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Some remarks on a pair of seemingly unrelated regression models

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Abstract: Linear regression models are foundation of current statistical theory and have been a prominent object of study in statistical data analysis and inference. A special class of linear regression models is called the seemingly unrelated regression models (SURMs) which allow correlated observations between different regression equations. In this article, we present a general approach to SURMs under some general assumptions, including establishing closed-form expressions of the best linear unbiased predictors (BLUPs) and the best linear unbiased estimators (BLUEs) of all unknown parameters in the models, establishing necessary and sufficient conditions for a family of equalities of the predictors and estimators under the single models and the combined model to hold. Some fundamental and valuable properties of the BLUPs and BLUEs under the SURM are also presented.

Keywords: SURM, BLUP, BLUE, covariance matrix, decomposition identity

MSC: 62H12, 62J05

1 Introduction

Linear regression models are foundation of current statistical theory and have been a prominent object of study in statistical data analysis and inference. A special class of linear regression models is called the seemingly unrelated regression model (SURM) which allows correlated observations between regression equations. In this article, we consider a SURM of the form:

$$\mathscr{L}_1 : \quad \mathbf{y}_1 = \mathbf{X}_1\boldsymbol{\beta}_1 + \boldsymbol{\varepsilon}_1, \tag{1.1}$$

$$\mathscr{L}_2 : \quad \mathbf{y}_2 = \mathbf{X}_2\boldsymbol{\beta}_2 + \boldsymbol{\varepsilon}_2, \tag{1.2}$$

where $\mathbf{y}_i \in \mathbb{R}^{n_i\times 1}$ are vectors of observable response variables, $\mathbf{X}_i \in \mathbb{R}^{n_i\times p_i}$ are known matrices of arbitrary ranks, $\boldsymbol{\beta}_i \in \mathbb{R}^{p_i\times 1}$ are fixed but unknown vectors, $i = 1,\ 2$, $\boldsymbol{\varepsilon}_1 \in \mathbb{R}^{n_1\times 1}$ and $\boldsymbol{\varepsilon}_2 \in \mathbb{R}^{n_2\times 1}$ are random error vectors satisfying

$$\mathrm{E}\begin{bmatrix}\boldsymbol{\varepsilon}_1\\ \boldsymbol{\varepsilon}_2\end{bmatrix} = \mathbf{0}, \quad \mathrm{Cov}\begin{bmatrix}\boldsymbol{\varepsilon}_1\\ \boldsymbol{\varepsilon}_2\end{bmatrix} = \begin{bmatrix}\boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12}\\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22}\end{bmatrix} := \boldsymbol{\Sigma}. \tag{1.3}$$

Under these assumptions, (1.1)–(1.3) can jointly be written as

$$\mathscr{L} : \quad \mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \quad \mathrm{E}(\boldsymbol{\varepsilon}) = \mathbf{0}, \quad \mathrm{Cov}(\boldsymbol{\varepsilon}) = \boldsymbol{\Sigma}, \tag{1.4}$$

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where

$$\mathbf{y} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} \mathbf{X}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_2 \end{bmatrix}, \quad \boldsymbol{\beta} = \begin{bmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \end{bmatrix}, \quad \boldsymbol{\varepsilon} = \begin{bmatrix} \boldsymbol{\varepsilon}_1 \\ \boldsymbol{\varepsilon}_2 \end{bmatrix}.$$

The two individual equations are in fact linked each other since the disturbance terms in the two models are correlated. Thus, such a pair of linear regression models are usually called a seemingly unrelated regression model (SURM). It is well known that there are two main motivations for using SURMs in statistical analysis: the first one is to gain efficiency in estimation of parameters by combining information on the given different equations; the second is to impose and/or test restrictions that involve parameters in the different equations. Some earlier and seminal work in this area was presented in [1–3], whereas there are relatively many papers and also chapters in monographs on econometrics that approached SURMs, e.g., a thorough treatment is given in [4], and a survey can be found in [5–7] among others.

In the statistical inference of $\mathscr{L}_1$ and $\mathscr{L}_2$, a main objects of study is to estimate $\boldsymbol{\beta}_i$ and predict $\boldsymbol{\varepsilon}_i$, where the traditional procedure is to establish estimators and predictors of $\boldsymbol{\beta}_i$ and $\boldsymbol{\varepsilon}_i$, respectively. It is, however, better to simultaneously identify estimators and predictors of all unknown parameters in $\mathscr{L}_1$ and $\mathscr{L}_2$. Some recent contributions on simultaneous estimators/predictors of combined unknown parameter vectors under linear regression models can be found, e.g., in [8–10]. In this article, we construct two general vectors of the unknown vectors $\boldsymbol{\beta}_i$ and $\boldsymbol{\varepsilon}_i$ in $\mathscr{L}_1$ and $\mathscr{L}_2$ as follows

$$\boldsymbol{\psi}_1 = \mathbf{G}_1\boldsymbol{\beta}_1 + \mathbf{H}_1\boldsymbol{\varepsilon}_1, \quad \boldsymbol{\psi}_2 = \mathbf{G}_2\boldsymbol{\beta}_2 + \mathbf{H}_2\boldsymbol{\varepsilon}_2, \tag{1.5}$$

where $\mathbf{G}_i$ and $\mathbf{H}_i$ are given $k_i \times p_i$ and $k_i \times n_i$ matrices, respectively, $i = 1, 2$. Furthermore, merging the two vectors gives

$$\boldsymbol{\psi} = \mathbf{G}\boldsymbol{\beta} + \mathbf{H}\boldsymbol{\varepsilon}, \quad \boldsymbol{\psi} = \begin{bmatrix} \boldsymbol{\psi}_1 \\ \boldsymbol{\psi}_2 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} \mathbf{G}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{G}_2 \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} \mathbf{H}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_2 \end{bmatrix}. \tag{1.6}$$

