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Research Article

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The properties of solutions for several types of Painlevé equations concerning fixed-points, zeros and poles

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Abstract: The purpose of this manuscript is to study some properties on meromorphic solutions for several types of q-difference equations. Some exponents of convergence of zeros, poles and fixed points related to meromorphic solutions for some q-difference equations are obtained. Our theorems are some extension and improvements to those results given by Qi, Peng, Chen, and Zhang.

Keywords: fixed point, q-difference equation, meromorphic solution

MSC: 39A13, 30D35

1 Introduction and main results

Around 2006, Halburd-Korhonen [1] and Chiang-Feng [2] established independently some important fundamental results of Nevanlinna theory about the complex difference and difference operators. After their wonderful work, considerable attention has been paid in studying complex difference equations, and a lot of important and interesting results (see [2–4]) focusing on complex difference equations and difference analogues of Nevanlinna theory were obtained. Halburd-Korhonen [1, 5, 6] studied the equation

$$f(z+1)+f(z-1)=R(z,f), \tag{1.1}$$

where $R(z,f)$ is rational in f and meromorphic in z, and they singled out the difference Painlevé I equation

$$f(z+1)+f(z-1)=\frac{az+b}{f(z)}+c, \tag{1.2}$$

and the difference Painlevé II equation

$$f(z+1)+f(z-1)=\frac{(az+b)f(z)+c}{1-f(z)^2}. \tag{1.3}$$

Later, Ronkainen [7] in 2010 further discussed the equation

$$f(z+1)f(z-1)=R(z,f) \tag{1.4}$$

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where $R(z, f)$ is rational and irreducible in f and meromorphic in z. He pointed out that either f satisfies the difference Riccati equation

$$f(z+1) = \frac{A(z)f(z) + B(z)}{f(z) + C(z)},$$

or equation (1.4) can be transformed to one of the following equations

$$f(z+1)f(z-1) = \frac{\eta(z)f(z)^2 - \lambda(z)f(z) + \mu(z)}{(f(z)-1)(f(z)-\upsilon(z))},$$
$$f(z+1)f(z-1) = \frac{\eta(z)f(z)^2 - \lambda(z)f(z)}{f(z)-1},$$
$$f(z+1)f(z-1) = \frac{\eta(z)(f(z) - \lambda(z))}{f(z)-1},$$
$$f(z+1)f(z-1) = h(z)f(z)^m,$$

where $\eta(z), \lambda(z), \upsilon(z)$ satisfy some conditions. The above four equations can be called as the difference Painlevé *III* equations.

In what follows, we should assume that the readers are familiar with the fundamental theorems and the standard notations in the theory of Nevanlinna value distribution (see Hayman [8], Yang [9] and Yi-Yang [10]). Let f be a meromorphic function, we denote $\sigma(f)$, $\lambda(f)$ and $\lambda(\frac{1}{f})$ to be the order, the exponent of convergence of zeros and the exponent of convergence of poles of $f(z)$, respectively, and denote $\tau(f)$ to be the exponent of convergence of fixed points of $f(z)$, which is defined by

$$\tau(f) = \limsup_{r \to +\infty} \frac{\log N(r, \frac{1}{f(z)-z})}{\log r}.$$

In 2010, Chen-Shon [11] considered the difference Painlevé *I,II* equation (1.2),(1.3) and obtained the following theorems.

Theorem 1.1. *(see [11, Theorem 4]). Let a, b, c be constants, where a, b are not both equal to zero. Then*

(i) if $a \neq 0$, then (1.2) *has no rational solution;*

(ii) if $a = 0$, and $b \neq 0$, then (1.2) *has a nonzero constant solution $w(z) = A$, where A satisfies $2A^2 - cA - b = 0$. The other rational solution $w(z)$ satisfies $w(z) = \frac{P(z)}{Q(z)} + A$, where $P(z)$ and $Q(z)$ are relatively prime polynomials and satisfy* $\deg P < \deg Q$.

Theorem 1.2. *(see [11, Theorem 1]). Let a, b, c be constants with $ac \neq 0$. If $f(z)$ is a finite order transcendental meromorphic solution of equation* (1.3)*, then*

(i) f has at most one nonzero finite Borel exceptional value for $\sigma(f) > 0$;

(ii) $\lambda(1/f) = \lambda(f) = \sigma(f)$.

