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Xiao Yun Cheng, Mei Wang, Wei Wang, and Jun Tao Wang*

Stabilizers in EQ-algebras

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Abstract: The main goal of this paper is to introduce the notion of stabilizers in EQ-algebras and develop stabilizer theory in EQ-algebras. In the paper, we introduce (fuzzy) left and right stabilizers and investigate some related properties of them. Then, we discuss the relations among (fuzzy) stabilizers, (fuzzy) prefilters (filters) and (fuzzy) co-annihilators. Also, we obtain that the set of all prefilters in a good EQ-algebra forms a relative pseudo-complemented lattice, where $St_r(F, G)$ is the relative pseudo-complemented of F with respect to G. These results will provide a solid algebraic foundation for the consequence connectives in higher fuzzy logic.

Keywords: EQ-algebra; (fuzzy) stabilizer; (fuzzy) prefilter; fuzzy congruence relation

MSC: 08A72; 03E72

1 Introduction

EQ-algebra was proposed by Nov$\acute{a}$k in [1]. One of the motivations was to introduce a special algebra as the correspondence of truth values for high-order fuzzy type theory (FTT). Another motivation is from the equational style of proof in logic. It has three connectives: meet $\wedge$, product $\otimes$ and fuzzy equality $\sim$. The implication operation $\rightarrow$ is the derived of the fuzzy equality $\sim$ and it together with $\otimes$ no longer strictly form the adjoint pair in general. But a special type of EQ-algebras called as a residuated EQ-algebra whence it is lattice-ordered with a bottom element 0, becomes a residuated lattice. Hence, EQ-algebra generalizes the residuated lattice. About EQ-algebras, one can see [1-6].

The filter theory of the logical algebras plays an important role in the study of these algebras and the completeness of the corresponding non-classical logics. From a logical point of view, various filters correspond to various sets of provable formulas. Until now, some types of filters on residuated lattices or other logical algebras based on residuated lattices, for instance, implicative (positive implicative, Boolean, prime, obstinate, normal, etc) filters, have been extensively studied including their relations and characterizations in [7, 8]. Correspondingly, the fuzzy filters and fuzzy implicative (positive implicative, Boolean, prime, obstinate, normal, etc) have been obtained [9-15]. Recently, the prefilters (equivalent to the filters in residuated lattices) and filter theory in EQ-algebras have been investigated. For example, Liu and Zhang introduced implicative and positive implicative prefilters (filters) in [5]. And then Xin et al. [6] proposed fuzzy implicative and fuzzy positive implicative prefilters (filters) based on [2] where they consider the relations among special fuzzy prefilters (filters). The notion of stabilizers, introduced from analytic theory, is helpful for studying structures and properties in algebraic systems. Haveshki and Mohamadhasani [16] introduced the stabilizers in BL-algebras and investigated some basic properties of them. Saeid and Mohtashamnia [17] introduced some new

Xiao Yun Cheng: School of Science, Xi'an Aeronautical University, Xi'an, 710077, Shaanxi, China
Mei Wang: School of Mathematics and Statistics, Weinan Normal University, 714099, P.R. China
Wei Wang: Department of Basic Courses, Shaanxi Railway institute, WeiNan, 714000, P.R. China
***Corresponding Author: Jun Tao Wang:** School of Science, Xi'an Shiyou University, Xi'an, 710065, Shaanxi, China, E-mail: wjt@xsyu.edu.cn

types of stabilizers in residuated lattices and discussed the relations between stabilizers and some types of filters such as Boolean, obstinate and fantastic filters.

Since EQ-algebras generalize residuated lattices, it is meaningful for us to extend some concepts from residuated lattices to EQ-algebras. Inspired by the above, in this paper, we introduce and develop the theory of stabilizers and fuzzy stabilizers. This paper is organized as follows: In section 2, we review some basic definitions and results about EQ-algebras. In section 3, we introduce two types of stabilizers of EQ-algebras and investigate their related properties. Also we discuss the relations between stabilizers and prefilters (filters) in EQ-algebras. We get that the collection of all prefilters in a good EQ-algebra forms a relative pseudo-complemented lattice. In section 4, we introduce two types of fuzzy stabilizers of EQ-algebras and discuss the relations between fuzzy stabilizers and fuzzy prefilters (filters) in EQ-algebras. Finally, we conclude that the set of all fuzzy filters constitutes a relative pseudo-complemented lattice in a prelinear and residuated lattice-ordered EQ-algebra.

2 Preliminaries

In this section, we recollect some definitions and results which will be used in the following.

Definition 2.1. *([1]) An EQ-algebra is an algebra* $(E, \wedge, \otimes, \sim, 1)$ *of type* $(2, 2, 2, 0)$ *such that for all* $x, y, z, t \in E$*:*
(E1) $(E, \wedge, 1)$ *is a commutative idempotent monoid (i.e.* $\wedge$*-semilattice with top element* 1*);*
(E2) $(E, \otimes, 1)$ *is a commutative monoid and* $\otimes$ *is isotone w.r.t.* $\leq$ *(where* $x \leq y$ *is defined as* $x \wedge y = x$*);*
(E3) $x \sim x = 1$ *(reflexivity axiom);*
(E4) $((x \wedge y) \sim z) \otimes (t \sim x) \leq z \sim (t \wedge y)$ *(substitution axiom);*
(E5) $(x \sim y) \otimes (z \sim t) \leq (x \sim z) \sim (y \sim t)$ *(congruence axiom);*
(E6) $(x \wedge y \wedge z) \sim x \leq (x \wedge y) \sim x$ *(monotonicity axiom);*
(E7) $(x \wedge y) \sim x \leq (x \wedge y \wedge z) \sim (x \wedge z)$ *(monotonicity axiom);*
(E8) $x \otimes y \leq x \sim y$ *(boundedness axiom).*

In what follows, by E we denote the universe of an EQ-algebra $(E, \wedge, \otimes, \sim, 1)$, unless specifically stated.

For any $x, y \in E$, define $x \to y = (x \wedge y) \sim x$, $\bar{x} = x \sim 1$, $\neg x = x \sim 0$ and $x^n \to y = x \otimes x^{n-1} \to y$ for $n \geq 1$.

Proposition 2.2. *([1]) Let E be an EQ-algebra. Then for any* $x, y, z \in E$*:*
(p1) $x \leq y$ *implies* $x \to y = 1, x \sim y = y \to x, \bar{x} \leq \bar{y}$*;*
(p2) $x \to x = 1, x \to 1 = 1, x \leq \bar{x}, \bar{1} = 1$*;*
(p3) $x \leq y \to x$*;*
(p4) $x \leq y$ *implies* $z \to x \leq z \to y$ *and* $y \to z \leq x \to z$*;*
(p5) $x \to y \leq (z \to x) \to (z \to y)$ *and* $x \to y \leq (y \to z) \to (x \to z)$*;*
(p6) $(x \to y) \otimes (y \to z) \leq x \to z$*.*

Definition 2.3. *([1, 3, 4]) An EQ-algebra E is said to be*
- *bounded if it has a a bottom element* 0*;*
- *good if* $\bar{x} = x$ *for all* $x \in E$*;*
- *separated if for all* $x, y \in E$, $x \sim y = 1$ *implies* $x = y$*;*
- *residuated if for all* $x, y, z \in E$, $(x \otimes y) \wedge z = x \otimes y$ *iff* $x \wedge ((y \wedge z) \sim y) = x$*;*
- *prelinear if for all* $x, y \in E$, 1 *is the unique upper bound in E of the set* $\{x \to y, y \to x\}$*;*
- *idempotent if for all* $x \in E$, $x \otimes x = x$*;*
- *involutive (IEQ-algebra) if it contains a bottom element* 0 *and for all* $x \in E$, $\neg\neg x = x$*;*
- *lattice-ordered if the underlying* $\wedge$*-semilattice is a lattice;*
- *a* ℓ*EQ-algebra if it is lattice-ordered and for all* $x, y, z, u \in E$, $((x \vee y) \sim z) \otimes (u \sim x) \leq ((u \vee y) \sim z)$*.*

Notice that every residuated EQ-algebra is good and every good EQ-algebra is separated.

Proposition 2.4. *([1,3]) Let E be a good EQ-algebra. Then for any $x, y, z \in E$:*
(1) $x \to (y \to z) = y \to (x \to z)$;
(2) $x \le (x \to y) \to y$;
(3) For all indexed families $\{a_i\}$ in E, provided that $\{a_i\}$ has supremum in E, we have $\vee_i a_i \to c = \wedge_i(a_i \to c)$;
(4) $x \otimes (x \to y) \le y$.

Proposition 2.5. *([1,3]) Let E be a residuated EQ-algebra. Then for any $x, y, z \in E$:*
(1) $x \otimes y \le z$ iff $x \le y \to z$;
(2) $(x \otimes y) \to z = x \to (y \to z)$;
(3) $x \le y \to (x \otimes y)$;
(4) $x \to y \le (x \otimes z) \to (y \otimes z)$.

Proposition 2.6. *([4]) (1) If E is a prelinear and separated lattice-ordered EQ-algebra, then $(x \wedge y) \to z = (x \to z) \vee (y \to z)$ for any $x, y, z \in E$.*
(2) If E is a prelinear and good lattice-ordered EQ-algebra, then E is an ℓEQ-algebra and $x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x)$ for any $x, y, z \in E$.

Definition 2.7. *([4]) A nonempty subset F of an EQ-algebra E is called a prefilter of E if it satisfies:*
(F1) $1 \in F$,
(F2) $x \in F$, $x \to y \in F$ imply $y \in F$ for all $x, y \in E$.
A prefilter F is called a filter if it satisfies:
(F3) $x \to y \in F$ implies $(x \otimes z) \to (y \otimes z) \in F$ for all $x, y, z \in E$.