In this setting,

$$\mathrm{E}(\boldsymbol{\psi}_i) = \mathbf{G}_i\boldsymbol{\beta}_i = \mathbf{G}_i\mathbf{S}_i\boldsymbol{\beta}, \quad \mathrm{E}(\boldsymbol{\psi}) = \mathbf{G}\boldsymbol{\beta}, \tag{1.7}$$

$$\mathrm{Cov}(\boldsymbol{\psi}_i) = \mathbf{H}_i\boldsymbol{\Sigma}_{ii}\mathbf{H}_i' = \mathbf{H}_i\mathbf{T}_i\boldsymbol{\Sigma}(\mathbf{H}_i\mathbf{T}_i)', \quad \mathrm{Cov}\{\boldsymbol{\psi}_i, \mathbf{y}_i\} = \mathbf{H}_i\boldsymbol{\Sigma}_{ii} = \mathbf{H}_i\mathbf{T}_i\boldsymbol{\Sigma}\mathbf{T}_i', \tag{1.8}$$

$$\mathrm{Cov}(\boldsymbol{\psi}) = \mathbf{H}\boldsymbol{\Sigma}\mathbf{H}', \quad \mathrm{Cov}\{\boldsymbol{\psi}, \mathbf{y}\} = \mathbf{H}\boldsymbol{\Sigma} \tag{1.9}$$

for $i = 1, 2$, where $\mathbf{S}_1 = [\mathbf{I}_{p_1}, \mathbf{0}]$, $\mathbf{S}_2 = [\mathbf{0}, \mathbf{I}_{p_2}]$, $\mathbf{T}_1 = [\mathbf{I}_{n_1}, \mathbf{0}]$, and $\mathbf{T}_2 = [\mathbf{0}, \mathbf{I}_{n_2}]$. When $\mathbf{G}_i = \mathbf{X}_i$ and $\mathbf{H}_i = \mathbf{I}_{n_i}$, (1.5) becomes $\boldsymbol{\psi}_i = \mathbf{X}_i\boldsymbol{\beta}_i + \boldsymbol{\varepsilon}_i = \mathbf{y}_i$, the observed response vector in $\mathscr{L}_1$ and $\mathscr{L}_2$. Hence, (1.5) includes all vector operations in $\mathscr{L}_1$ and $\mathscr{L}_2$ as its special cases.

Throughout out this article, $\mathbb{R}^{m\times n}$ denotes the collection of all $m \times n$ real matrices, and use $\mathbf{A}'$, $r(\mathbf{A})$, and $\mathscr{R}(\mathbf{A})$ to stand for the transpose, the rank, and the range of a matrix $\mathbf{A} \in \mathbb{R}^{m\times n}$, respectively; $\mathbf{I}_m$ denotes the identity matrix of order m. The Moore–Penrose inverse of $\mathbf{A}$, denoted by $\mathbf{A}^+$, is defined to be the unique solution $\mathbf{G}$ satisfying the four matrix equations $\mathbf{AGA} = \mathbf{A}$, $\mathbf{GAG} = \mathbf{G}$, $(\mathbf{AG})' = \mathbf{AG}$, and $(\mathbf{GA})' = \mathbf{GA}$. $\mathbf{P_A}$, $\mathbf{E_A}$, and $\mathbf{F_A}$ stand for the three orthogonal projectors (symmetric idempotent matrices) $\mathbf{P_A} = \mathbf{AA}^+$, $\mathbf{E_A} = \mathbf{A}^\perp = \mathbf{I}_m - \mathbf{AA}^+$, and $\mathbf{F_A} = \mathbf{I}_n - \mathbf{A}^+\mathbf{A}$ induced from $\mathbf{A}^+$. For two symmetric matrices $\mathbf{A}$ and $\mathbf{B}$ of the same size, $\mathbf{A} \succcurlyeq \mathbf{B}$ means that $\mathbf{A} - \mathbf{B}$ is nonnegative definite.

Concerning the predictability of $\boldsymbol{\psi}$ in (1.6), we need the following definition.

Definition 1.1. The vector $\boldsymbol{\psi}$ in (1.6) is said to be *predictable* under $\mathscr{L}$ if there exists a matrix $\mathbf{L} \in \mathbb{R}^{k\times n}$ such that $\mathrm{E}(\mathbf{Ly} - \boldsymbol{\psi}) = \mathbf{0}$. In particular, the $\mathbf{G}\boldsymbol{\beta}$ is said to be *estimable* under $\mathscr{L}$ if there exists a matrix $\mathbf{L} \in \mathbb{R}^{k\times n}$ such that $\mathrm{E}(\mathbf{Ly} - \mathbf{G}\boldsymbol{\beta}) = \mathbf{0}$.

Definition 1.2. Let $\boldsymbol{\psi}$ be defined in (1.6). If there exists a matrix $\mathbf{L}$ such that

$$\mathrm{Cov}(\mathbf{Ly} - \boldsymbol{\psi}) = \min \quad \text{s.t.} \quad \mathrm{E}(\mathbf{Ly} - \boldsymbol{\psi}) = \mathbf{0} \tag{1.10}$$

holds in the Löwner partial ordering, the linear statistic $\mathbf{Ly}$ is defined to be the best linear unbiased predictor(BLUP) of $\boldsymbol{\psi}$ under $\mathscr{L}$, and is denoted by

$$\mathbf{Ly} = \mathrm{BLUP}_{\mathscr{L}}(\boldsymbol{\psi}) = \mathrm{BLUP}_{\mathscr{L}}(\mathbf{G}\boldsymbol{\beta} + \mathbf{H}\boldsymbol{\varepsilon}). \tag{1.11}$$

If $\mathbf{H} = \mathbf{0}$, or $\mathbf{G} = \mathbf{0}$ in (1.6), then the $\mathbf{Ly}$ satisfying (1.10) is called the best linear unbiased estimator(BLUE) and the BLUP of $\mathbf{G}\boldsymbol{\beta}$ and $\mathbf{H}\boldsymbol{\varepsilon}$ under $\mathscr{L}$, respectively, and are denoted by

$$\mathbf{Ly} = \mathrm{BLUE}_{\mathscr{L}}(\mathbf{G}\boldsymbol{\beta}), \quad \mathbf{Ly} = \mathrm{BLUP}_{\mathscr{L}}(\mathbf{H}\boldsymbol{\varepsilon}), \tag{1.12}$$

respectively.