In 2013 and 2014, Zhang-Yi [12], Zhang-Yang [13] studied the difference Painlevé *III* equations with the constant coefficients, and obtained

Theorem 1.3. *(see [13]). If $f(z)$ is a transcendental finite order meromorphic solution of*

$$f(z+1)f(z-1)(f(z)-1) = \eta f(z) \quad or \quad f(z+1)f(z-1)(f(z)-1) = f(z)^2 - \lambda f(z),$$

where $\eta(\neq 0)$, $\lambda(\neq 0, 1)$ are constants, then

(i) $\lambda(f) = \sigma(f)$;

(ii) f has at most one nonzero Borel exceptional value for $\sigma(f) > 0$.

Theorem 1.4. *(see [12, Theorem 4.3]). If $f(z)$ is a transcendental finite order meromorphic solution of*

$$f(z+1)f(z-1)(f(z)-1)^2 = f(z)^2 - \lambda f(z) + \mu,$$

where λ, μ are constants and $\lambda\mu \neq 0$, then $\lambda(f) = \sigma(f)$.

In 2007, Barnett, Halburd, Korhonen and Morgan [14] first established the Logarithmic Derivative Lemma on complex q-difference operators. Then by applying those fundamental results, many mathematicians have done a lot of work about the value distribution of complex q-difference operators, solutions for complex q-difference equations, by replacing the difference $f(z+c)$ with the q-difference $f(qz)$, $q \in \mathbb{C} \setminus \{0, 1\}$ for the meromorphic function $f(z)$ in some expression concerning complex difference equations and complex difference operators (see [15–28]).

In 2015, Qi and Yang [29] considered the following equation

$$f(qz) + f\left(\frac{z}{q}\right) = \frac{az+b}{f(z)} + c, \tag{1.5}$$

which can be seen as q-difference analogues of (1.2), and obtained the result as follows.

Theorem 1.5. *[29, Theorem 1.1]. Let $f(z)$ be a transcendental meromorphic solution with zero order of equation* (1.5)*, and a, b, c be three constants such that a, b cannot vanish simultaneously. Then,*

(i) $f(z)$ has infinitely many poles.

(ii) if $a \neq 0$, then $f(z) - d$ has infinitely many zeros for any $d \in \mathbb{C}$;

(iii) if $a = 0$ and $f(z)$ takes a finite value A finitely often, then A is a solution of $2z^2 - cz - b = 0$.

Motivated by the idea from [29] and [13], our main aim of this aritcle is further to investigate some properties of meromorphic solutions for some q-difference equations, which can be called as q-difference Painlevé *III* equations. We obtain the following four results.

Theorem 1.6. *Let $q \in \mathbb{C}\backslash\{0\}$ and $|q| \neq 1$, and let $f(z)$ be a zero order transcendental meromorphic solution of the following equation*

$$f(qz)f(\frac{z}{q})(f(z)-1)^2 = f(z)^2 - \lambda f(z) + \mu, \tag{1.6}$$

where λ, μ are constants satisfying $\lambda\mu \neq 0$. Then

(i) for any $\eta \in \mathbb{C} - \{0, 1\}$, $f(\eta z)$ has infinitely many fixed-points and $\tau(f(\eta z)) = \sigma(f)$, especially, $f(q^j z)$ has infinitely many fixed-points and $\tau(f(q^j z)) = \sigma(f)$, where j is a positive integer;

(ii) $\Delta_q f$, $\frac{\Delta_q f}{f}$ have infinitely many poles, and

$$\lambda\left(\frac{1}{\Delta_q f}\right) = \lambda\left(\frac{1}{\frac{\Delta_q f}{f}}\right);$$

(iii) $f(z)$ has infinitely many zeros and poles, and the Nevanlinna exceptional value of $f(z)$ can only come from a set $E = \{z \,|z^4 - 2z^3 + \lambda z - \mu = 0\}$.