Given an EQ-algebra E and $x, y \in E$, if F is a prefilter of E, then $x \in F$ and $x \le y$ imply $y \in F$. Also, define $x \to^0 y = y$, $x \to^n y = x \to (x \to^{n-1} y)$, then for $\emptyset \neq A \subseteq E$, the prefilter generated by A is $< A >= \{x \in E : a_1 \to (a_2 \to (\cdots \to (a_n \to x)\cdots)) = 1$ for some $a_i \in A, n \ge 1\}$. In particular, $< a >=< \{a\} >= \{x \in E : a \to^n x = 1$ for some $n \ge 1\}$.

Definition 2.8. *([5]) A prefilter F of an EQ-algebra E is called*
- *a positive implicative prefilter of E if for any $x, y, z \in E$, $x \to (y \to z) \in F$ and $x \to y \in F$ imply $x \to z \in F$;*
- *an obstinate prefilter of E if for any $x, y \in E$, $x, y \notin F$ implies $x \to y \in F$ and $y \to x \in F$.*

Definition 2.9. *([6]) Let μ be a fuzzy set of an EQ-algebra E. Then μ is called a fuzzy prefilter of E if it satisfies:*
(FF1) $\mu(1) \ge \mu(x)$ for all $x \in E$,
(FF2) $\mu(y) \ge \mu(x) \wedge \mu(x \to y)$ for all $x, y \in E$.
A fuzzy prefilter μ is called a fuzzy filter if it satisfies:
(FF3) $\mu((x \otimes z) \to (y \otimes z)) \ge \mu(x \to y)$ for all $x, y, z \in E$.

Proposition 2.10. *([6]) Let μ be a fuzzy prefilter of an EQ-algebra E. Then for all $x, y \in E$:*
(1) $x \le y$ implies $\mu(x) \le \mu(y)$;
Furthermore, if μ is a fuzzy filter, we have:
(2) $\mu(x \otimes y) = \mu(x) \wedge \mu(y)$;
(3) $\mu(x \wedge y) \ge \mu(x) \wedge \mu(y)$.

The following theorem provides a method for determining the fuzzy prefilter of an good EQ-algebra.

Theorem 2.11. *([18]) Let μ be a fuzzy set of a good EQ-algebra E. If μ satisfied (FF1),*
(FF4) $x \le y$ implies $\mu(x) \le \mu(y)$,
(FF5) $\mu(x \otimes y) \ge \mu(x) \wedge \mu(y)$,
for any $x, y \in E$, then μ is a fuzzy prefilter of E.

Definition 2.12. *([6]) Let μ be a fuzzy set of an EQ-algebra E. For all $x, y, z \in E$, μ is called a*
• *fuzzy positive implicative prefilter of E if μ is a prefilter of E and $\mu(x \to z) \geq \mu(x \to (y \to z)) \wedge \mu(x \to y)$;*
• *fuzzy implicative prefilter of E if $\mu(x \to z) \geq \mu(x \to (\neg z \to y)) \wedge \mu(y \to z)$.*

Theorem 2.13. *([6]) Let μ be a fuzzy prefilter of E. Then*
(1) μ is a fuzzy positive implicative filter of E if and only if $\mu(x \wedge (x \to y) \to y) = \mu(1)$ for all $x, y \in E$;
(2) μ is a fuzzy implicative filter of E if and only if $\mu(x) = \mu(\neg x \to x)$ for all $x \in E$.

Definition 2.14. *([18]) A fuzzy relation R on E is called a fuzzy congruence relation if R is a fuzzy equivalence relation satisfying $R(x \triangle u, y \triangle v) \geq R(x, y) \wedge R(u, v)$ for any $x, y, u, v \in E$, where $\triangle \in \{\otimes, \wedge, \sim\}$.*

Lemma 2.15. *([18]) Let μ be a fuzzy filter of E and $\mu(1) = 1$. Define a fuzzy relation R on E by $R(x, y) = \mu(x \sim y)$ for $x, y \in E$. Then R is a fuzzy congruence relation on E, which is called the generated fuzzy congruence relation by μ.*

Lemma 2.16. *([18]) Assume that R is a fuzzy congruence relation on a good EQ-algebra E. Define a fuzzy set $\mu(x) = R(1, x)$ for $x \in E$. Then μ is a fuzzy prefilter of E.*

3 Stabilizers in EQ-algebras

In this section, we introduce two types of stabilizers in an EQ-algebra and investigate the related properties of them. Also, we discuss the relationships between stabilizers and prefilters in EQ-algebras. Moreover, we get that $\mathcal{PF}(E)$ forms a relative pseudo-complemented lattice whence E is a good EQ-algebra.

Definition 3.1. *Let E be an EQ-algebra and A be a nonempty subset of E. The right stabilizer and left stabilizer of A are defined as follows:*

$$St_r(A) = \{a \in E : a \to x = x, \forall x \in A\},$$

$$St_l(A) = \{a \in E : x \to a = a, \forall x \in A\}.$$

Example 3.2. *Let $E = \{0, a, b, c, d, 1\}$ with $0 < a < b < d < 1$, $a < c < d$. Define operations $\otimes$ and $\sim$ on E as follows:*

$\otimes$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	0	0	a
b	0	0	0	0	0	b
c	0	0	0	0	0	c
d	0	0	0	0	d	d
1	0	a	b	c	d	1

$\sim$	0	a	b	c	d	1
0	1	1	a	a	a	a
a	1	1	a	a	a	a
b	a	a	1	c	c	c
c	a	a	c	1	c	c
d	a	a	c	c	1	d
1	a	a	c	c	d	1

Then $(E, \wedge, \otimes, \sim, 1)$ is an EQ-algebra. One can calculate that the right stabilizer and the left stabilizer of A are $St_r(A) = \{d, 1\}$, $St_l(A) = \{a, 1\}$ by taking $A = \{c, 1\}$.

Proposition 3.3. *Let $A, B, A_i, B_i (i \in I)$ be nonempty subsets of E. We have:*
(1) If $A \subseteq B$, then $St_r(B) \subseteq St_r(A)$ and $St_l(B) \subseteq St_l(A)$;
(2) $St_l(E) = \{1\}$ and $St_r(\{1\}) = E$;
(3) If E is good, then $St_r(E) = \{1\}$ and $St_l(\{1\}) = E$;

(4) $St_r(A) = \bigcap\{St_r(\{x\}) : x \in A\}$ *and* $St_l(A) = \bigcap\{St_l(\{x\}) : x \in A\}$;
(5) $\bigcap St_r(B_i) \subseteq St_r(\bigcap B_i)$ *and* $\bigcap St_l(B_i) \subseteq St_l(\bigcap B_i)$;
(6) If $f : E \to E$ *is a homomorphism and* $x \in E$, *then* $f(St_r(\{x\})) \subseteq St_r(\{f(x)\})$ *and* $f(St_l(\{x\})) \subseteq St_l(\{f(x)\})$;
(7) If E is residuated, then $a \in St_r(A)$ *implies* $a^n \in St_r(A)$ *for any* $n \in N$.

Proof. (1) and (4) are straightforward.
(2) $\{1\} \subseteq St_l(E)$ follows from $x \to 1 = 1$ for all $x \in E$. Let $a \in St_r(E)$. Then $x \to a = a$ for all $x \in E$. Hence $a = 1$ by taking $x = a$. Similarly, $St_l(\{1\}) = E$.
(3) $\{1\} \subseteq St_r(E)$ follows from $1 \to x = x$ for all $x \in E$. Let $a \in St_r(E)$. Then $a \to x = x$ for all $x \in E$. Hence $a = 1$ by taking $x = a$. Similarly, $St_l(\{1\}) = E$.
(5) It is clear by (1).
(6) Let $b \in f(St_r(\{x\}))$. Then there is $a \in St_r(\{x\})$, namely, $a \to x = x$ such that $b = f(a)$. Hence $f(a) \to f(x) = f(a \to x) = f(x)$, which implies $b = f(a) \in St_r(\{f(x)\})$. The proof of $f(St_l(\{x\})) \subseteq St_l(\{f(x)\})$ is similar.
(7) Let $a \in St_r(A)$. Then for all $x \in A$, $a \to x = x$. Hence $a^2 \to x = a \otimes a \to x = a \to (a \to x) = a \to x = x$, which implies $a^2 \in St_r(A)$. Now let $a^{n-1} \in St_r(A)$. Then $a^n \to x = a \otimes a^{n-1} \to x = a \to (a^{n-1} \to x) = a \to x = x$. It follows that $a^n \in St_r(A)$. □

Theorem 3.4. *Let E be a good EQ-algebra and* $A \subseteq E$. *Then* $St_r(A)$ *is a prefilter of E.*

Proof. It follows from $1 \to x = x$ for all $x \in A$ that $1 \in St_r(A)$. Let $a, a \to b \in St_r(A)$ for any $a, b \in E$. Then $a \to x = x$ and $(a \to b) \to x = x$ for all $x \in A$. Hence $b \to x \le (a \to b) \to (a \to x) = a \to ((a \to b) \to x) = a \to x = x$. On the other hand, $x \le b \to x$. So we obtain $x = b \to x$ for any $x \in A$. That is, $b \in St_r(A)$. Therefore $St_r(A)$ is a prefilter of E. □

Notice that the result of Theorem 3.4 is not true in general for any EQ-algebra. For example, let E be an EQ-algebra given in Example 3.2, it is clear that E is not good. By taking $A = \{d, 1\}$ we can calculate that $St_r(A) = \{a, c, 1\}$ is not a prefilter of E since $a \in St_r(A)$, $a \to b \in St_r(A)$, but $b \notin St_r(A)$.