BLUPs/BLUEs are well known objects of study in regression analysis because of their simple and optimality properties in statistical inferences, and are one of the prominent research objects in the field of statistics and applications. Because the BLUPs of $\boldsymbol{\psi}_i$ under $\mathscr{L}_1$ and $\mathscr{L}_2$, and the BLUPs of $\boldsymbol{\psi}_i$ under $\mathscr{L}$ are not necessarily the same, it is natural to compare the BLUPs under these models, and establish possible connections for the BLUPs, such as,

$$\mathrm{BLUP}_{\mathscr{L}}(\boldsymbol{\psi}_i) = \mathrm{BLUP}_{\mathscr{L}_i}(\boldsymbol{\psi}_i), \ \ i = 1, 2, \tag{1.13}$$

$$\mathrm{BLUP}_{\mathscr{L}}(\boldsymbol{\psi}) = \begin{bmatrix} \mathrm{BLUP}_{\mathscr{L}_1}(\boldsymbol{\psi}_1) \\ \mathrm{BLUP}_{\mathscr{L}_2}(\boldsymbol{\psi}_2) \end{bmatrix}. \tag{1.14}$$

This article aims at establishing necessary and sufficient conditions for the equalities to hold, and presents some consequences and applications of these equalities.

2 Preliminary results

We need the following tools in the analysis of (1.1)–(1.14).

Lemma 2.1 ([11]). *Let* $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{B} \in \mathbb{R}^{m\times k}$, $\mathbf{C} \in \mathbb{R}^{l\times n}$ *and* $\mathbf{D} \in \mathbb{R}^{l\times k}$. *Then*

$$r[\,\mathbf{A},\ \mathbf{B}\,] = r(\mathbf{A}) + r(\mathbf{E_A B}) = r(\mathbf{B}) + r(\mathbf{E_B A}), \tag{2.1}$$

$$r\begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix} = r(\mathbf{A}) + r(\mathbf{CF_A}) = r(\mathbf{C}) + r(\mathbf{AF_C}). \tag{2.2}$$

If $\mathscr{R}(\mathbf{B}) \subseteq \mathscr{R}(\mathbf{A})$ *and* $\mathscr{R}(\mathbf{C}') \subseteq \mathscr{R}(\mathbf{A}')$, *then*

$$r\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = r(\mathbf{A}) + r(\mathbf{D} - \mathbf{CA}^{+}\mathbf{B}). \tag{2.3}$$

In addition, the following results hold.

(a) $r[\,\mathbf{A},\ \mathbf{B}\,] = r(\mathbf{A}) \Leftrightarrow \mathscr{R}(\mathbf{B}) \subseteq \mathscr{R}(\mathbf{A}) \Leftrightarrow \mathbf{AA}^{+}\mathbf{B} = \mathbf{B} \Leftrightarrow \mathbf{E_A B} = \mathbf{0}.$

(b) $r\begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix} = r(\mathbf{A}) \Leftrightarrow \mathscr{R}(\mathbf{C}') \subseteq \mathscr{R}(\mathbf{A}') \Leftrightarrow \mathbf{CA}^{+}\mathbf{A} = \mathbf{C} \Leftrightarrow \mathbf{CF_A} = \mathbf{0}.$

Lemma 2.2 ([12]). *The linear matrix equation* $\mathbf{AX} = \mathbf{B}$ *is solvable for* $\mathbf{X}$ *if and only if* $r[\mathbf{A},\ \mathbf{B}] = r(\mathbf{A})$, *or equivalently,* $\mathbf{AA}^{+}\mathbf{B} = \mathbf{B}$. *In this case, the general solution of the equation can be written as* $\mathbf{X} = \mathbf{A}^{+}\mathbf{B} + (\mathbf{I} - \mathbf{A}^{+}\mathbf{A})\mathbf{U}$, *where* $\mathbf{U}$ *is an arbitrary matrix.*

In order to directly solve the matrix minimization problem in (1.10), we need the following known result.

Lemma 2.3 ([9]). *Let*

$$f(\mathbf{L}) = (\mathbf{LC} + \mathbf{D})\mathbf{M}(\mathbf{LC} + \mathbf{D})' \ \ s.t. \ \ \mathbf{LA} = \mathbf{B},$$

where $\mathbf{A} \in \mathbb{R}^{p\times q}$, $\mathbf{B} \in \mathbb{R}^{n\times q}$, $\mathbf{C} \in \mathbb{R}^{p\times m}$ *and* $\mathbf{D} \in \mathbb{R}^{n\times m}$ *are given*, $\mathbf{M} \in \mathbb{R}^{m\times m}$ *is nnd, and the matrix equation* $\mathbf{LA} = \mathbf{B}$ *is solvable. Then there always exists a solution* $\mathbf{L}_0$ *of* $\mathbf{L}_0\mathbf{A} = \mathbf{B}$ *such that*

$$f(\mathbf{L}) \succcurlyeq f(\mathbf{L}_0)$$

holds for all solutions of $\mathbf{LA} = \mathbf{B}$. *In this case, the matrix* $\mathbf{L}_0$ *satisfying the above inequality is determined by the following solvable matrix equation*

$$\mathbf{L}_0\left[\mathbf{A},\ \mathbf{CMC}'\mathbf{A}^{\perp}\right] = \left[\mathbf{B},\ -\mathbf{DMC}'\mathbf{A}^{\perp}\right].$$