Theorem 1.7. *For $q(\neq 0) \in \mathbb{C}$ and $|q| \neq 1$, and let $f(z)$ be a zero order transcendental meromorphic solution of the following equation*

$$f(qz)f(\frac{z}{q})(f(z)-1)^2 = f(z)^2. \tag{1.7}$$

Then

(i) for any $\eta \in \mathbb{C} - \{0, 1\}$, $f(\eta z)$ has infinitely many fixed-points and $\tau(f(\eta z)) = \sigma(f)$;

(ii) $f(z)$, $\Delta_q f$, $\frac{\Delta_q f}{f}$ have infinitely many poles, and

$$\lambda\left(\frac{1}{f}\right) = \lambda\left(\frac{1}{\Delta_q f}\right) = \lambda\left(\frac{1}{\frac{\Delta_q f}{f}}\right).$$

Theorem 1.8. *Let $q \in \mathbb{C}\backslash\{0\}$ and $|q| \neq 1$, and let $f(z)$ be a zero order transcendental meromorphic solution of the following equation*

$$f(qz)f(\frac{z}{q})(f(z)-1) = \lambda(z)f(z), \tag{1.8}$$

where $\lambda(z)$ is a nonconstant polynomial. Then

(i) for any $\eta \in \mathbb{C}\backslash\{0,1\}$, $f(\eta z)$ has infinitely many fixed-points and $\tau(f(\eta z)) = \sigma(f)$;

(ii) $f(z)$, $\Delta_q f$ have infinitely many zeros and poles, and $\frac{\Delta_q f}{f}$ has infinitely many poles, and

$$\lambda(f) = \lambda(\Delta_q f) = \lambda(\frac{1}{f}) = \lambda\left(\frac{1}{\Delta_q f}\right) = \lambda\left(\frac{1}{\frac{\Delta_q f}{f}}\right);$$

(iii) $f(z)$ has no Nevanlinna exceptional value.

Theorem 1.9. *Let $q \in \mathbb{C}\backslash\{0\}$ and $|q| \neq 1$, and let $f(z)$ be a zero order transcendental meromorphic solution of the following equation*

$$f(qz)f(\frac{z}{q})f(z)^2 = h(z), \tag{1.9}$$

where $h(z)$ is a nonconstant rational function. Then

(i) for any $\eta \in \mathbb{C} - \{0,1\}$, $f(\eta z)$ has infinitely many fixed-points and $\tau(f(\eta z)) = \sigma(f)$;

(ii) $f(z)$, $\Delta_q f$, $\frac{\Delta_q f}{f}$ have infinitely many zeros and poles, and

$$\lambda(f) = \lambda(\frac{1}{f}) = \lambda(\Delta_q f) = \lambda\left(\frac{1}{\Delta_q f}\right) = \lambda\left(\frac{\Delta_q f}{f}\right) = \lambda\left(\frac{1}{\frac{\Delta_q f}{f}}\right);$$

(iii) $f(z)$ has no Nevanlinna exceptional value.

2 Proofs of Theorems 1.6 and 1.7

2.1 The proof of Theorem 1.6

Suppose that $f(z)$ is a zero order transcendental meromorphic solution of equation (1.6).

(*i*) For any $\eta \in \mathbb{C}\backslash\{0,1\}$, substituting ηz into (1.6), we have

$$f(q\eta z)f(\frac{\eta z}{q})(f(\eta z)-1)^2 = f(\eta z)^2 - \lambda f(\eta z) + \mu. \tag{2.1}$$

Denote $g(z) = f(\eta z)$, then (2.1) can be represented as

$$g(qz)g(\frac{z}{q})(g(z)-1)^2 = g(z)^2 - \lambda g(z) + \mu.$$

Let

$$P_1(z,g) := g(qz)g(\frac{z}{q})(g(z)-1)^2 - g(z)^2 + \lambda g(z) - \mu = 0.$$

Thus, it follows

$$P_1(z,z) = z^2(z-1)^2 - z^2 + \lambda z - \mu \not\equiv 0.$$

In view of $P_1(z,z) \not\equiv 0$ and by Theorem 2.5 in [14], it follows that

$$m\left(r, \frac{1}{g(z)-z}\right) = S(r,g).$$

Thus, since f is of zero order, from Theorems 1.1 and 1.3 in [25], it yields

$$\begin{aligned} N\left(r, \frac{1}{f(\eta z)-z}\right) &= N\left(r, \frac{1}{g(z)-z}\right) = T(r,g) + S(r,g) \\ &= T(r, f(\eta z)) + S(r, f(\eta z)) = T(r,f) + S(r,f). \end{aligned}$$

Therefore, for any $\eta \in \mathbb{C}\backslash\{0,1\}$, $\tau(f(\eta z)) = \sigma(f)$.