Theorem 3.5. *Let E be an idempotent and residuated EQ-algebra. Then for* $A \subseteq E$, $St_r(A)$ *is a positive implicative prefilter of E.*

Proof. According to Theorem 3.4, $St_r(A)$ is a prefilter of E. Now let $a \to (b \to c), a \to b \in St_r(A)$ for any $a, b, c \in E$. Then $(a \to (b \to c)) \to x = x$ and $(a \to b) \to x = x$ for all $x \in A$. Hence from Proposition 2.4 and Proposition 2.5, $(a \to b) \otimes (b \to (a \to c)) \to x = (b \to (a \to c)) \to ((a \to b) \to x) = (a \to (b \to c)) \to ((a \to b) \to x) = (a \to (b \to c)) \to x = x$. So by (p6) $(a \to c) \to x = (a \otimes a \to c) \to x = (a \to (a \to c)) \to x \le (a \to b) \otimes (b \to (a \to c)) \to x = x$, and we have $(a \to c) \to x \le x$. Combining $(a \to c) \to x \ge x$, it follows that $(a \to c) \to x = x$. This shows $a \to c \in St_r(A)$ and therefore $St_r(A)$ is a positive implicative prefilter of E. □

Definition 3.6. *([18]) Let E be a lattice-ordered EQ-algebra. A proper prefilter F of E is called prime if* $x \vee y \in F$ *implies* $x \in F$ *or* $y \in F$ *for any* $x, y \in E$.

Theorem 3.7. *Let E be a good lattice-ordered EQ-algebra and a be a co-atom in E. Then* $St_r(\{a\})$ *is a prime prefilter of E.*

Proof. According to Theorem 3.4 $St_r(\{a\})$ is a proper prefilter of E. Since a is a co-atom, we have $a \neq 1$. Now we put $x \vee y \in St_r(\{a\})$, but $x \notin St_r(\{a\})$ and $y \notin St_r(\{a\})$. That is, $x, y \to a \neq a$. Hence by (p3) $a < x \to a$ and $a < y \to a$. Thus, $x \to a = 1$ and $y \to a = 1$ as a is a co-atom. Combining (3) of Proposition 2.4, we get $x \vee y \to a = (x \to a) \wedge (y \to a) = 1$, which contracts to $x \vee y \to a = 1$ and $a \neq 1$. Therefore $St_r(\{a\})$ is a prime prefilter of E. □

We see that $St_r(\{0\}) = \{x \in E : \neg x = 0\}$ is the set of all dense elements in E. In what follows, we give some results of $St_r(\{0\})$.

Theorem 3.8. *(1) If E is an IEQ-algebra, then $St_r(\{0\}) = \{1\}$;*
(2) If E is a good EQ-algebra, then $St_r(\{0\}) = \{1\}$ if and only if $x \to y, y \to x \in St_r(\{0\})$ implies $x = y$ for any $x, y \in E$.
(3) $St_r(\{0\}) = A$ if and only if $0 \in St_l(A)$.

Proof. (1) Let $x \in St_r(\{0\})$. Then $x \to 0 = 0$ and thus $x = \neg\neg x = (x \to 0) \to 0 = 0 \to 0 = 1$. This implies $St_r(\{0\} = \{1\}$.
(2) Let $St_r(\{0\}) = \{1\}$ and $x \to y, y \to x \in St_r(\{0\})$ for any $x, y \in E$. Then $x \to y, y \to x = 1$ and so $x = y$. Conversely, let $a \in St_r(\{0\})$. Since $a \to 1 = 1 \in St_r(\{0\})$ and $1 \to a = a \in St_r(\{0\})$, by hypothesis we have $a = 1$ and therefore $St_r(\{0\}) = \{1\}$.
(3) $St_r(\{0\}) = A$ iff $\neg x = x \to 0 = 0$ for all $x \in A$ iff $0 \in St_l(A)$. □

Definition 3.9. *A prefilter F of E is called a normal prefilter of E if $z \in F$ and $z \to ((y \to x) \to x) \in F$ imply $(x \to y) \to y \in F$ for all $x, y, z \in E$.*

Example 3.10. *Let $E = \{0, a, b, 1\}$ with $0 < a < b < 1$. Define operations $\otimes$ and $\sim$ as follows:*

$\otimes$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	0	b
1	0	a	b	1

$\sim$	0	a	b	1
0	1	a	0	0
a	a	1	a	a
b	0	a	1	b
1	0	a	b	1

Then $(E, \wedge, \otimes, \sim, 1)$ is an EQ-algebra [5]. Routine calculation shows that $\{b, 1\}$ is a normal prefilter of E.

Lemma 3.11. *Let F be a prefilter of a good EQ-algebra E. Then F is a normal prefilter of E if and only if $(y \to x) \to x \in F$ implies $(x \to y) \to y \in F$ for all $x, y \in E$.*

Proof. Let F be a normal prefilter of E and $(y \to x) \to x \in F$ for any $x, y \in E$. As $1 \to ((y \to x) \to x) = (y \to x) \to x \in F$ and $1 \in F$, we have $(x \to y) \to y \in F$. Conversely, let $z \in F$ and $z \to ((y \to x) \to x) \in F$. Since F is a prefilter of E, then $(y \to x) \to x \in F$ and by hypothesis $(x \to y) \to y \in F$. □

Theorem 3.12. *Let E be a good EQ-algebra.*
(1) If F is a normal prefilter of E, then $St_r(\{0\}) \subseteq F$;
(2) If $F = \{1\}$ is a normal prefilter of E, then $St_r(\{0\}) = \{1\}$;
(3) $St_r(\{0\})$ is a normal prefilter if and only if $St_r(\{0\})$ is the intersection of all normal prefilters of E;
(4) If $St_r(\{0\}) = E - \{0\}$, then $St_r(\{0\})$ is a normal prefilter of E.

Proof. (1) Let $x \in St_r(\{0\})$. Then $(x \to 0) \to 0 = 1 \in F$. Since F is a normal prefilter of E, by Lemma 3.11 we have $(0 \to x) \to x = x \in F$.
(2) and (3) By (1).
(4) Since $St_r(\{0\}) = E - \{0\}$ and $St_r(\{0\})$ is a prefilter of E, we can easily prove that $(x \to y) \to y, (y \to x) \to x \in St_r(\{0\})$ for all $x, y \in E$. □

Definition 3.13. *An EQ-algebra E is said to be simple if it has no non-trivial prefilter.*

Example 3.14. *Let $E = \{0, a, b, c, d, 1\}$ with $0 < a < c < d < 1, 0 < b < c < d < 1$. Define operations $\otimes$ and $\sim$ on E as follows:*

$\otimes$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	0	0	a
b	0	0	0	0	0	b
c	0	0	0	0	0	c
d	0	0	0	0	c	d
1	0	a	b	c	d	1

$\sim$	0	a	b	c	d	1
0	1	d	d	d	c	0
a	d	1	c	d	c	a
b	d	c	1	d	c	b
c	d	d	d	1	d	c
d	c	c	c	d	1	d
1	0	a	b	c	d	1

Then one can easily verify that $(E, \wedge, \otimes, \sim, 1)$ *is a simple EQ-algebra.*

Theorem 3.15. *Let E be an idempotent and residuated EQ-algebra. Then the following are equivalent:*
(1) E is simple;
(2) $St_r(\{a\}) = \{1\}$ *for any* $a \neq 1$ *in E.*

Proof. (1) $\Rightarrow$ (2) Suppose that E is simple. If for some $a \neq 1$ such that $St_r(\{a\}) \neq \{1\}$, then there exists $x \neq 1$ such that $x \in St_r(\{a\})$. So, we have $x \to a = a$. By the simplicity of E, we get $< x >= E$ and hence for some $n \geq 1$, $x \to^n a = 1$ as $a \in E$. On the other hand, since E is idempotent and residuated, it follows from Proposition 2.5 that $x \to^n a = x \to (x \to^{n-1} a) = (x \otimes x) \to^{n-1} a = x \to^{n-1} a = \cdots = x \to a$. This shows $a = 1$, which is a contradiction. Therefore $St_r(\{a\}) = \{1\}$ for any $a \neq 1$.
(2) $\Rightarrow$ (1) Suppose $St_r(\{a\}) = \{1\}$ for all $a \neq 1$ in E. To prove that E is simple, it suffice to show that $< y >= E$ for any $y \neq 1$. Now if $x \in E$, $x \neq 1$ such that $y \to^n x \neq 1$, then by hypothesis $y \to (y \to^n x) \neq y \to^n x$. That is, $y \to^{n+1} x \neq y \to^n x$ which contradicts to the fact that E is idempotent and residuated. This indicates $y \to^n x = 1$ and hence $x \in < y >$. Therefore $< y >= E$ for any $y \neq 1$. □

Definition 3.16. *Let* A, B *be two nonempty subsets of* E. *The right, left stabilizer of* A *with respect to* B *are defined by:*

$$St_r(A, B) = \{a \in E : (a \to x) \to x \in B, \forall x \in A\},$$

$$St_l(A, B) = \{a \in E : (x \to a) \to a \in B, \forall x \in A\}.$$

Example 3.17. *Consider the EQ-algebra E in Example 3.2. It is not difficult to check that the right, left stabilizer of A with respect to B are* $St_r(A, B) = \{b, d, 1\}$, $St_l(A, B) = \{0, a, d, 1\}$ *by taking* $A = \{c, 1\}$, $B = \{d, 1\}$.