In this case, the general expression of $\mathbf{L}_0$ *and the corresponding* $f(\mathbf{L}_0)$ *and* $f(\mathbf{L})$ *are given by*

$$\begin{aligned}
\mathbf{L}_0 &= \underset{\mathbf{LA}=\mathbf{B}}{\operatorname{argmin}} f(\mathbf{L}) = \left[\mathbf{B},\ -\mathbf{DMC}'\mathbf{A}^{\perp}\right]\left[\mathbf{A},\ \mathbf{CMC}'\mathbf{A}^{\perp}\right]^{+} + \mathbf{U}\left[\mathbf{A},\ \mathbf{CMC}'\right]^{\perp},\\
f(\mathbf{L}_0) &= \min_{\mathbf{LA}=\mathbf{B}} f(\mathbf{L}) = \mathbf{FMF}' - \mathbf{FMC}'\mathbf{TCMF}',\\
f(\mathbf{L}) &= f(\mathbf{L}_0) + (\mathbf{LC}+\mathbf{D})\,\mathbf{MC}'\mathbf{TCM}\,(\mathbf{LC}+\mathbf{D})'\\
&= f(\mathbf{L}_0) + \left(\mathbf{LCMC}'\mathbf{A}^{\perp} + \mathbf{DMC}'\mathbf{A}^{\perp}\right)\mathbf{T}\left(\mathbf{LCMC}'\mathbf{A}^{\perp} + \mathbf{DMC}'\mathbf{A}^{\perp}\right)',
\end{aligned}$$

where $\mathbf{F} = \mathbf{BA}^{+}\mathbf{C} + \mathbf{D}$, $\mathbf{T} = \left(\mathbf{A}^{\perp}\mathbf{CMC}'\mathbf{A}^{\perp}\right)^{+}$, *and* $\mathbf{U} \in \mathbb{R}^{n\times p}$ *is arbitrary.*

3 Exact formulas for BLUPs of all parameters under SURMs

Classic estimation/prediction problems of unknown parameters in SURMs were considered, e.g., in [13–15]. Some effective algebraic methods for deriving analytical formulas of BLUPs/BLUEs under general linear regression models have recently been proposed and used in [8–10, 16–26]. In this section, we first give a new derivation of exact formulas for calculating the BLUPs of $\boldsymbol{\psi}_i$ in (1.5), and show a variety of algebraic and statistical properties of the BLUPs. It can be seen from (1.1), (1.2), and (1.5) that

$$\begin{aligned}
\mathbf{L}_i\mathbf{y}_i - \boldsymbol{\psi}_i &= \mathbf{L}_i\mathbf{X}_i\boldsymbol{\beta}_i + \mathbf{L}_i\boldsymbol{\varepsilon}_i - \mathbf{G}_i\boldsymbol{\beta}_i - \mathbf{H}_i\boldsymbol{\varepsilon}_i\\
&= (\mathbf{L}_i\mathbf{X}_i - \mathbf{G}_i)\boldsymbol{\beta}_i + (\mathbf{L}_i - \mathbf{H}_i)\boldsymbol{\varepsilon}_i\\
&= (\mathbf{L}_i\mathbf{X}_i - \mathbf{G}_i)\boldsymbol{\beta}_i + (\mathbf{L}_i - \mathbf{H}_i)\mathbf{T}_i\boldsymbol{\varepsilon},\quad i = 1,\ 2.
\end{aligned}\tag{3.1}$$

Then, the expectations and covariance matrices of $\mathbf{L}_i\mathbf{y}_i - \boldsymbol{\psi}_i$ can be written as

$$\mathrm{E}(\mathbf{L}_i\mathbf{y}_i - \boldsymbol{\psi}_i) = (\mathbf{L}_i\mathbf{X}_i - \mathbf{G}_i)\boldsymbol{\beta}_i,\tag{3.2}$$

$$\begin{aligned}
\mathrm{Cov}(\mathbf{L}_i\mathbf{y}_i - \boldsymbol{\psi}_i) &= \mathrm{Cov}[(\mathbf{L}_i\mathbf{X}_i - \mathbf{G}_i)\boldsymbol{\beta}_i + (\mathbf{L}_i\mathbf{T}_i - \mathbf{H}_i\mathbf{T}_i)\boldsymbol{\varepsilon}]\\
&= (\mathbf{L}_i\mathbf{T}_i - \mathbf{H}_i\mathbf{T}_i)\boldsymbol{\Sigma}(\mathbf{L}_i\mathbf{T}_i - \mathbf{H}_i\mathbf{T}_i)' \triangleq f_i(\mathbf{L}_i)
\end{aligned}\tag{3.3}$$

for $i = 1,\ 2$. Hence, the constrained covariance matrix minimization problems in (1.10) convert to mathematical problems of minimizing the quadratic matrix-valued functions $f_i(\mathbf{L}_i)$ subject to $(\mathbf{L}_i\mathbf{X}_i - \mathbf{G}_i)\boldsymbol{\beta}_i = \mathbf{0}$, $i = 1,\ 2$. Our first main result is presented below.

Theorem 3.1. *Let* $\mathscr{L}_1$ *and* $\mathscr{L}_2$ *be as given in* (1.1) *and* (1.2), *respectively, and denote*

$$\mathbf{C}_i = \mathrm{Cov}\{\boldsymbol{\psi}_i,\ \mathbf{y}_i\} = \mathbf{H}_i\mathbf{T}_i\boldsymbol{\Sigma}\mathbf{T}_i',\quad i = 1, 2.\tag{3.4}$$

Then, the parameter vectors $\boldsymbol{\psi}_i$ *in* (1.5) *are predictable by* $\mathbf{y}_i$ *in* $\mathscr{L}_1$ *and* $\mathscr{L}_2$, *respectively, if and only if*

$$\mathscr{R}(\mathbf{X}_i') \supseteq \mathscr{R}(\mathbf{G}_i'),\quad i = 1, 2.\tag{3.5}$$

