitution $v = \varphi(x, u)$ such that $\varphi(x, u) \neq 0$, which is equivalent to the following identity holding for the undetermined functions λ^{β}_{α}*

$$(R^{\alpha})^{*}(x, u, u_{(1)}, v_{(1)}, \cdots, u_{(k)}, v_{(k)})|_{v=\varphi(x,u^{\alpha})} = \sum_{\beta=1}^{m} \lambda^{\beta}_{\alpha} R^{\beta}. \tag{5.5}$$

In this paper $\alpha = 1$ and $R^{\alpha}(x, u, u_{(1)}, \cdots, u_{(k)}) = \Delta(x, t, u_t, u_x, u_{xx}, u_{xxx}) = Eq.(2.9)$. The formal Lagrangian is

$$\mathcal{L} = v(u_t + 4u_{xxx} + 12u^2 u_x + 12u_x^2 + 12uu_{xx}). \tag{5.6}$$

Substituting it into $(5.2) = 0$, we have the adjoint equation to Eq. (1.3)

$$4v_{xxx} - 12uv_{xx} + 12u^2 v_x + v_t = 0. \tag{5.7}$$

By means of

$$(4v_{xxx} - 12uv_{xx} + 12u^2 v_x + v_t)|_{v=\varphi(x,t,u)} = \lambda \cdot \Delta, \tag{5.8}$$

it leads us to

$$
\begin{aligned}
&-12u^2\varphi_x - 12u^2\varphi_u u_x - \varphi_t - \varphi_u u_t + 12u\varphi_{xx} + 24u\varphi_{ux}u_x + 12u\varphi_u u_{xx} - 12\varphi_{uu}u_x^2\\
&-4\varphi_{xxx} - 12\varphi_{uxx}u_x - 12\varphi_{uux}u_x^2 - 12\varphi_{ux}u_{xx} - 4\varphi_{uuu}u_x^3 - 12\varphi_{uu}u_x u_{xx} - 4\varphi_u u_{xx}\\
&= \lambda(u_t + 4u_{xxx} + 12u^2u_x + 12u_x^2 + 12uu_{xx})
\end{aligned} \tag{5.9}
$$

Firstly we obtain $\lambda = -\varphi_u$ by comparing the terms with the third-order derivative of u. And then

$$
\begin{cases} \varphi_{uux} + \varphi_{uu} - \varphi_u = 0 \\ \varphi_{uu} = 0. \end{cases} \tag{5.10}
$$

We therefore get $\varphi_u = 0$ and $4\varphi_{xxx} - 12u\varphi_{xx} + 12u^2\varphi_x + \varphi_t = 0$. In view of $\varphi_u = 0$, one has $\varphi_x = \varphi_t = 0$, so $\varphi = c \neq 0$, which proves that Eq. (1.3) is self-adjoint.

We are now in the position to construct the conservation law of Eq. (1.3).

Theorem 3 (Ibragimov's method)**.** *Let the system of differential Eq. (5.1) be nonlinearly self-adjoint. Then every Lie point, Lie-Bäcklund, nonlocal symmetry*

$$
X = \xi^i(x, u, u_{(1)}, \cdots)\frac{\partial}{\partial x^i} + \eta^\alpha(x, u, u_{(1)}, \cdots)\frac{\partial}{\partial u^\alpha} \tag{5.11}
$$

admitted by the system of Eq. (5.1)gives rise to a conservation law, where the components $\mathcal{C}^i$ of the conserved vector $\mathcal{C} = (\mathcal{C}^1, \cdots, \mathcal{C}^n)$ are determined by

$$
\begin{aligned}
\mathcal{C}^i = W^\alpha &\left[\frac{\partial \mathcal{L}}{\partial u_i^\alpha} - \sum_{j=1}^n D_j\left(\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha}\right) + \sum_{j,k=1}^n D_jD_k\left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha}\right)\right]\\
&+\sum_{j=1}^n D_j(W^\alpha)\left[\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} - \sum_{k=1}^n D_k\left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha}\right)\right] + \sum_{j,k=1}^n D_jD_k(W^\alpha)\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha},
\end{aligned} \tag{5.12}
$$

with $W^\alpha = \eta^\alpha - \sum_{j=1}^n \xi^j u_j^\alpha$.