Proposition 3.18. *Let* A, B, A_i, B_i *be nonempty subsets and F be a prefilter of E. We have:*
(1) If E is good and $St_r(A, B) = E$ $(St_l(A, B) = E)$*, then* $A \subseteq B$*;*
(2) $F \subseteq B$ *iff* $St_r(F, B) = E$*;*
(3) If E is good, then $F \subseteq B$ *iff* $St_l(F, B) = E$*;*
(4) $St_r(F, F) = E$*. Also, if E is good, then* $St_l(F, F) = E$*;*
(5) $St_r(A) \subseteq St_r(A, F)$ *and* $St_l(A) \subseteq St_l(A, F)$*;*
(6) If $A_i \subseteq B_i$ *and* $A_j \subseteq B_j$*, then* $St_r(B_i, A_j) \subseteq St_r(A_i, B_j)$ *and* $St_l(B_i, A_j) \subseteq St_l(A_i, B_j)$*;*
(7) If E is separated, then $St_r(A, \{1\}) = St_r(A)$ *and* $St_l(A, \{1\}) = St_l(A)$*;*
(8) $St_r(A, \bigcap B_i) \subseteq \bigcap St_r(A, B_i)$ *and* $St_l(A, \bigcap B_i) \subseteq \bigcap St_l(A, B_i)$*.*

Proof. (1) Let $St_r(A, B) = E$ $(St_l(A, B) = E)$. Then for any $x \in A$, $(x \to x) \to x = 1 \to x \in B$. Hence $A \subseteq B$.
(2) Let $F \subseteq B$ and $a \in E$. Then for any $x \in F$, $x \leq (a \to x) \to x$. Since F is a prefilter of E, we have $(a \to x) \to x \in F$ and so $(a \to x) \to x \in B$. It follows that $St_r(F, B) = E$.
(3) Let $F \subseteq B$ and $a \in E$. Since E is good, then for any $x \in F$, $x \leq (x \to a) \to a$. Considering that F is a prefilter of E, we obtain that $(x \to a) \to a \in F$ and thus $(x \to a) \to a \in B$. It follows that $St_l(F, B) = E$.
(4) By (2) and (3).
(5) Let $a \in St_r(A)$. Then for all $x \in A$, $a \to x = x$ and hence $(a \to x) \to x = 1 \in F$. This implies $a \in St_r(A, F)$,

that is, $St_r(A) \subseteq St_r(A, F)$. Similarly, $St_l(A) \subseteq St_l(A, F)$.
(6) Let $a \in St_r(B_i, A_j)$. Then $(a \to x) \to x \in A_j$ for all $x \in B_i$. Hence $(a \to x) \to x \in B_j$ for all $x \in A_i$, which shows $St_r(B_i, A_j) \subseteq St_r(A_i, B_j)$. Similarly, $St_l(B_i, A_j) \subseteq St_l(A_i, B_j)$.
(7) Let $a \in St_r(A, \{1\})$. Then for all $x \in A$, $(a \to x) \to x = 1$ and so $a \to x \leq x$. On the other hand, $x \leq a \to x$. It follows that $a \to x = x$ for all $x \in A$. Further $a \in St_r(A)$. Conversely, let $a \in St_r(A)$. Then $a \to x = x$ for all $x \in A$. It implies that $(a \to x) \to x = 1$ and hence $a \in St_r(A, \{1\})$. Therefore $St_r(A, \{1\}) = St_r(A)$. The other part is similar.
(8) Let $a \in St_r(A, \bigcap B_i)$. Then $(a \to x) \to x \in \bigcap B_i$ for all $x \in A$. Hence $(a \to x) \to x \in B_i$ for every i and all $x \in A$. This shows $a \in \bigcap St_r(A, B_i)$. Similarly, $St_l(A, \bigcap B_i) \subseteq \bigcap St_l(A, B_i)$. □

The following are the relationships between the left stabilizer $St_l(A, B)$ and the right stabilizer $St_r(A, B)$ in an EQ-algebra.

Theorem 3.19. *Let $A \subseteq E$ and F be a prefilter of a good EQ-algebra E. If F is a normal prefilter of E, then $St_r(A, F) = St_l(A, F)$.*

Proof. By Lemma 3.11. □

Theorem 3.20. *Let $F(x) = \{y \in E : x \leq y\}$ be a normal prefilter of a good EQ-algebra E for all $x \in E$. Then $St_l(A, B) = St_r(A, B)$ for all $A, B \subseteq E$,*

Proof. Let $F(x)$ be a normal prefilter of E for all $x \in E$. We have $(b \to a) \to a \in F((a \to b) \to b)$ as $(a \to b) \to b \in F((a \to b) \to b)$. It implies $(b \to a) \to a \leq (a \to b) \to b$. Symmetrically, $(a \to b) \to b \leq (b \to a) \to a$. Therefore $(a \to b) \to b = (b \to a) \to a$ for all $a, b \in E$ and so $St_r(A, B) = St_l(A, B)$. □

Next we discuss the relationships between stabilizers and some types of prefilters (filters) in EQ-algebras.

Lemma 3.21. *In any good EQ-algebra E, $y \to x = ((y \to x) \to x) \to x$ and $\neg x = \neg\neg\neg x$ for all $x, y \in E$.*

Proof. Let $x, y \in E$. By Proposition 2.4, $y \to x \leq ((y \to x) \to x) \to x$ and $y \leq (y \to x) \to x$. Hence by (p4) $((y \to x) \to x) \to x \leq y \to x$. Therefore $y \to x = ((y \to x) \to x) \to x$. □

Theorem 3.22. *Let E be a good EQ-algebra and A, B be two prefilters of E. Then $St_r(A, B)$ is a prefilter of E.*

Proof. Let $a, a \to b \in St_r(A, B)$ for any $a, b \in E$. Then for all $x \in A$, $(a \to x) \to x, ((a \to b) \to x) \to x \in B$. Since $x \leq a \to x$, we have $a \to x \in A$, and hence $((a \to b) \to (a \to x)) \to (a \to x) \in B$. Again by (p5), $b \to x \leq (a \to b) \to (a \to x)$. It follows that $((a \to b) \to (a \to x)) \to (a \to x) \leq (b \to x) \to (a \to x)$. This implies $(b \to x) \to (a \to x) \in B$. Considering $a \to x = ((a \to x) \to x) \to x$ from Lemma 3.21, we obtain that $((a \to x) \to x) \to ((b \to x) \to x) = (b \to x) \to ((a \to x) \to x) \to x) = (b \to x) \to (a \to x)$. Therefore $(b \to x) \to x \in B$ and so $b \in St_r(A, B)$. That is, $St_r(A, B)$ is a prefilter of E. □

Definition 3.23. *A nonempty subset F of a bounded lattice-ordered EQ-algebra E is called a Boolean prefilter of E if F is a prefilter and $x \vee \neg x \in F$ for all $x \in E$.*

Example 3.24. *Let E be the EQ-algebra given in Example 3.2. Then one can check that $F = \{b, c, d, 1\}$ is a Boolean prefilter of E.*

Theorem 3.25. *Let E be a bounded good lattice-ordered EQ-algebra. If A is a prefilter and B is a Boolean prefilter of E, then $St_r(A, B)$ is a Boolean prefilter of E.*

Proof. By Theorem 3.22 $St_r(A, B)$ is a prefilter of E. Let $a \in E$. Then from Proposition 2.4, for all $x \in A$, $a \vee \neg a \leq ((a \to x) \to x) \vee ((\neg a \to x) \to x) \leq ((a \to x) \wedge (\neg a \to x)) \to x = ((a \vee \neg a) \to x) \to x$. Since B is a Boolean prefilter, we have $a \vee \neg a \in B$. It follows that $((a \vee \neg a) \to x) \to x \in B$, which shows $St_r(A, B)$ is a Boolean prefilter of E. □

Theorem 3.26. *Let E be a good EQ-algebra. If A is a prefilter and B is an obstinate prefilter of E, then $St_r(A, B)$ is an obstinate prefilter of E.*

Proof. By Theorem 3.22 $St_r(A, B)$ is a prefilter of E. Now let $a, b \notin St_r(A, B)$. Then $(a \to x) \to x \notin B$ and $(b \to x) \to x \notin B$ for all $x \in A$. Considering $a \le (a \to x) \to x, b \le (b \to x) \to x$, we have $a, b \notin B$. Since B is an obstinate prefilter, then $a \to b, b \to a \in B$. Thus $((a \to b) \to b) \to b = a \to b \in B$ and $((b \to a) \to a) \to a = b \to a \in B$ for all $a, b \in A$. This shows that $St_r(A, B)$ is an obstinate prefilter of E. □

We denote by $\mathcal{PF}(E)$ the set of all prefilters in an EQ-algebra E. For any $F, G \in \mathcal{PF}(E)$, define $F \sqcap G = F \cap G$, $F \sqcup G =< F \cup G >$. Then $(\mathcal{PF}(E), \sqcap, \sqcup, \{1\}, E)$ is a bounded lattice.

Theorem 3.27. *Let E be a good EQ-algebra. Then $(\mathcal{PF}(E), \sqcap, \sqcup, \{1\}, E)$ is a relative pseudo-complement lattice, where $St_r(F, G)$ is the relative pseudo-complement of F with respect to G in $\mathcal{PF}(E)$.*

Proof. By Theorem 3.22 $St_r(F, G) \in \mathcal{PF}(E)$. Now we prove $St_r(F, G) \cap F \subseteq G$ for any $F, G \in \mathcal{PF}(E)$. Let $a \in St_r(F, G) \cap F$. Then $a \in F$ and $(a \to x) \to x \in G$ for all $x \in F$. Taking $x = a$, we have $(a \to a) \to a = 1 \to a = a \in G$. Suppose that $M \in \mathcal{PF}(E)$ such that $M \cap F \subseteq G$. Let $b \in M$. Since $b, x \le (b \to x) \to x$ for all $x \in F$ and F, M are prefilters of E, we obtain $(b \to x) \to x \in F \cap M$ for all $x \in F$, which implies $(b \to x) \to x \in G$ and so $b \in St_r(F, G)$. That is, $M \subseteq St_r(F, G)$. Therefore $St_r(F, G)$ is the relative pseudo-complement of F with respect to G in $\mathcal{PF}(E)$. □

Corollary 3.28. *Let E be a good EQ-algebra. Then $(\mathcal{PF}(E), \sqcap, \sqcup, \{1\}, E)$ is a pseudo-complement lattice, where $St_r(F)$ is the pseudo-complement of F in $\mathcal{PF}(E)$.*

4 Fuzzy stabilizers in EQ-algebras

In this section, we introduce two types of fuzzy stabilizers in EQ-algebras. We mainly investigate their properties and discuss the relations between fuzzy stabilizers and some types of fuzzy prefilters (filters). Moreover, we find the condition that $\mathcal{FF}(E)$ constitutes a relative pseudo-complemented lattice.