In these cases,

$$\mathrm{Cov}(\widehat{\mathbf{L}}_i\mathbf{y}_i - \boldsymbol{\psi}_i) = \min\ s.t.\ \mathrm{E}(\widehat{\mathbf{L}}_i\mathbf{y}_i - \boldsymbol{\psi}_i) = \mathbf{0} \Leftrightarrow \widehat{\mathbf{L}}_i[\mathbf{X}_i,\ \boldsymbol{\Sigma}_{ii}\mathbf{X}_i^{\perp}] = [\mathbf{G}_i,\ \mathbf{C}_i\mathbf{X}_i^{\perp}],\quad i = 1,\ 2.\tag{3.6}$$

The matrix equations in (3.6) *are solvable under* (3.5), *and the general solutions* $\widehat{\mathbf{L}}_i$ *and the corresponding* $\mathrm{BLUP}_{\mathscr{L}_i}(\boldsymbol{\psi}_i)$ *can be written as*

$$\mathrm{BLUP}_{\mathscr{L}_i}(\boldsymbol{\psi}_i) = \widehat{\mathbf{L}}_i\mathbf{y}_i = \widehat{\mathbf{L}}_i\mathbf{T}_i\mathbf{y} = \left([\,\mathbf{G}_i,\ \mathbf{C}_i\mathbf{X}_i^{\perp}\,][\,\mathbf{X}_i,\ \boldsymbol{\Sigma}_{ii}\mathbf{X}_i^{\perp}\,]^{+} + \mathbf{U}_i[\,\mathbf{X}_i,\ \boldsymbol{\Sigma}_{ii}\mathbf{X}_i^{\perp}\,]^{\perp}\right)\mathbf{T}_i\mathbf{y}, \tag{3.7}$$

where $\mathbf{U}_i \in \mathbb{R}^{k_i\times n_i}$ *are arbitrary,* $i = 1,\ 2$. *The corresponding* $f_i(\widehat{\mathbf{L}}_i)$ *and* $f_i(\mathbf{L}_i)$ *under* (3.1)–(3.3) *are given by*

$$\begin{aligned} f_i(\widehat{\mathbf{L}}_i) &= \mathrm{Cov}[\,\mathrm{BLUP}_{\mathscr{L}_i}(\boldsymbol{\psi}_i) - \boldsymbol{\psi}_i\,] \\ &= \left([\,\mathbf{G}_i,\ \mathbf{C}_i\mathbf{X}_i^{\perp}\,][\,\mathbf{X}_i,\ \boldsymbol{\Sigma}_{ii}\mathbf{X}_i^{\perp}\,]^{+}\mathbf{T}_i - \mathbf{H}_i\mathbf{T}_i\right)\boldsymbol{\Sigma}\left([\,\mathbf{G}_i,\ \mathbf{C}_i\mathbf{X}_i^{\perp}\,][\,\mathbf{X}_i,\ \boldsymbol{\Sigma}_{ii}\mathbf{X}_i^{\perp}\,]^{+}\mathbf{T}_i - \mathbf{H}_i\mathbf{T}_i\right)', \end{aligned} \tag{3.8}$$

$$\begin{aligned} f_i(\mathbf{L}_i) &= f_i(\widehat{\mathbf{L}}_i) + \left(\mathbf{L}_i\mathbf{T}_i\boldsymbol{\Sigma}\mathbf{T}_i' - \mathbf{H}_i\mathbf{T}_i\boldsymbol{\Sigma}\mathbf{T}_i'\right)\left(\mathbf{X}_i^{\perp}\mathbf{T}_i\boldsymbol{\Sigma}\mathbf{T}_i'\mathbf{X}_i^{\perp}\right)^{+}\left(\mathbf{L}_i\mathbf{T}_i\boldsymbol{\Sigma}\mathbf{T}_i' - \mathbf{H}_i\mathbf{T}_i\boldsymbol{\Sigma}\mathbf{T}_i'\right)' \\ &= f_i(\widehat{\mathbf{L}}_i) + (\mathbf{L}_i\boldsymbol{\Sigma}_{ii} - \mathbf{C}_i)\left(\mathbf{X}_i^{\perp}\boldsymbol{\Sigma}_{ii}\mathbf{X}_i^{\perp}\right)^{+}(\mathbf{L}_i\boldsymbol{\Sigma}_{ii} - \mathbf{C}_i)' \end{aligned} \tag{3.9}$$

for $i = 1,\ 2$. *Further, the following results hold.*

(a) $r[\,\mathbf{X}_i,\ \boldsymbol{\Sigma}_{ii}\mathbf{X}_i^{\perp}\,] = r[\,\mathbf{X}_i,\ \boldsymbol{\Sigma}_{ii}\,]$, $\mathscr{R}[\,\mathbf{X}_i,\ \boldsymbol{\Sigma}_{ii}\mathbf{X}_i^{\perp}\,] = \mathscr{R}[\,\mathbf{X}_i,\ \boldsymbol{\Sigma}_{ii}\,]$ *and* $\mathscr{R}(\mathbf{X}_i) \cap \mathscr{R}(\boldsymbol{\Sigma}_{ii}\mathbf{X}_i^{\perp}) = \{\mathbf{0}\}$, $i = 1,\ 2$.

(b) $\widehat{\mathbf{L}}_i$ *are unique if and only if* $r[\,\mathbf{X}_i,\ \boldsymbol{\Sigma}_{ii}\,] = n_i$, $i = 1,\ 2$.

(c) $\mathrm{BLUP}_{\mathscr{L}_i}(\boldsymbol{\psi}_i)$ *are unique with probability* 1 *if and only if* $\mathbf{y}_i \in \mathscr{R}[\,\mathbf{X}_i,\ \boldsymbol{\Sigma}_{ii}\,]$ *hold with probability* 1, $i = 1,\ 2$.