For the generator $V = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \phi\frac{\partial}{\partial u}$, we have $W = \phi - \xi u_x - \tau u_t$, we therefore obtain the following components of conserved vector

$$
\begin{aligned}
\mathcal{C}^x = W &\left[\frac{\partial \mathcal{L}}{\partial u_x} - D_x\left(\frac{\partial \mathcal{L}}{\partial u_{xx}}\right) + D_x^2\left(\frac{\partial \mathcal{L}}{\partial u_{xxx}}\right)\right]\\
&+ D_x(W)\left[\frac{\partial \mathcal{L}}{\partial u_{xx}} - D_x\left(\frac{\partial \mathcal{L}}{\partial u_{xxx}}\right)\right] + D_x^2(W)\frac{\partial \mathcal{L}}{\partial u_{xxx}},
\end{aligned} \tag{5.13}
$$

$$
\mathcal{C}^t = W\frac{\partial \mathcal{L}}{\partial u_t}. \tag{5.14}
$$

Taking the formal Lagrangian $\mathcal{L}$ given by (5.6) into (5.13) and (5.14), we can simplify the expressions of $\mathcal{C}^x$ and $\mathcal{C}^t$ as follows

$$
\mathcal{C}^x = W\left[12u^2v + 24vu_x - D_x(12uv) + D_x^2(4v)\right] + D_x(W)\left[12uv - D_x(4v)\right] + D_x^2(W)(4v), \tag{5.15}
$$

$$
\mathcal{C}^t = Wv. \tag{5.16}
$$

For the generator $V_1 = \frac{\partial}{\partial x}$, it has the Lie characteristic function $W = -u_x$. By using of the formulae (5.15) and (5.16), it can give rise to the following components of the conserved vector

$$
\begin{aligned}
\mathcal{C}^x &= -u_x(12u^2v + 12vu_x - 12uv_x + 4v_{xx}) - u_{xx}(12uv - 4v_x) - 4vu_{xxx},\\
\mathcal{C}^t &= -u_x v.
\end{aligned}
$$

For the generator $V_2 = \frac{\partial}{\partial t}$, we have $W = -u_t$, the formulae (5.15) and (5.16) yield the following components of the conserved vector

$$
\mathcal{C}^x = -u_t(12u^2v + 12vu_x - 12uv_x + 4v_{xx}) - u_{xt}(12uv - 4v_x) - 4vu_{xxt},
$$

$$\mathcal{C}^t = -u_t v.$$

For the generator $V_3 = x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}$, we have $W = -u - xu_x - 3tu_t$, the formulae (5.15) and (5.16) imply the following components of the conserved vector

$$\begin{aligned}\mathcal{C}^x &= -(u + xu_x + 3tu_t)(12u^2v + 12vu_x - 12uv_x + 4v_{xx}) - (2u_x + xu_{xx} + 3tu_{xt})(12uv - 4v_x)\\ &\quad - 4v(3u_{xx} + xu_{xxx} + 3tu_{xxt}),\\ \mathcal{C}^t &= -(u + xu_x + 3tu_t)v.\end{aligned}$$

These vectors involve an arbitrary solution v of the adjoint equation (5.7) and hence provide an infinite number of conservation laws.

6 Conclusions

In this paper, we have obtained the symmetries and the corresponding Lie algebras of Li-Mei system by using Lie symmetry analysis method. Meanwhile, the optimal system and its similarity reductions are investigated. Furthermore, we proved that it is nonlinearly self-adjoint. Finally, the conserved vectors were constructed via the Ibragimov's method.

The vector fields generate the equation under consideration supply a representation of a Lie algebra. However, for a given finitely dimensional Lie algebra, such as nine types of simply Lie algebras, how to get its representation via vector fields? If we have already obtained the vector fields, can we get the differential equation which generates the vector field? If the differential equation is obtained, is it unique? All of them are the aims that we will study in the near future.

Authors' contributions

Hengtai Wang denotes to studying of the whole system and writing the main body of the article. Huiwen Chen contributes the graphs of the paper. Zigen Ouyang checks all the errors of this paper. Fubin Li denotes some calculations of this article.