Definition 4.1. *Let μ be a fuzzy set of E and R be a fuzzy congruence relation on E. Define the fuzzy right-stabilizer and left-stabilizer of μ with respect to R as follows: for $x \in E$*

$$St_R^r(\mu)(x) = \wedge_{z \in E}\{\mu(z) \to R(x \to z, z)\},$$

$$St_R^l(\mu)(x) = \wedge_{z \in E}\{\mu(z) \to R(z \to x, x)\},$$

where $\to$ is the residuated implication with respect to a left-continuous t-norm.

Notice that if A is a classic subsets of E and R is the identity relation Id, then

$$St_r(A) = \{a \in E : a \to x = x, \forall x \in A\}$$

$$St_l(A) = \{a \in E : x \to a = a, \forall x \in A\}$$

are the right-stabilizer and left-stabilizer of A, respectively.

Example 4.2. *Consider the EQ-algebra E in Example 3.2. We define a fuzzy set μ by $\mu(0) = \mu(a) = \mu(b) = \mu(c) = \mu(d) = 0.4, \mu(1) = 1$. One can check that μ is a fuzzy filter of E and also the fuzzy congruence relation R is generated by μ. Then the fuzzy right-stabilizer and left-stabilizer of μ with respect to R are $St_R^r(\mu) = E$; $St_R^l(\mu)(0) = St_R^l(\mu)(a) = St_R^l(\mu)(c) = St_R^l(\mu)(d) = St_R^l(\mu)(1), St_R^l(\mu)(b) = 0.4$, where the residuated implication $\to$ is taken as ffjukasiewicz implication.*

Proposition 4.3. *Let R be a fuzzy congruence relation on E and μ, ν be fuzzy sets of E. We have:*
(1) If $\mu \subseteq \nu$, then $St_R^r(\nu) \subseteq St_R^r(\mu)$ and $St_R^l(\nu) \subseteq St_R^l(\mu)$;
(2) $St_R^r(\mu \cup \nu) = St_R^r(\mu) \cap St_R^r(\nu)$ and $St_R^l(\mu \cup \nu) = St_R^l(\mu) \cap St_R^l(\nu)$;
(3) $St_R^r(\mu \cap \nu) = St_R^r(\mu) \cup Ann_R(\nu)$ and $St_R^l(\mu \cap \nu) = St_R^l(\mu) \cup St_R^l(\nu)$;
(4) $St_R^r(\chi_{\{1\}}) = E$ and $St_R^l(\chi_{\{1\}}) = R(\bar{\cdot}, \cdot)$, where $\chi_{\{1\}}$ is defined by
$\chi_{\{1\}}(1) = 1, \chi_{\{1\}}(otherwise) = 0$;
(5) If E is bounded, then $St_R^r(\chi_{\{0\}}) = R(\neg\cdot, 0)$ and $St_R^l(\chi_{\{0\}}) = R(\cdot, 1)$, where $\chi_{\{0\}}$ is defined by $\chi_{\{0\}}(1) = 1, \chi_{\{0\}}(otherwise) = 0$.

Proof. (1) Let $\mu \subseteq \nu$. For $x \in E$, it follows from (p4) that $St_l^r(\nu)(x) = \wedge_{z\in E}\{\nu(z) \to R(z \to x, x)\} \le \wedge_{z\in E}\{\mu(z) \to R(z \to x, x)\} = St_l^r(\mu)(x)$. The proof of $St_R^r(\nu) \subseteq St_R^r(\mu)$ is similar.
(2) For $x \in E$, $St_l^r(\mu \cup \nu)(x) = \wedge_{z\in E}\{\mu(z) \vee \nu(z) \to R(z \to x, x)\} = \wedge_{z\in E}\{(\mu(z) \to R(z \to x, x)) \wedge (\nu(z) \to R(z \to x, x))\} = St_l^r(\mu) \cap St_l^r(\nu)$. Similarly, $St_R^r(\mu \cup \nu) = St_R^r(\mu) \cap St_R^r(\nu)$.
(3) For $x \in E$, $St_R^l(\mu \cap \nu)(x) = \wedge_{z\in E}\{\mu(z) \wedge \nu(z) \to R(z \to x, x)\} = \wedge_{z\in E}\{(\mu(z) \to R(z \to x, x)) \vee (\nu(z) \to R(z \to x, x))\} = St_R^l(\mu) \cup St_R^l(\nu)$. Similarly, $St_R^r(\mu \cap \nu) = St_R^r(\mu) \cup St_R^r(\nu)$.
(4) For $x \in E$, $St_R^r(\chi_{\{1\}}) = E$ follows from $St_R^r(\chi_{\{1\}})(x) = \wedge_{z\in E}\{\chi_{\{1\}}(z) \to R(x \to z, z)\} = 1 \to R(1, x \to 1) = R(1, 1) = 1$ and $St_R^l(\chi_{\{1\}}) = R(\bar{\cdot}, \cdot)$ follows from $St_R^l(\chi_{\{1\}})(x) = \wedge_{z\in E}\{\chi_{\{1\}}(z) \to R(z \to x, x)\} = 1 \to R(1 \to x, x) = R(\bar{x}, x)$.
(5) For $x \in E$, $St_R^r(\chi_{\{0\}}) = R(\neg\cdot, 0)$ follows from $St_R^r(\chi_{\{0\}})(x) = \wedge_{z\in E}\{\chi_{\{0\}}(z) \to R(x \to z, z)\} = 1 \to R(x \to 0, 0) = R(\neg x, 0)$ and $St_R^l(\chi_{\{0\}}) = R(\cdot, 1)$ follows from $St_R^l(\chi_{\{0\}})(x) = \wedge_{z\in E}\{\chi_{\{0\}}(z) \to R(z \to x, x)\} = 1 \to R(0 \to x, 1) = R(1, x)$. □

Lemma 4.4. *Let R be a fuzzy congruence relation on a good lattice-ordered EQ-algebra E. Then $R(x, y) = R(x \leftrightarrow y, 1)$ for any $x, y \in E$, where $x \leftrightarrow y = (x \to y) \wedge (y \to x)$.*

Proof. It is clear that $R(x \leftrightarrow y, 1) = R(x \leftrightarrow y, y \leftrightarrow y) \ge R(x, y) \wedge R(y, y) = R(x, y)$. On the other hand, since $x \otimes (x \to y) \le y$, we have $R(x \leftrightarrow y, 1) = R(x \leftrightarrow y, 1) \wedge R(x, x) = R(x \otimes (x \leftrightarrow y), x \otimes 1) = R(x \otimes (x \leftrightarrow y), x) \wedge R(y, y) \le R(((x \otimes (x \leftrightarrow y)) \vee y), x \vee y) = R(y, x \vee y)$. By the similar way, $R(x \leftrightarrow y, 1) \le R(x, x \vee y)$. Hence $R(x \leftrightarrow y, 1) \le R(x, x \vee y) \wedge R(y, x \vee y) \le R(x \wedge (x \vee y), y \wedge (x \vee y))$. It follows that $R(x, y) = R(x \leftrightarrow y, 1)$. □

Theorem 4.5. *Suppose that E is a prelinear and good lattice-ordered EQ-algebra satisfying $x \wedge (x \to y) \le y$ for all $x, y \in E$, and μ is a fuzzy set of E and R is a fuzzy congruence relation on E. Then $St_R^r(\mu)$ is a fuzzy prefilter of E.*

Proof. Let E be a prelinear and good EQ-algebra. Clearly, $St_R^r(\mu)(1) = \wedge_{z\in E}\{\mu(z) \to R((1 \to z, z)\} = 1 \ge St_R^r(\mu)(x)$. Now let $x, y \in E$. Since $R(\cdot, 1)$ is a fuzzy prefilter of E, from (3) of Proposition 2.4, 2.6, 2.10 and Lemma 2.16, 4.4 we have:

$$
\begin{aligned}
& St_R^r(\mu)(x) \wedge St_R^r(\mu)(x \to y) \\
&= \wedge_{z\in E}\{\mu(z) \to R((x \to z, z)\} \wedge \wedge_{z\in E}\{\mu(z) \to R((x \to y) \to z, z)\} \\
&= \wedge_{z\in E}\{(\mu(z) \to R((x \to z, z)) \wedge (\mu(z) \to R((x \to y) \to z, z))\} \\
&= \wedge_{z\in E}\{(\mu(z) \to R((x \to z) \to z, 1)) \wedge (\mu(z) \to R(((x \to y) \to z) \to z), 1))\} \\
&= \wedge_{z\in E}\{(\mu(z) \to R((x \to z) \to z, 1)) \wedge R(((x \to y) \to z) \to z), 1)\} \\
&\le \wedge_{z\in E}\{(\mu(z) \to R((x \to z) \to z, 1) \wedge (((x \to y) \to z) \to z), 1 \wedge 1)\} \\
&= \wedge_{z\in E}\{(\mu(z) \to R((x \to z) \vee ((x \to y) \to z) \to z, 1)\} \\
&= \wedge_{z\in E}\{(\mu(z) \to R((x \wedge (x \to y) \to z) \to z, 1)\}
\end{aligned}
$$

$\leq \wedge_{z\in E}\{(\mu(z) \to R((y \to z) \to z, 1)\}$
$= \wedge_{z\in E}\{(\mu(z) \to R(y \to z, z)\} = St^r_R(\mu)(y).$

Therefore $St^r_R(\mu)$ is a fuzzy prefilter of E. □

Example 4.6. *Let $E = \{0, a, b, c, d, e, f, 1\}$ with $0 < a, b < d, a < c, d < f, c < e, f < 1$. Define operations $\otimes$ and $\sim$ on E as follows:*

$\otimes$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	0	0	0	0	a	0	a
b	0	0	b	0	b	0	b	b
c	0	0	0	c	0	c	c	c
d	0	0	b	0	b	a	b	d
e	0	a	0	c	a	e	c	e
f	0	0	b	c	b	c	f	f
1	0	a	b	c	d	e	f	1

$\sim$	0	a	b	c	d	e	f	1
0	1	b	e	b	0	b	0	0
a	b	1	0	d	e	d	a	a
b	e	0	1	0	b	0	b	b
c	b	d	0	1	a	f	e	c
d	0	e	b	a	1	a	d	d
e	b	d	0	f	a	1	c	e
f	0	a	b	e	d	c	1	f
1	0	a	b	c	d	e	f	1

Then $(E, \wedge, \otimes, \sim, 1)$ is a prelinear and good EQ-algebra. Also we obtain the implication $\to$ on E below:

$\to$	0	a	b	c	d	e	f	1
0	1	1	1	1	1	1	1	1
a	b	1	b	1	1	1	1	1
b	e	e	1	e	1	e	1	1
c	b	d	b	1	d	1	1	1
d	0	e	b	e	1	e	1	1
e	b	d	b	f	d	1	f	1
f	0	a	b	e	d	e	1	1
1	0	a	b	c	d	e	f	1

It is easy to verify that $x \wedge (x \to y) \leq y$ for all $x, y \in E$.