(*ii*) Next, we divide the proof into three cases: ii_1). $\lambda-\mu\neq 1$; ii_2). $\lambda-\mu=1,\mu=1$; ii_3). $\lambda-\mu=1,\mu\neq 1$.

Case ii_1): $\lambda-\mu\neq 1$. We can rewrite (1.6) as the following form

$$\frac{f(qz)f(\frac{z}{q})}{f(z)^2}=\frac{f(z)^2-\lambda f(z)+\mu}{f(z)^2(f(z)-1)^2}. \tag{2.2}$$

Thus, in view of Theorems 1.1 and 1.3 in [25], Theorem 1.1 in [14] and Valiron-Mohon'ko Lemma [30], and from (2.2) and $\lambda-\mu\neq 1$, it follows that

$$\begin{aligned}4T(r,f)&=T\left(r,\frac{f(z)^2-\lambda f(z)+\mu}{f(z)^2(f(z)-1)^2}\right)+O(1)\\&=T\left(r,\frac{f(qz)f(\frac{z}{q})}{f(z)^2}\right)+O(1)\\&\leq 2T\left(r,\frac{f(qz)}{f(z)}\right)+S(r,f)+O(1)\\&\leq 2T\left(r,\frac{\Delta_q f}{f}\right)+S(r,f),\end{aligned}$$

that is,

$$2T(r,f)\leq T\left(r,\frac{\Delta_q f}{f}\right)+S(r,f). \tag{2.3}$$

Then, we have

$$N\left(r,\frac{\Delta_q f}{f}\right)=T\left(r,\frac{\Delta_q f}{f}\right)-m\left(r,\frac{\Delta_q f}{f}\right)\geq 2T(r,f)+S(r,f). \tag{2.4}$$

Hence, we can conclude that $\frac{\Delta_q f}{f}$ has infinitely many poles and $\lambda\left(\frac{1}{\frac{\Delta_q f}{f}}\right)=\sigma(f)$ by combining with

$$N\left(r,\frac{\Delta_q f}{f}\right)\leq T\left(r,\frac{\Delta_q f}{f}\right)\leq 2T(r,f)+S(r,f). \tag{2.5}$$

Since $\Delta_q f(z)=f(qz)-f(z)$ and $\Delta_{q^{-1}}f(z)=f(\frac{z}{q})-f(z)$, then it follow from (1.6) that

$$f(qz)f(\frac{z}{q})=(\Delta_q f+f)(\Delta_{q^{-1}}f+f)=\frac{f(z)^2-\lambda f(z)+\mu}{(f(z)-1)^2},$$

that is,

$$(\Delta_q f+\Delta_{q^{-1}}f)f+\Delta_q f\Delta_{q^{-1}}f=\frac{-f^4+2f^3-\lambda f+\mu}{(f-1)^2}. \tag{2.6}$$

Since $\lambda\mu\neq 0$ and $\lambda-\mu\neq 1$, then in view of Theorems 1.1 and 1.3 in [25],Theorem 1.1 in [14] and Valiron-Mohon'ko Lemma [30], it yields

$$\begin{aligned}4T(r,f)&=T\left(r,\frac{-f^4+2f^3-\lambda f+\mu}{(f-1)^2}\right)+O(1)\\&=T\left(r,(\Delta_q f+\Delta_{q^{-1}}f)f+\Delta_q f\Delta_{q^{-1}}f\right)+O(1)\\&\leq T(r,f)+2T(r,\Delta_q f)+2T(r,\Delta_{q^{-1}}f)+O(1)\\&\leq T(r,f)+4T(r,\Delta_q f)+S(r,f),\end{aligned}$$

that is,

$$\frac{3}{4}T(r,f)\leq T(r,\Delta_q f)+S(r,f). \tag{2.7}$$

On the other hand, we can rewrite (1.6) as

$$f(qz)f(\frac{z}{q})f(z)^2=2f(qz)f(\frac{z}{q})f(z)-f(qz)f(\frac{z}{q})+f(z)^2-\lambda f(z)+\mu. \tag{2.8}$$