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References

[1] Gardner C.S., Greene J.M., Kruskal M.D., Miura R.M., Method for solving the Korteweg-de Vries equation, Phys. Rev. Lett., 1967, 19, 1095-1097.
[2] Li Y.S., Soliton and integrable systems, Advanced Series in Nonlinear Science, Shanghai Scientific and Technological Education Publishing House, Shang Hai, 1999 (in Chinese).
[3] Olver P.J., Applications of Lie groups to differential equations, Grauate Texts in Mathematics, Springer, New York, 1993, 107.
[4] Bluman G.W., Kumei S., Symmetries and Differential Equations, Springer-Verlag, World Publishing Corp., 1989.
[5] Cantwell B.J., Introduction to Symmetry Analysis, Cambridge University Press, 2002.
[6] Noether E., Invariante Variationsprobleme, Königliche Gesellschaft der Wissenschaften zu Göttingen, Nachrichten, Mathematisch-Physikalische Klasse Heft 1918, 2, 235-257, English transl.: Transport Theory Statist. Phys. 1971, 1, 186-207.
[7] Zhao Z.L., Han B., Lie symmetry analysis of the Heisenberg equation, Commun. Nonlinear Sci. Numer. Simulat., 2017, 45, 220-234.
[8] Ibragimov N.H., A new conservation theorem, J. Math. Anal. Appl., 2007, 333, 311-328.

[9] Liu H.Z., Li J.B., Lie symmetry analysis and exact solutions for the short pulse equation, Nonlinear. Anal., 2009, 71, 2126-2133.
[10] Liu H.Z., Li J.B., Liu L., Lie group classifications and exact solutions for two variable-coefficient equations, Appl. Math. Comput., 2009, 215, 2927-2935.
[11] Liu H.Z., Li J.B., Lie symmetries, conservation laws and exact solutions for two Rod equations, Acta. Appl. Math., 2010, 110, 573-587.
[12] Liu H.Z., Li J.B., Liu L., Lie symmetry analysis, optimal systems and exact solutions to the fifth-order KdV types of equations, J. Math. Anal. Appl., 2010, 368, 551-558.
[13] Liu H.Z., Li J.B., Liu L., Group classifications, symmetry reductions and exact solutions to the nonlinear elastic Rod equations, Adv. Appl. Clifford Algebras, 2012, 22, 107-122.
[14] Nadjafikhah M., Ahangari F., Symmetry analysis and conservation laws for the Hunter-Saxton equation, Commun. Theor. Phys., 2013, 59, 335-348.
[15] Paliathanasis A., Tsamparlis M., Lie symmetries for systems of evolution equations, J. Geom. Phys., 2018, 124, 165-169.
[16] Rashidi S., Hejazi S.R., Lie symmetry approach for the Vlasov-Maxwell system of equations, J. Geom. Phys., 2018, 132, 1-12.
[17] Bansal A., Biswas A., Zhou Q., Babatin M. M., Lie symmetry analysis for cubic-quartic nonlinear Schrodinger's equation, Optik, 2018, 169, 12-15.
[18] Dorjgotov K., Ochiai H., Zunderiya U., Lie symmetry analysis of a class of time fractional nonlinear evolution systems, Appl. Math. Comput., 2018, 329, 105-117.
[19] Kumar M., Tiwari A.K., Soliton solutions of BLMP equation by Lie symmetry approach, Comput. Math. Appl., 2018, 75, 1434-1442.
[20] Zhang Y., Zhai X.H., Perturbation to Lie symmetry and adiabatic invariants for Birkhoffian systems on time scales, Commun. Nonlinear Sci. Numer. Simulat., 2019, 75, 251-261.
[21] Kumar M., Tanwar D.V., On Lie symmetries and invariant solutions of $(2 + 1)$-dimensional Gardner equation, Commun. Nonlinear Sci. Numer. Simulat., 2019, 69, 45-57.
[22] Li D., Gao F., Zhang H., Solving the $(2 + 1)$-dimensional highter order Broer-Kaup system via a transformation and tanh-function method, Chaos Solit. Fract., 2004, 20, 1021-1025.
[23] Mei J., Li D., Zhang H., New soliton-like and periodic solution of $(2 + 1)$-dimesional highter order Broer-Kaup system, Chaos Solit. Fract., 2004, 22, 669-674.
[24] Li J.B., On the exact traveling wave solutions of $(2+1)$-dimensional higher order Broer-Kaup equation, Int. J. Bifurcat. Chaos, 2014, 24, 1450007.
[25] Humphreys J.E., Introduction to Lie algebras and representation theory, Springer-Verlag, New York, 1972.
[26] Ovsiannikov L.V., Group analysis of differential equations, Academic Press, 1982.
[27] Grigoriev Y.N., Ibragimov N.H., Kovalev V.F., Meleshko S.V., Symmmetry of integro-differential equations: with applications in mechanics and plasma physica, Springer, 2010.