Definition 4.7. *Let μ, ν be two fuzzy sets of E. Define the fuzzy right-stabilizer and the fuzzy left-stabilizer of μ with respect to ν by*

$$St^r(\mu, \nu)(x) = \wedge_{z\in E}\{\mu(z) \to \nu((x \to z) \to z)\},$$

$$St^l(\mu, \nu)(x) = \wedge_{z\in E}\{\mu(z) \to \nu((z \to x) \to x)\},$$

where $\to$ is the residuated implication with respect to a left-continuous t-norm.

Notice that if A, B are two classic subsets of E, then

$$St_r(A, B) = \{a \in E : \forall x \in A, (a \to x) \to x \in B\},$$

$$St_l(A, B) = \{a \in E : \forall x \in A, (x \to a) \to a \in B\},$$

are the right-stabilizer and left-stabilizer of A with respect to B of E.

Example 4.8. *Consider the EQ-algebra E in Example 3.2. Take two fuzzy sets μ, ν as follows: $\mu(0) = \mu(a) = 0.4, \mu(b) = 0.5, \mu(c) = 0.8, \mu(d) = \mu(1) = 1; \nu(0) = \nu(a) = \nu(b) = 0.2, \nu(c) = \nu(d) = \nu(1) = 0.9.$*

Then the fuzzy right-stabilizer and the left-stabilizer of μ *with respect to* ν *are* $St^r(\mu,\nu)(0) = St^r(\mu,\nu)(a) = 0.8, St^r(\mu,\nu)(b) = St^r(\mu,\nu)(c) = 0.9, St^r(\mu,\nu)(d) = St^r(\mu,\nu)(1) = 0.7$*;* $St^l(\mu,\nu)(0) = St^l(\mu,\nu)(a) = 0.8, St^l(\mu,\nu)(b) = 0.2, St^l(\mu,\nu)(c) = St^l(\mu,\nu)(d) = St^r(\mu,\nu)(1) = 0.9$*, where the residuated implication* $\to$ *is taken as ﬃukasiewicz implication.*

Proposition 4.9. *Let* $\mu, \nu, \mu_i, \nu_i (i \in I)$ *be fuzzy sets of* E*. Then we have:*
(1) $St^r(\chi_{\{1\}}, \nu) = E$ *if and only if* $\nu(1) = 1$*;*
(2) If E *is bounded, then* $St^l(\chi_{\{0\}}, \nu) = E$ *if and only if* $\nu(1) = 1$*;*
(3) If E *is bounded, then* $St^r(\chi_{\{0\}}, \nu) = \nu$ *iff* $\nu(\neg\neg x) = \nu(x)$*;*
(4) If $\mu_1 \subseteq \mu_2$ *and* $\nu_1 \subseteq \nu_2$*, then* $St^r(\mu_2, \nu_1) \subseteq St^r(\mu_1, \nu_2)$ *and* $St^l(\mu_2, \nu_1) \subseteq St^l(\mu_1, \nu_2)$*;*
(5) $St^r(\mu, \bigcap_{i\in I} \nu_i) = \bigcap_{i\in I} St^r(\mu, \nu_i)$ *and* $St^l(\mu, \bigcap_{i\in I} \nu_i) = \bigcap_{i\in I} St^l(\mu, \nu_i)$*;*
(6) $St^r(\bigcup_{i\in I} \mu_i, \nu) = \bigcap_{i\in I} St^r(\mu_i, \nu)$ *and* $St^l(\bigcup_{i\in I} \mu_i, \nu) = \bigcap_{i\in I} St^l(\mu_i, \nu)$*;*
(7) If E *is bounded, then* $D(E) = St^r_R(\chi_{\{0\}}, \chi_{\{1\}})$*, where* $D(E) = \{x \in E : \neg x = 0\}$*.*

Proof. (1) It follows from $St^r(\chi_{\{1\}}, \nu)(x) = \wedge_{y\in E}\{\chi_{\{1\}}(y) \to \nu((x \to y) \to y)\} = \nu(1)$.
(2) It follows from $St^l(\chi_{\{0\}}, \nu)(x) = \wedge_{y\in E}\{\chi_{\{0\}}(y) \to \nu((y \to x) \to x)\} = \nu(1)$.
(3) It follows from $St^r(\chi_{\{0\}}, \nu)(x) = \wedge_{y\in E}\{\chi_{\{0\}}(y) \to \nu((x \to y) \to y)\} = \nu(\neg\neg x)$.
(4) Since $St^r(\mu_2, \nu_1)(x) = \wedge_{y\in E}\{\mu_2(y) \to \nu_1((x \to y) \to y)\} \le \wedge_{y\in E}\{\mu_1(y) \to \nu_2((x \to y) \to y)\} = St^r(\mu_1, \nu_2)(x)$, we get $St^r(\mu_2, \nu_1) \subseteq St^r(\mu_1, \nu_2)$. Similarly, $St^l(\mu_2, \nu_1) \subseteq St^l(\mu_1, \nu_2)$.
(5) Since $St^r(\mu, \bigcap_{i\in I} \nu_i) = \wedge_{y\in E}\{\mu(y) \to \wedge_{i\in I}\nu_i((x \to y) \to y)\} = \wedge_{y\in E} \wedge_{i\in I} (\mu(y) \to \nu_i((x \to y) \to y)) = \wedge_{i\in I} \wedge_{y\in E} \{\mu(y) \to \nu_i((x \to y) \to y)\} = \wedge_{i\in I} St^r(\mu, \nu_i)(x)$, then $St^r(\mu, \bigcap_{i\in I} \nu_i) = \bigcap_{i\in I} St^r(\mu, \nu_i)$.
(6) Since $St^l(\bigcup_{i\in I} \mu_i, \nu) = \wedge_{y\in E}\{\vee_{i\in I}\mu_i(y) \to \nu((y \to x) \to x)\} = \wedge_{y\in E} \wedge_{i\in I} (\mu_i(y) \to \nu((y \to x) \to x) = \wedge_{i\in I} \wedge_{y\in E} \{\mu_i(y) \to \nu((y \to x) \to x)\} = \wedge_{i\in I} St^l(\mu_i, \nu)(x)$, then $St^l(\bigcup_{i\in I} \mu_i, \nu) = \bigcap_{i\in I} St^l(\mu_i, \nu)$.
(7) For $x \in D(E)$, $St^r_R(\chi_{\{0\}}, \chi_{\{1\}}) = \wedge_{y\in E}\{\chi_{\{0\}}(y) \to \chi_{\{1\}}((x \to y) \to y)\} = \chi_{\{1\}}((x \to 0) \to 0) = \chi_{\{1\}}(1) = 1$. □

Proposition 4.10. *Let* μ, ν *be two fuzzy sets of a good EQ-algebra* E*. Then we have:*
(1) $St^l(\chi_{\{1\}}, \nu) = \nu$*;*
(2) If $St^r(\mu, \nu) = E$ *or* $St^l(\mu, \nu) = E$*, then* $\mu \subseteq \nu$*;*
(3) If ν *is a fuzzy prefilter such that* $\mu \subseteq \nu$*, then* $St^l(\mu, \nu) = E$*. In particular,* $St^l(\nu, \nu) = E$*.*
(4) If μ *is a fuzzy prefilter such that* $\mu \subseteq \nu$*, then* $St^l(\mu, \nu) = E$*;*
(5) If ν *is a fuzzy prefilter, then* $\nu \subseteq St^r(\mu, \nu)$*.*

Proof. (1) Since $St^l(\chi_{\{1\}}, \nu)(x) = \wedge_{y\in E}\{\chi_{\{1\}}(y) \to \nu((y \to x) \to x)\} = \nu(x)$, we have $St^\wedge(\chi_{\{1\}}, \nu) = \nu$.
(2) If $St^r(\mu, \nu) = E$, then for all $x \in E$, $St^r(\mu, \nu)(x) = \wedge_{y\in E}\{\mu(y) \to \nu((x \to y) \to y)\} = 1$. Hence $\mu(x) \to \nu((x \to x) \to x) = \mu(x) \to \nu(x) = 1$. This implies $\mu(x) \le \nu(x)$ for all $x \in E$, that is, $\mu \subseteq \nu$. The other part is similar.
(3) If $\mu \subseteq \nu$ and ν is a fuzzy prefilter, then $St^l(\mu, \nu)(x) = \wedge_{y\in E}\{\mu(y) \to \nu((y \to x) \to x)\} \ge \wedge_{y\in E}\{\mu(y) \to \nu(y)\} = 1$. This shows $St^l(\mu, \nu) = E$.
(4) If $\mu \subseteq \nu$ and μ is a fuzzy prefilter, then $St^l(\mu, \nu)(x) = \wedge_{y\in E}\{\mu(y) \to \nu((y \to x) \to x)\} \ge \wedge_{y\in E}\{\mu((y \to x) \to x) \to \nu((y \to x) \to x)\} = 1$. This implies $St^l(\mu, \nu) = E$.
(5) Since ν is a fuzzy prefilter, then $St^r(\mu, \nu) = \wedge_{y\in E}\{\mu(y) \to \nu((x \to y) \to y)\} \ge \wedge_{y\in E}\nu((x \to y) \to y)) \ge \wedge_{y\in E}\nu(x) = \nu(x)$. Hence $\nu \subseteq St^r(\mu, \nu)$. □

The following are the relationships among fuzzy right-stabilizers, fuzzy left-stabilizers and fuzzy co-annihilators in EQ-algebras. To do this, we introduce the notion of fuzzy normal prefilters in EQ-algebras.