By applying Theorem 2.5 in [17] for (2.8), it yields

$$m(r,f)=S(r,f). \tag{2.9}$$

Thus, in view of (2.7) and (2.9), and Theorem 1.1 in [14], it follows

$$\begin{aligned}N(r,\Delta_q f) &= T(r,\Delta_q f)-, m(r,\Delta_q f)\\ &\geq T(r,\Delta_q f)-\left(m\left(r,\frac{\Delta_q f}{f}\right)+m(r,f)\right)\geq \frac{3}{4}T(r,f)+S(r,f).\end{aligned}$$

Hence, we conclude that $\Delta_q f$ has infinitely many poles and $\lambda\left(\frac{1}{\Delta_q f}\right)=\sigma(f)$ by combining with

$$N(r,\Delta_q f)\leq 2T(r,f)+S(r,f). \tag{2.10}$$

Case ii_2). $\lambda-\mu=1$ and $\mu=1$. Thus, $\lambda=2$. Then equation (1.6) becomes

$$f(qz)f(\frac{z}{q})=1,$$

and

$$\frac{1}{f(z)^2}=\frac{f(qz)f(\frac{z}{q})}{f(z)^2}.$$

Thus, in view of Theorems 1.1 and 1.3 in [25],Theorem 1.1 in [14] and Valiron-Mohon'ko Lemma [30], we deduce

$$2T(r,f)=T\left(r,\frac{1}{f^2}\right)+O(1)=T\left(r,\frac{f(qz)f(\frac{z}{q})}{f(z)^2}\right)+O(1)\leq 2T\left(r,\frac{\Delta_q f}{f}\right)+S(r,f),$$

that is,

$$T(r,f)\leq T\left(r,\frac{\Delta_q f}{f}\right)+S(r,f). \tag{2.11}$$

So, in view of Theorem 1.1 in [14] and (2.11), it yields

$$N\left(r,\frac{\Delta_q f}{f}\right)=T\left(r,\frac{\Delta_q f}{f}\right)-m\left(r,\frac{\Delta_q f}{f}\right)\geq T(r,f)+S(r,f),$$

which implies that $\frac{\Delta_q f}{f}$ has infinitely many poles and $\lambda(\frac{1}{\frac{\Delta_q f}{f}})=\sigma(f)$ by combining with (2.5).

For $\lambda=2$ and $\mu=1$, we can rewrite (1.6) as

$$(\Delta_q f+f)(\Delta_{q^{-1}}f+f)=1,$$

that is,

$$(\Delta_q f+\Delta_{q^{-1}}f)f+\Delta_q f\Delta_{q^{-1}}f=1-f(z)^2. \tag{2.12}$$

Thus, by applying Theorems 1.1 and 1.3 in [25] for (2.12), it follows

$$\begin{aligned}2T(r,f) &= T(r,1-f^2)+O(1)\\ &= T(\Delta_q f+\Delta_{q^{-1}}f)f+\Delta_q f\Delta_{q^{-1}}f)+O(1)\\ &\leq T(r,f)+4T(r,\Delta_q f)+S(r,f),\end{aligned}$$

that is,

$$\frac{1}{4}T(r,f)\leq T(r,\Delta_q f)+S(r,f). \tag{2.13}$$

In view of (1.6) and Theorem 1.1 in [14], it follows

$$2m(r,f)=m(r,f^2)=m\left(r,\frac{f(z)^2}{f(qz)f(\frac{z}{q})}\right)\leq m\left(r,\frac{f(z)}{f(qz)}\right)+m\left(r,\frac{f(z)}{f(\frac{z}{q})}\right)=S(r,f),$$

that is,

$$m(r,f) = S(r,f). \tag{2.14}$$

In view of (2.13) and (2.14) and Theorem 1.1 in [14], we can deduce

$$\begin{aligned} N(r,\Delta_q f) &= T(r,\Delta_q f) - m(r,\Delta_q f) \\ &\geq T(r,\Delta_q f) - \left(m\left(r,\frac{\Delta_q f}{f}\right) + m(r,f)\right) \geq \frac{1}{4}T(r,f) + S(r,f). \end{aligned}$$

Hence, we conclude that $\Delta_q f$ has infinitely many poles and $\lambda\left(\frac{1}{\Delta_q f}\right) = \sigma(f)$ by combining with (2.10).