(d) *The covariance matrices of* $\mathrm{BLUP}_{\mathscr{L}_i}(\boldsymbol{\psi}_i)$, *as well as the covariance matrices between* $\mathrm{BLUP}_{\mathscr{L}_i}(\boldsymbol{\psi}_i)$ *and* $\boldsymbol{\psi}_i$ *are unique, and satisfy the formulas*

$$\mathrm{Cov}[\mathrm{BLUP}_{\mathscr{L}_i}(\boldsymbol{\psi}_i)] = \left([\,\mathbf{G}_i,\ \mathbf{C}_i\mathbf{X}_i^{\perp}\,][\,\mathbf{X}_i,\ \boldsymbol{\Sigma}_{ii}\mathbf{X}_i^{\perp}\,]^{+}\right)\boldsymbol{\Sigma}_{ii}\left([\,\mathbf{G}_i,\ \mathbf{C}_i\mathbf{X}_i^{\perp}\,][\,\mathbf{X}_i,\ \boldsymbol{\Sigma}_{ii}\mathbf{X}_i^{\perp}\,]^{+}\right)', \tag{3.10}$$

$$\mathrm{Cov}\{\mathrm{BLUP}_{\mathscr{L}_i}(\boldsymbol{\psi}_i),\ \boldsymbol{\psi}_i\} = [\,\mathbf{G}_i,\ \mathbf{C}_i\mathbf{X}_i^{\perp}\,][\,\mathbf{X}_i,\ \boldsymbol{\Sigma}_{ii}\mathbf{X}_i^{\perp}\,]^{+}\mathbf{C}_i', \tag{3.11}$$

$$\begin{aligned} \mathrm{Cov}(\boldsymbol{\psi}_i) - \mathrm{Cov}[\mathrm{BLUP}_{\mathscr{L}_i}(\boldsymbol{\psi}_i)] &= \mathbf{H}_i\mathbf{T}_i\boldsymbol{\Sigma}(\mathbf{H}_i\mathbf{T}_i)' \\ &\quad - \left([\,\mathbf{G}_i,\ \mathbf{C}_i\mathbf{X}_i^{\perp}\,][\,\mathbf{X}_i,\ \boldsymbol{\Sigma}_{ii}\mathbf{X}_i^{\perp}\,]^{+}\right)\boldsymbol{\Sigma}_{ii}\left([\,\mathbf{G}_i,\ \mathbf{C}_i\mathbf{X}_i^{\perp}\,][\,\mathbf{X}_i,\ \boldsymbol{\Sigma}_{ii}\mathbf{X}_i^{\perp}\,]^{+}\right)' \end{aligned} \tag{3.12}$$

for $i = 1,\ 2$.

(e) *The BLUPs of* $\boldsymbol{\psi}_i$ *can be decomposed as the sums*

$$\mathrm{BLUP}_{\mathscr{L}_i}(\boldsymbol{\psi}_i) = \mathrm{BLUE}_{\mathscr{L}_i}(\mathbf{G}_i\boldsymbol{\beta}_i) + \mathrm{BLUP}_{\mathscr{L}_i}(\mathbf{H}_i\boldsymbol{\varepsilon}_i),\quad i = 1,\ 2. \tag{3.13}$$

(f) *If* $\boldsymbol{\psi}_1$ *and* $\boldsymbol{\psi}_2$ *are predictable under* $\mathscr{L}_1$ *and* $\mathscr{L}_2$, *respectively, then* $\mathbf{P}_1\boldsymbol{\psi}_1$ *and* $\mathbf{P}_2\boldsymbol{\psi}_2$ *are predictable under* $\mathscr{L}_1$ *and* $\mathscr{L}_2$, *respectively, and* $\mathrm{BLUP}_{\mathscr{L}_i}(\mathbf{P}_i\boldsymbol{\psi}_i) = \mathbf{P}_i\mathrm{BLUP}_{\mathscr{L}_i}(\boldsymbol{\psi}_i)$ *hold for any matrices* $\mathbf{P}_i \in \mathbb{R}^{t_i\times k_i}$, $i = 1,\ 2$.

Proof. It can be seen from (1.1), (1.2), and (1.5) that

$$\mathrm{E}\left(\mathbf{L}_i\mathbf{y}_i - \boldsymbol{\psi}_i\right) = \mathbf{0} \Leftrightarrow \mathbf{L}_i\mathbf{X}_i\boldsymbol{\beta}_i - \mathbf{G}_i\boldsymbol{\beta}_i = \mathbf{0} \text{ for all } \boldsymbol{\beta}_i \Leftrightarrow \mathbf{L}_i\mathbf{X}_i = \mathbf{G}_i,\quad i = 1,\ 2.$$

From Lemma 2.2, the matrix equations are solvable respectively if and only if (3.5) hold. In these cases, we see from Lemma 2.2 that the first parts of (3.6) are equivalent to finding solutions $\widehat{\mathbf{L}}_i$ of the solvable matrix equations $\widehat{\mathbf{L}}_i\mathbf{X}_i = \mathbf{G}_i$ such that

$$f_i(\mathbf{L}_i) \succcurlyeq f_i(\widehat{\mathbf{L}}_i) \ \text{ s.t. } \ \mathbf{L}_i\mathbf{X}_i = \mathbf{G}_i,\quad i = 1,\ 2 \tag{3.14}$$

hold in the Löwner partial ordering. Further from Lemma 2.3, there always exist solutions $\widehat{\mathbf{L}}_i$ of $\widehat{\mathbf{L}}_i\mathbf{X}_i = \mathbf{G}_i$ such that (3.14) hold, and the $\widehat{\mathbf{L}}_i$ are determined by the matrix equations

$$\widehat{\mathbf{L}}_i[\,\mathbf{X}_i,\ \mathbf{T}_i\boldsymbol{\Sigma}\mathbf{T}_i'\mathbf{X}_i^{\perp}\,] = [\,\mathbf{G}_i,\ \mathbf{H}_i\mathbf{T}_i\boldsymbol{\Sigma}\mathbf{T}_i'\mathbf{X}_i^{\perp}\,],\quad i = 1,\ 2,$$

thus establishing the matrix equations in (3.6). Solving the matrix equations by Lemma 2.2 gives the $\widehat{\mathbf{L}}_i$ in (3.7). Also from (3.3),