Definition 4.11. *A fuzzy prefilter* μ *of* E *is said to be a fuzzy normal prefilter of* E *if* $\mu(z) \wedge \mu(z \to ((y \to x) \to x) \le \mu((x \to y) \to y)$ *for all* $x, y, z \in E$*.*

Example 4.12. *Let* E *be the EQ-algebra in Example 3.2. Then the fuzzy set* μ *defined by* $\mu(0) = m, \mu(a) = \mu(b) = \mu(c) = \mu(d) = \mu(1) = n, 0 \le m < n \le 1$*, is a fuzzy normal prefilter of* E*.*

Theorem 4.13. *Let μ be a fuzzy prefilter of a good EQ-algebra E. Then μ is a fuzzy normal prefilter of E if and only if $\mu((y \to x) \to x) \leq \mu((x \to y) \to y)$ for all $x, y \in E$.*

Proof. Let μ be a fuzzy normal prefilter of E. Then $\mu(1) \wedge \mu(1 \to ((y \to x) \to x)) = \mu((y \to x) \to x) \leq \mu((x \to y) \to y)$. Conversely, let $x, y, z \in E$ such that $\mu((y \to x) \to x) \leq \mu((x \to y) \to y)$. Since μ is a fuzzy prefilter, we have $\mu(z) \wedge \mu(z \to ((y \to x) \to x)) \leq \mu((y \to x) \to x) \leq \mu((x \to y) \to y)$. □

Corollary 4.14. *Assume that E is a good EQ-algebra and μ is a fuzzy set of E. If ν be a fuzzy normal prefilter of E. Then $St^l(\mu, \nu) = St^r(\mu, \nu)$.*

Theorem 4.15. *Assume μ is a fuzzy set of a good EQ-algebra E. If ν is a fuzzy implicative prefilter of E, then $St^r(\mu, \nu) = St^l(\mu, \nu)$.*

Proof. Let ν be a fuzzy implicative prefilter of E. From $x \leq (y \to x) \to x$, we have $\neg((y \to x) \to x) \leq \neg x \leq x \to y$ and hence $(x \to y) \to y \leq \neg((y \to x) \to x) \to y \leq \neg((y \to x) \to x) \to ((y \to x) \to x)$. It follows $\nu(\neg((y \to x) \to x) \to ((y \to x) \to x) \geq \nu((x \to y) \to y)$. From Theorem 2.13, we have $\nu((y \to x) \to x) \geq \nu((x \to y) \to y)$. Similarly, $\nu((y \to x) \to x) \leq \nu((x \to y) \to y)$. Therefore $St^r(\mu, \nu) = St^l(\mu, \nu)$. □

From Theorem 4.13 and the proof of Theorem 4.15 we also see that every fuzzy implicative prefilter is a fuzzy normal prefilter in a bounded good EQ-algebra.

Let μ, ν be two fuzzy sets of E. The fuzzy co-annihilator of μ with respect to ν is defined as: for $x \in E$, $Ann(\mu, \nu)(x) = \wedge_{z \in E}\{\mu(z) \to \nu(z \vee x)\}$, where $\to$ is the residuated implication with respect to a left-continuous t-norm (see [18]).

Theorem 4.16. *Let μ, ν be two fuzzy sets of a prelinear and good EQ-algebra E. Then $Ann(\mu, \nu) \subseteq St^r(\mu, \nu)$ and $Ann(\mu, \nu) \subseteq St^l(\mu, \nu)$.*

Proof. By Theorem 4 of [3], $x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x)$. □

In what follows, we discuss the relations between fuzzy stabilizers and fuzzy prefilters (filters) in EQ-algebras.

Definition 4.17. *([18]) For all $x, y \in E$, a fuzzy prefilter μ of a lattice-ordered EQ-algebra E is called a*
- *fuzzy prime prefilter if E is nonconstant and $\mu(x \vee y) = \mu(x) \vee \mu(y)$;*
- *fuzzy Boolean prefilter if E is bounded and $\mu(x \vee \neg x) = \mu(1)$.*

Definition 4.18. *For all $x, y \in E$, a fuzzy prefilter μ of E is called a fuzzy obstinate prefilter if $\mu(x) \neq \mu(1)$, $\mu(y) \neq \mu(1)$ imply $\mu(x \to y) = \mu(1)$, $\mu(y \to x) = \mu(1)$.*

Example 4.19. *Let E be the EQ-algebra in Example 3.2 and μ be a fuzzy set defined by $\mu(0) = \mu(a) = 0.2$, $\mu(b) = \mu(c) = \mu(d) = \mu(1) = 0.8$. Then it is easy to verify that μ is a fuzzy obstinate prefilter of E.*

Theorem 4.20. *Suppose that E is a prelinear and good lattice-ordered EQ-algebra such that $x \wedge (x \to y) \leq y$ for all $x, y \in E$, and μ, ν are two fuzzy sets of E. We have:*
(1) If ν is a fuzzy filter of E, then $St^r(\mu, \nu)$ is a fuzzy prefilter of E;
(2) If ν is a fuzzy obstinate filter of E, then $St^r(\mu, \nu)$ is a fuzzy obstinate prefilter of E;
(3) If ν is a fuzzy prime filter of E, then $St^r(\mu, \nu)$ is a fuzzy prime prefilter of E;
(4) If ν is a fuzzy Boolean filter of E, then $St^r(\mu, \nu)$ is a fuzzy Boolean prefilter of E;
(5) If ν is a fuzzy positive implicative filter of E, then $St^r(\mu, \nu)$ is a fuzzy positive implicative prefilter of E.

Proof. Let E be a prelinear and good EQ-algebra.
(1) Since ν is a fuzzy filter of E, from Proposition 2.4 (3), Proposition 2.6, 2.10 we have:

$$\begin{aligned}&St^r(\mu, \nu)(x) \wedge St^r(\mu, \nu)(x \to y)\\ &= \wedge_{z \in E}\{\mu(z) \to \nu((x \to z) \to z)\} \wedge \wedge_{z \in E}\{\mu(z) \to \nu(((x \to y) \to z) \to z)\}\end{aligned}$$

$= \wedge_{z\in E}\{(\mu(z) \to \nu((x \to z) \to z)) \wedge (\mu(z) \to \nu(((x \to y) \to z) \to z))\}$
$= \wedge_{z\in E}\{(\mu(z) \to \nu((x \to z) \to z) \wedge \nu(((x \to y) \to z) \to z)\}$
$= \wedge_{z\in E}\{(\mu(z) \to \nu(((x \to z) \to z) \wedge ((x \to y) \to z) \to z)\}$
$= \wedge_{z\in E}\{(\mu(z) \to \nu((x \to z) \vee ((x \to y) \to z) \to z\}$
$= \wedge_{z\in E}\{(\mu(z) \to \nu((x \wedge (x \to y) \to z) \to z)\}$
$\leq \wedge_{z\in E}\{(\mu(z) \to \nu((y \to z) \to z)\}$
$= St^r(\mu, \nu)(y).$

On the other hand, $St^r(\mu,\nu)(1) = \wedge_{z\in E}\{\mu(z) \to \nu((1 \to z) \to z)\} = \wedge_{z\in E}\{\mu(z) \to \nu(1)\} \geq \wedge_{z\in E}\{\mu(z) \to \nu((x \to z) \to z)\} = St^r(\mu,\nu)(x)$.

Therefore $St^($$\mu, \nu)$ is a fuzzy prefilter of E.

(2) Assume that ν is a fuzzy obstinate filter of E. Let $St^r(\mu,\nu)(x) \neq St^r(\mu,\nu)(1)$ and $St^r(\mu,\nu)(y) \neq St^r(\mu,\nu)(1)$ for $x, y \in E$. Then $\nu((x \to z) \to z) \neq \nu(1)$ and $\nu((y \to z) \to z) \neq \nu(1)$. Since $\nu(x) \leq \nu((x \to z) \to z)$ and $\nu(y) \leq \nu((x \to z) \to z)$, we have $\nu(x) \neq \nu(1)$ and $\nu(y) \neq \nu(1)$. So $\nu(x \to y) = \nu(1)$ and $\nu(y \to x) = \nu(1)$, which shows $\nu(((x \to y) \to z) \to z) \geq \nu(x \to y) = \nu(1)$ and $\nu(((y \to x) \to z) \to z) \geq \nu(y \to x) = \nu(1)$. Thus $\nu(((x \to y) \to z) \to z) = \nu(1)$ and $\nu(((y \to x) \to z) \to z) = \nu(1)$. This implies $\wedge_{z\in E}\{\mu(z) \to \nu(((x \to y) \to z) \to z)\} = \wedge_{z\in E}\{\mu(z) \to \nu(1)\}$ and $\wedge_{z\in E}\{\mu(z) \to \nu(((y \to x) \to z) \to z)\} = \wedge_{z\in E}\{\mu(z) \to \nu(1)\}$. That is, $St^r(\mu,\nu)(x \to y) = St^r(\mu,\nu)(1)$ and $St^r(\mu,\nu)(y \to x) = St^r(\mu,\nu)(1)$. Therefore $St^r(\mu,\nu)$ is a fuzzy obstinate filter of E.