Case ii_3). $\lambda - \mu = 1$ and $\mu \neq 1$. Thus, $\lambda = \mu + 1$ and (1.6) becomes

$$f(qz)f(\frac{z}{q})(f(z)-1) = f(z) - \mu, \tag{2.15}$$

and

$$\frac{f(qz)f(\frac{z}{q})}{f(z)^2} = \frac{f(z)-\mu}{f(z)^2(f(z)-1)}. \tag{2.16}$$

By applying Valiron-Mohon'ko Lemma [30], and in view of Theorems 1.1 and 1.3 in [25] and $\mu \neq 1$, it follows

$$\begin{aligned} 3T(r,f) &= T\left(r,\frac{f-\mu}{f^2(f-1)}\right) + O(1) \\ &= T\left(r,\frac{f(qz)f(\frac{z}{q})}{f(z)^2}\right) + O(1) \leq 2T\left(r,\frac{\Delta_q f}{f}\right) + S(r,f), \end{aligned}$$

that is,

$$\frac{3}{2}T(r,f) \leq T\left(r,\frac{\Delta_q f}{f}\right) + S(r,f). \tag{2.17}$$

Thus, we can conclude from (2.17) and Theorem 1.1 in [14] that

$$N\left(r,\frac{\Delta_q f}{f}\right) \geq \frac{3}{2}T(r,f) + S(r,f).$$

By combining with (2.5), it means that $\frac{\Delta_q f}{f}$ has infinitely many poles and $\lambda\left(\frac{1}{\frac{\Delta_q f}{f}}\right) = \sigma(f)$.

In view of $\lambda - \mu = 1$ and $\mu \neq 1$, we can rewrite (1.6) as the following

$$f(qz)f(\frac{z}{q}) = (\Delta_q f + f)(\Delta_{q^{-1}} f + f) = \frac{f-\mu}{f-1},$$

that is,

$$(\Delta_q f + \Delta_{q^{-1}} f)f + \Delta_q f \Delta_{q^{-1}} f = \frac{-f^3 + f^2 + f - \mu}{f-1}. \tag{2.18}$$

Thus, by applying Valiron-Mohon'ko Lemma [30] and Theorem 1.1 in [14] for (2.18), we have

$$\begin{aligned} 3T(r,f) &= T\left(r,\frac{-f^3+f^2+f-\mu}{f-1}\right) + O(1) \\ &= T(r,(\Delta_q f + \Delta_{q^{-1}} f)f + \Delta_q f \Delta_{q^{-1}} f) + O(1) \\ &\leq T(r,f) + 4T(r,\Delta_q f) + S(r,f), \end{aligned}$$

that is,

$$\frac{1}{2}T(r,f) \leq T(r,\Delta_q f) + S(r,f). \tag{2.19}$$

And in view of (1.6), we have

$$f(qz)f(\frac{z}{q})f(z) = f(qz)f(\frac{z}{q}) + f(z) - \mu. \tag{2.20}$$

By Theorem 2.5 in [14], it yields

$$m(r,f) = S(r,f).$$

Thus, it follows from Theorem 1.1 in [14] that

$$\begin{aligned} N(r,\Delta_q f) &= T(r,\Delta_q f) - m(r,\Delta_q f) \\ &\geq T(r,\Delta_q f) - \left(m\left(r,\frac{\Delta_q f}{f}\right) + m(r,f)\right) \geq \frac{1}{2}T(r,f) + S(r,f). \end{aligned}$$

Hence, by combining with (2.10), we conclude that $\Delta_q f$ has infinitely many poles and $\lambda\left(\frac{1}{\Delta_q f}\right) = \sigma(f)$.

Therefore, this completes the proof of Theorem 1.6 (ii).

(*iii*) By the process of the proof of Theorem 1.6 (ii), we have $m(r,f) = S(r,f)$, this means $N(r,f) = T(r,f) + S(r,f)$, that is, ∞ is not a Nevanlinna exceptional value of $f(z)$.

Besides, set

$$P_2(z,f(z)) := f(qz)f(\frac{z}{q})(f(z)-1)^2 - f(z)^2 + \lambda f(z) - \mu \equiv 0.$$

Since $\mu \neq 0$, then it follows $P_2(z,0) = \mu \neq 0$. Thus, in view of Theorem 2.5 in [14], we have

$$m\left(r,\frac{1}{f}\right) = S(r,f),$$

which implies that 0 is not a Nevanlinna exceptional value of f.