$$\begin{aligned} f_i(\widehat{\mathbf{L}}_i) &= \mathrm{Cov}(\widehat{\mathbf{L}}_i\mathbf{y}_i - \boldsymbol{\psi}_i) \\ &= \left([\,\mathbf{G}_i,\ \mathbf{C}_i\mathbf{X}_i^{\perp}\,][\,\mathbf{X}_i,\ \boldsymbol{\Sigma}_{ii}\mathbf{X}_i^{\perp}\,]^{+}\mathbf{T}_i - \mathbf{H}_i\mathbf{T}_i\right)\boldsymbol{\Sigma}\left([\,\mathbf{G}_i,\ \mathbf{C}_i\mathbf{X}_i^{\perp}\,][\,\mathbf{X}_i,\ \boldsymbol{\Sigma}_{ii}\mathbf{X}_i^{\perp}\,]^{+}\mathbf{T}_i - \mathbf{H}_i\mathbf{T}_i\right)', \end{aligned}$$

as required for (3.8) for $i = 1, 2$. Eq. (3.9) follows from Lemma 2.3.

Result (a) is well known. Results (b) and (c) follow directly from (3.7). Taking covariance matrices of (3.7) yields (3.10). From (3.4) and (3.7),

$$\mathrm{Cov}\{\mathrm{BLUP}_{\mathscr{L}_i}(\boldsymbol{\psi}_i),\ \boldsymbol{\psi}_i\} = \mathrm{Cov}\{\widehat{\mathbf{L}}_i\mathbf{y}_i,\ \boldsymbol{\psi}_i\} = \widehat{\mathbf{L}}_i\mathrm{Cov}\{\mathbf{y}_i,\ \boldsymbol{\psi}_i\} = [\,\mathbf{G}_i,\ \mathbf{C}_i\mathbf{X}_i^{\perp}\,][\,\mathbf{X}_i,\ \boldsymbol{\Sigma}_{ii}\mathbf{X}_i^{\perp}\,]^{+}\mathbf{C}_i',$$

thus establishing (3.11) for $i = 1, 2$. The two equalities in (3.12) follow from (1.8) and (3.10). Results (e) and (f) are direct consequences of (3.7). □

We next derive the BLUPs of $\boldsymbol{\psi}$ and $\boldsymbol{\psi}_i$ under $\mathscr{L}$, respectively. Note from $\mathscr{L}$, (1.5) and (1.6) that

$$\mathbf{K}\mathbf{y} - \boldsymbol{\psi} = \mathbf{K}\mathbf{X}\boldsymbol{\beta} + \mathbf{K}\boldsymbol{\varepsilon} - \mathbf{G}\boldsymbol{\beta} - \mathbf{H}\boldsymbol{\varepsilon} = (\mathbf{K}\mathbf{X} - \mathbf{G})\boldsymbol{\beta} + (\mathbf{K} - \mathbf{H})\boldsymbol{\varepsilon},$$
$$\mathbf{K}_i\mathbf{y} - \boldsymbol{\psi}_i = \mathbf{K}_i\mathbf{X}\boldsymbol{\beta} + \mathbf{K}_i\boldsymbol{\varepsilon} - \mathbf{G}_i\boldsymbol{\beta}_i - \mathbf{H}_i\boldsymbol{\varepsilon}_i = (\mathbf{K}_i\mathbf{X} - \mathbf{G}_i\mathbf{S}_i)\boldsymbol{\beta} + (\mathbf{K}_i - \mathbf{H}_i\mathbf{T}_i)\boldsymbol{\varepsilon},\quad i = 1, 2.$$

Then the expectations and covariance matrices of $\mathbf{K}\mathbf{y} - \boldsymbol{\psi}$ and $\mathbf{K}_i\mathbf{y} - \boldsymbol{\psi}_i$ can be written as

$$\mathrm{E}(\,\mathbf{K}\mathbf{y} - \boldsymbol{\psi}\,) = (\mathbf{K}\mathbf{X} - \mathbf{G})\boldsymbol{\beta},\quad \mathrm{E}(\,\mathbf{K}_i\mathbf{y} - \boldsymbol{\psi}_i\,) = (\mathbf{K}_i\mathbf{X} - \mathbf{G}_i\mathbf{S}_i)\boldsymbol{\beta}, \tag{3.15}$$

$$\mathrm{Cov}(\,\mathbf{K}\mathbf{y} - \boldsymbol{\psi}\,) = \mathrm{Cov}[\,(\mathbf{K}\mathbf{X} - \mathbf{G})\boldsymbol{\beta} + (\mathbf{K} - \mathbf{H})\boldsymbol{\varepsilon}\,] = (\mathbf{K} - \mathbf{H})\boldsymbol{\Sigma}(\mathbf{K} - \mathbf{H})' \triangleq g(\mathbf{K}), \tag{3.16}$$

$$\mathrm{Cov}(\,\mathbf{K}_i\mathbf{y} - \boldsymbol{\psi}_i\,) = \mathrm{Cov}[\,(\mathbf{K}_i\mathbf{X} - \mathbf{G}_i\mathbf{S}_i)\boldsymbol{\beta} + (\mathbf{K}_i - \mathbf{H}_i\mathbf{T}_i)\boldsymbol{\varepsilon}\,] = (\mathbf{K}_i - \mathbf{H}_i\mathbf{T}_i)\boldsymbol{\Sigma}(\mathbf{K}_i - \mathbf{H}_i\mathbf{T}_i)' \triangleq g_i(\mathbf{K}_i) \tag{3.17}$$

for $i = 1, 2$. Our second main result is presented below.