(3) Assume that ν is a fuzzy prime filter of E. From Proposition 2.4 and Proposition 2.6, $St^r(\mu,\nu)(x \vee y) = \wedge_{z\in E}[(\mu(z) \to \nu(((x \vee y) \to z) \to z)] = \wedge_{z\in E}[(\mu(z) \to \nu((x \to z) \wedge (y \to z) \to z)] = \wedge_{z\in E}[(\mu(z) \to \nu(((x \to z) \to z) \vee ((y \to z) \to z))] = \wedge_{z\in E}[(\mu(z) \to \nu((x \to z) \to z)) \vee \nu((y \to z) \to z))] = \wedge_{z\in E}[\mu(z) \to \nu((x \to z) \to z)] \vee \wedge_{z\in E}[\mu(z) \to \nu((y \to z) \to z)] = St^r(\mu,\nu)(x) \vee St^r(\mu,\nu)(y)$. Therefore $St^r(\mu,\nu)$ is a fuzzy prime filter of E.

(4) Assume that ν is a fuzzy Boolean filter of E. Then for $x \in E$, $St^r(\mu,\nu)(x \vee \neg x) = \wedge_{z\in E}[\mu(z) \to \nu(((x \vee \neg x) \to z) \to z)] \geq \wedge_{z\in E}[\mu(z) \to \nu(x \vee \neg x)] = \wedge_{z\in E}[\mu(z) \to \nu(1)] = St^r(\mu,\nu)(1)$. This implies that $St^r(\mu,\nu)$ is a fuzzy Boolean prefilter of E.

(5) Assume that ν is a fuzzy prime filter of E. By Theorem 2.13, we have that $St^r(\mu,\nu)(x \wedge (x \to y) \to y) = \wedge_{z\in E}[\mu(z) \to \nu(((x \wedge (x \to y) \to y) \to z) \to z)] \geq \wedge_{z\in E}[\mu(z) \to \nu(x \wedge (x \to y) \to y)] = \wedge_{z\in E}[\mu(z) \to \nu(1)] = \wedge_{z\in E}[\mu(z) \to \nu((1 \to z) \to z)] = St^r(\mu,\nu)(1)$. Therefore using Theorem 2.13 we see that $St^r(\mu,\nu)$ is a fuzzy positive implicative prefilter of E. □

Corollary 4.21. *Suppose that E is a linearly ordered good EQ-algebra such that $x \wedge (x \to y) \leq y$ for all $x, y \in E$. If μ is a fuzzy set of E and ν is a non-constant fuzzy filter of E, then $St^r(\mu,\nu)$ is a fuzzy prime prefilter of E.*

Lemma 4.22. *Let E be a prelinear and residuated EQ-algebra. Then $((x \otimes y) \to z) \to z \geq ((x \to z) \to z) \otimes ((y \to z) \to z)$ for any $x, y, z \in E$.*

Proof. Let $x, y, z \in E$. Then from (p5) and Proposition 2.5, $x \leq y \to (x \otimes y) \leq ((x \otimes y) \to z) \to (y \to z) \leq ((y \to z) \to z) \to (((x \otimes y) \to z) \to z)$ and $z \leq ((x \otimes y) \to z) \to z \leq ((y \to z) \to z) \to (((x \otimes y) \to z) \to z)$. Since E is prelinear, by Proposition 2.6 we have $(x \to z) \to z \leq x \vee z \leq ((y \to z) \to z) \to (((x \otimes y) \to z) \to z)$. Therefore $((x \otimes y) \to z) \to z \geq ((x \to z) \to z) \otimes ((y \to z) \to z)$. □

Theorem 4.23. *Let E be a prelinear and residuated lattice-ordered EQ-algebra, and μ be a fuzzy set and ν is a fuzzy filter of E. Then $St^r(\mu,\nu)$ is a fuzzy filter of E.*

Proof. By Theorem 4.20, we see $St^r(\mu,\nu)(1) = 1$. Let $x, y \in E$ such that $x \leq y$. Then $(x \to z) \to z \leq (y \to z) \to z$. Hence $St^r(\mu,\nu)(y) = \wedge_{z\in E}\{\mu(z) \to \nu((y \to z) \to z)\} \geq \wedge_{z\in E}\{\mu(z) \to \nu((x \to z) \to z)\} = St^r(\mu,\nu)(x)$. This shows (FF4). Next, set $x, y \in E$. Since ν is a fuzzy filter of E, by Proposition 2.10 and Lemma 4.22 we have that $St^r(\mu,\nu)(x \otimes y) = \wedge_{z\in E}\{\mu(z) \to \nu(((x \otimes y) \to z) \to z)\} \geq \wedge_{z\in E}\{\mu(z) \to \nu((x \to z) \to z) \otimes ((y \to z) \to z))\} \geq \wedge_{z\in E}\{\mu(z) \to \nu((x \to z) \to z) \wedge \nu((y \to z) \to z)\} = \wedge_{z\in E}\{(\mu(z) \to \nu((x \to z) \to z)) \wedge (\mu(z) \to \nu((y \to z) \to z))\} = \wedge_{z\in E}\{\mu(z) \to \nu((x \to z) \to z)\} \wedge \{\mu(z) \to \nu((y \to z) \to z)\} = St^r(\mu,\nu)(x) \wedge St^r(\mu,\nu)(y)$. This proves (FF5) and hence from Theorem 2.11 $St^r(\mu,\nu)$ is a fuzzy prefilter of E. Finally, let $x, y, z \in E$. We have

$St^r(\mu, \nu)(x \otimes z \to y \otimes z) = \wedge_{w \in E}\{\mu(w) \to \nu(((x \otimes z) \to (y \otimes z)) \to w) \to w)\} \geq \wedge_{w \in E}\{\mu(w) \to \nu(((x \to y) \to w) \to w)\} = St^r(\mu, \nu)(x \to y)$. Therefore $St^r(\mu, \nu)$ is a fuzzy filter of E. □

The set of all fuzzy prefilters (filters) of E is denoted by $\mathcal{FPF}(E)$ ($\mathcal{FF}(E)$). If μ be a fuzzy set of E, then the fuzzy prefilter generated by μ is defined as $< \mu >= \bigcap_{\nu \in \mathcal{FPF}(E), \mu \subseteq \nu} \nu$ and for $x \in E$, μ can be further represented as $< \mu > (x) = \vee\{\mu(a_1) \wedge \cdots \wedge \mu(a_n) | a_1, \cdots, a_n \in E, a_1 \to (a_2 \to (\cdots \to (a_n \to x)\cdots)) = 1\}$. For any $\mu, \nu \in \mathcal{FPF}(E)$, define $\mu \leq \nu$ if and only if $\mu \subseteq \nu$, $\mu \wedge \nu = \mu \cap \nu$, $\mu \vee \nu =< \mu \cup \nu >$. Then $(\mathcal{FPF}(E), \wedge, \vee, \emptyset, E)$ is a bounded lattice (see [2]), and $(\mathcal{FF}(E), \wedge, \vee, \emptyset, E)$ is a bounded lattice whence E is a residuated EQ-algebra (see [18]). Let us consider the residuated implication $\to$ as the Gödel residuated implication in the definition of $St^r(\mu, \nu)$. We have the following theorem.

Theorem 4.24. *Suppose that E is a prelinear and residuated lattice-ordered EQ-algebra. Then $(\mathcal{FF}(E), \wedge, \vee, \emptyset, E)$ is a relative pseudo-complemented lattice, where $St^r(\mu, \nu)$ is the relative pseudo-complement of μ with respect to ν in $\mathcal{FF}(E)$.*

Proof. According to Theorem 4.23 we only prove that $St^r(\mu, \nu) \cap \mu \subseteq \nu$. Let $\mu, \nu \in \mathcal{FF}(E)$. Indeed, for $x \in E$, $(St^r(\mu, \nu) \cap \mu)(x) = St^r(\mu, \nu)(x) \wedge \mu(x) = \wedge_{z \in E}\{\mu(z) \to \nu((x \to z) \to z)\} \wedge \mu(x) \leq \mu(x) \wedge (\mu(x) \to \nu(x)) \leq \nu(x)$. Now let $\lambda \in \mathcal{FF}(E)$ such that $\lambda \cap \mu \subseteq \nu$. Since (4), (5) of Proposition 4.10 and λ is a fuzzy prefilter of E, we have $St^r(\mu, \nu) = \wedge_{z \in E}\{\mu(z) \to \nu((x \to z) \to z)\} \geq \wedge_{z \in E}\{\mu(z) \to \lambda((x \to z) \to z) \wedge \mu((x \to z) \to z)\} = \wedge_{z \in E}\{\mu(z) \to \lambda((x \to z) \to z)\} \wedge \wedge_{z \in E}\{\mu(z) \to \mu((x \to z) \to z)\} = St^r(\mu, \lambda)(x) \wedge St^r(\mu, \mu)(x) = St^r(\mu, \lambda)(x) \geq \lambda(x)$. This shows $\lambda \subseteq St^r(\mu, \nu)$. Therefore $St^r(\mu, \nu)$ is the relative pseudo-complement of μ with respect to ν in $\mathcal{FF}(E)$. □

5 Conclusions

In this paper, motivated by the previous research of stabilizers in logic algebras, we introduce two types of (fuzzy) stabilizers in an EQ-algebra and several important results have been obtained. In particular, we prove that the set of all prefilters (fuzzy filters) $\mathcal{PF}(E)$ ($\mathcal{FF}(E)$) constitutes a relative pseudo-complemented lattice whence E is a particular EQ-algebra.

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