Now, let $\beta \notin E$, then it follow that

$$P_2(z,\beta) = \beta^4 - 2\beta^3 + \lambda\beta - \mu \neq 0.$$

From Theorem 2.5 in [14], it yields $m\left(r,\frac{1}{f-\beta}\right) = S(r,f)$, which implies that β is not a Nevanlinna exceptional value of $f(z)$.

Hence, the conclusion of Theorem 1.6 (iii) is true.

Therefore, this completes the proof of Theorem 1.6.

2.2 The proof of Theorem 1.7

By using the similar argument as in the proof of Theorem 1.6, it is easy to get the conclusions of Theorem 1.7.

3 Proofs of Theorems 1.8 and 1.9

3.1 The proof of Theorem 1.8

Similar to the argument as in the proof of Theorem 1.6, we can prove that $\tau(f(\eta z)) = \sigma(f)$ and $\Delta_q f, \frac{\Delta_q f}{f}$ have infinitely many poles and $\lambda\left(\frac{1}{\Delta_q f}\right) = \lambda\left(\frac{1}{\frac{\Delta_q f}{f}}\right)$. Now, we only need to prove that $\Delta_q f$ has infinitely many zeros and $\lambda(\Delta_q f) = \sigma(f)$. We rewrite (1.8) as the form

$$f(qz)f(\frac{z}{q})f(z) = f(qz)f(\frac{z}{q}) + \lambda(z)f(z). \tag{3.1}$$

So, it yields from Lemma Theorem 2.5 in [17] that

$$m(r,f) = S(r,f), \tag{3.2}$$

which leads to $N(r,f) = T(r,f) + S(r,f)$. This means that $f(z)$ has infinitely many poles and $\lambda(\frac{1}{f}) = \sigma(f)$.

Besides, we can rewrite (1.8) again as the form

$$\frac{\lambda(z)}{f(z)} = \frac{f(qz)}{f(z)}\frac{f(\frac{z}{q})}{f(z)}(f(z)-1). \tag{3.3}$$

Thus, in view of (3.2) and (3.3), and by Theorem 1.1 in [14], we can deduce

$$m\left(r,\frac{1}{f}\right) \le m\left(r,\frac{f(qz)}{f(z)}\right) + m\left(r,\frac{f(z)}{f(\frac{z}{q})}\right) + m(r,f) + m(r,\frac{1}{\lambda}) = S(r,f), \tag{3.4}$$

that is, $N(r,\frac{1}{f}) = T(r,f) + S(r,f)$, which implies that $f(z)$ has infinitely many zeros and $\lambda(f) = \sigma(f)$.

Since $\Delta_q f(z) = f(qz) - f(z)$ and $\Delta_{q^{-1}} f(z) = f(\frac{z}{q}) - f(z)$, thus, substituting these into (1.8), it follows that

$$(f(z) + \Delta_q f(z))(f(z) + \Delta_{q^{-1}} f(z)) = \frac{\lambda(z)f(z)}{f(z)-1}. \tag{3.5}$$

Assume that z_0 is a zero of $f(z)$, in view of (3.5), then z_0 must be a zero of $\Delta_q f$ or $\Delta_{q^{-1}} f$. Thus, in view of (3.4) and Theorem 1.1 in [14], it follows

$$\begin{aligned} T(r,f) = N\left(r,\frac{1}{f}\right) + S(r,f) &\le N(r,\frac{1}{\Delta_q f}) + N(r,\frac{1}{\Delta_{q^{-1}} f}) + S(r,f) \\ &\le 2N(r,\frac{1}{\Delta_q f}) + S(r,f), \end{aligned}$$

which implies that $\Delta_q f$ has infinitely many zeros and $\lambda(\Delta_q f) \ge \sigma(f)$. By combining with (2.10), we have $\lambda(\Delta_q f) = \sigma(f)$.

Hence, this completes the proof of Theorem 1.8 *(i)* and *(ii)*.

(iii) In view of (3.2) and (3.4), we have $\delta(0,f) = 0$ and $\delta(\infty,f) = 0$, it means that 0 and ∞ are not Nevanlinna exceptional values of $f(z)$.

Set

$$P_3(z,f) := f(qz)f(\frac{z}{q})(f(z)-1) - \lambda(z)f(z).$$

Then for any $a \in C\backslash\{0\}$, and since $\lambda(z)$ is a nonconstant polynomial, we have $P_3(z,a) = a^2(a-1) - a\lambda(z) \not\equiv 0$. Thus, in view of Theorem 2.5 in [14], it yields that $m\left(r,\frac{1}{f-a}\right) = S(r,f)$, which means that $\delta(a,f) = 0$. Hence, it follows that $f(z)$ has no Nevanlinna exceptional value.

Therefore, this completes the proof of Theorem 1.8.

3.2 The proof of Theorem 1.9

By using the similar argument as in the proof of Theorem 1.8, we can get the conclusions of Theorem 1.9 (i), (iii), $\Delta_q f$ has infinitely many zeros and poles, and $\frac{\Delta_q f}{f}$ has infinitely many poles, and

$$\lambda(\Delta_q f) = \lambda\left(\frac{1}{\Delta_q f}\right) = \lambda\left(\frac{1}{\frac{\Delta_q f}{f}}\right) = \sigma(f),$$

and f also has infinitely many poles and zeros, and $\lambda(f) = \lambda(\frac{1}{f})$. Thus, we only need to prove that $\frac{\Delta_q f}{f}$ has infinitely many zeros and $\lambda\left(\frac{\Delta_q f}{f}\right) = \sigma(f)$.

In view of (1.9), we have

$$f(q^2z)f(z)f(qz)^2 = h(qz). \tag{3.6}$$

Thus, from (1.9) and (3.6), it follows

$$\frac{f(q^2z)}{f(qz)}\frac{f(z)}{f(\frac{z}{q})}\frac{f(qz)^2}{f(z)^2} = \frac{h(qz)}{h(z)}. \tag{3.7}$$

Denote $g(z) = \frac{f(qz)}{f(z)}$. Thus, we can rewrite (3.7) as

$$g(qz)g(\frac{z}{q})g(z)^2 = \frac{h(qz)}{h(z)}.$$

Set

$$P_4(z,g) = g(qz)g(\frac{z}{q})g(z)^2 - \frac{h(qz)}{h(z)}.$$

Since $h(z)$ is a nonconstant rational function and $|q| \neq 1$, then $P_4(z,1) = 1 - \frac{h(qz)}{h(z)} \not\equiv 0$. Thus, in view of Theorem 2.5 in [14] and Theorems 1.1 and 1.3 in [25], it yields

$$m\left(r, \frac{1}{g-1}\right) = S(r,g) = S\left(r, \frac{f(qz)}{f(z)}\right) \le S(r,f). \tag{3.8}$$

Hence, it follows

$$m\left(r, \frac{1}{\frac{\Delta_q f}{f}}\right) = m\left(r, \frac{1}{\frac{f(qz)}{f(z)} - 1}\right) = m\left(r, \frac{1}{g(z)-1}\right) = S(r,f). \tag{3.9}$$

Thus, by applying Theorem 1.1 in [14] and Valiron-Mohon'ko Lemma [30] for (1.9), it follows

$$\begin{aligned} 4T(r,f) = T\left(r, \frac{h(z)}{f(z)^4}\right) + O(\log r) &= T\left(r, \frac{f(qz)}{f(z)}\frac{f(z)}{f(\frac{z}{q})}\right) + O(1) \\ &\le 2T\left(r, \frac{f(qz)}{f(z)}\right) + S\left(r, \frac{f(qz)}{f(z)}\right) + O(\log r) \\ &\le 4T(r,f) + S(r,f), \end{aligned}$$

that is,

$$T\left(r, \frac{\Delta_q f}{f}\right) = T\left(r, \frac{f(qz)}{f(z)}\right) + O(1) = 2T(r,f) + S(r,f). \tag{3.10}$$

Hence, in view of (3.9) and (3.10), it follows

$$\begin{aligned} N\left(r, \frac{1}{\frac{\Delta_q f}{f}}\right) &= T\left(r, \frac{\Delta_q f}{f}\right) - m\left(r, \frac{1}{\frac{\Delta_q f}{f}}\right) + O(1) \\ &= T\left(r, \frac{\Delta_q f}{f}\right) + S(r,f) = 2T(r,f) + S(r,f), \end{aligned}$$

which means that $\frac{\Delta_q f}{f}$ has infinitely many zeros and $\lambda\left(\frac{\Delta_q f}{f}\right) = \sigma(f)$.

Therefore, this completes the proof of Theorem 1.9.

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