Theorem 3.2. *Let $\mathscr{L}$ be as given in* (1.4), *and denote*

$$\mathbf{J} = \mathrm{Cov}\{\boldsymbol{\psi},\ \mathbf{y}\} = \mathbf{H}\boldsymbol{\Sigma},\quad \mathbf{J}_i = \mathrm{Cov}\{\boldsymbol{\psi}_i,\ \mathbf{y}\} = \mathbf{H}_i\mathbf{T}_i\boldsymbol{\Sigma},\quad i = 1, 2. \tag{3.18}$$

Then, the parameter vector $\boldsymbol{\psi}$ in (1.6) *is predictable by* $\mathbf{y}$ *in $\mathscr{L}$ if and only if*

$$\mathscr{R}(\mathbf{X}') \supseteq \mathscr{R}(\mathbf{G}'),\ \ i.e.,\ \mathscr{R}(\mathbf{X}_i') \supseteq \mathscr{R}(\mathbf{G}_i'),\quad i = 1, 2. \tag{3.19}$$

In this case,

$$\mathrm{E}(\widehat{\mathbf{K}}\mathbf{y} - \boldsymbol{\psi}\,) = \mathbf{0}\ \ and\ \ \mathrm{Cov}(\,\widehat{\mathbf{K}}\mathbf{y} - \boldsymbol{\psi}\,) = \min \Leftrightarrow \widehat{\mathbf{K}}[\,\mathbf{X},\ \boldsymbol{\Sigma}\mathbf{X}^{\perp}\,] = [\,\mathbf{G},\ \mathbf{J}\mathbf{X}^{\perp}\,]. \tag{3.20}$$

The matrix equation in (3.20) *is solvable as well under* (3.19), *while the general forms of* $\widehat{\mathbf{K}}$ *and the corresponding* $\mathrm{BLUP}_{\mathscr{L}}(\boldsymbol{\psi})$ *can be written as*

$$\mathrm{BLUP}_{\mathscr{L}}(\boldsymbol{\psi}) = \widehat{\mathbf{K}}\mathbf{y} = \Big(\,[\,\mathbf{G},\ \mathbf{J}\mathbf{X}^{\perp}\,][\,\mathbf{X},\ \boldsymbol{\Sigma}\mathbf{X}^{\perp}\,]^{+} + \mathbf{U}[\,\mathbf{X},\ \boldsymbol{\Sigma}\mathbf{X}^{\perp}\,]^{\perp}\Big)\,\mathbf{y}, \tag{3.21}$$

where $\mathbf{U} \in \mathbb{R}^{k\times n}$ *is arbitrary. The corresponding* $g(\widehat{\mathbf{K}})$ *and* $g(\mathbf{K})$ *in* (3.16) *are given by*

$$g(\widehat{\mathbf{K}}) = \mathrm{Cov}[\,\mathrm{BLUP}_{\mathscr{L}}(\boldsymbol{\psi}) - \boldsymbol{\psi}\,] = \Big([\,\mathbf{G},\ \mathbf{J}\mathbf{X}^{\perp}\,][\,\mathbf{X},\ \boldsymbol{\Sigma}\mathbf{X}^{\perp}\,]^{+} - \mathbf{H}\Big)\,\boldsymbol{\Sigma}\,\Big([\,\mathbf{G},\ \mathbf{J}\mathbf{X}^{\perp}\,][\,\mathbf{X},\ \boldsymbol{\Sigma}\mathbf{X}^{\perp}\,]^{+} - \mathbf{H}\Big)', \tag{3.22}$$

$$g(\mathbf{K}) = g(\widehat{\mathbf{K}}) + (\mathbf{K}\boldsymbol{\Sigma} - \mathbf{H}\boldsymbol{\Sigma})\Big(\mathbf{X}^{\perp}\boldsymbol{\Sigma}\mathbf{X}^{\perp}\Big)^{+}(\mathbf{K}\boldsymbol{\Sigma} - \mathbf{H}\boldsymbol{\Sigma})' = g(\widehat{\mathbf{K}}) + (\mathbf{K}\boldsymbol{\Sigma} - \mathbf{J})\,\Big(\mathbf{X}^{\perp}\boldsymbol{\Sigma}\mathbf{X}^{\perp}\Big)^{+}(\mathbf{K}\boldsymbol{\Sigma} - \mathbf{J})'\,. \tag{3.23}$$

In particular, the parameter vectors $\boldsymbol{\psi}_i$ *in* (1.5) *is predictable by* $\mathbf{y}$ *in $\mathscr{L}$ if and only if*

$$\mathscr{R}(\mathbf{X}') \supseteq \mathscr{R}[(\mathbf{G}_i\mathbf{S}_i)'],\quad i = 1, 2. \tag{3.24}$$

In this case,

$$\mathrm{E}(\widehat{\mathbf{K}}_i\mathbf{y} - \boldsymbol{\psi}_i\,) = \mathbf{0}\ \ and\ \ \mathrm{Cov}(\,\widehat{\mathbf{K}}_i\mathbf{y} - \boldsymbol{\psi}_i\,) = \min \Leftrightarrow \widehat{\mathbf{K}}_i[\,\mathbf{X},\ \boldsymbol{\Sigma}\mathbf{X}^{\perp}\,] = [\,\mathbf{G}_i\mathbf{S}_i,\ \mathbf{J}_i\mathbf{X}^{\perp}\,],\quad i = 1, 2. \tag{3.25}$$

The matrix equation in (3.25) *is solvable as well under* (3.24), *while the general forms of* $\widehat{\mathbf{K}}_i$ *and the corresponding* $\mathrm{BLUP}_{\mathscr{L}}(\boldsymbol{\psi}_i)$ *can be written as*

$$\mathrm{BLUP}_{\mathscr{L}}(\boldsymbol{\psi}_i) = \widehat{\mathbf{K}}_i\mathbf{y} = \Big(\,[\,\mathbf{G}_i\mathbf{S}_i,\ \mathbf{J}_i\mathbf{X}^{\perp}\,][\,\mathbf{X},\ \boldsymbol{\Sigma}\mathbf{X}^{\perp}\,]^{+} + \mathbf{U}_i[\,\mathbf{X},\ \boldsymbol{\Sigma}\mathbf{X}^{\perp}\,]^{\perp}\Big)\,\mathbf{y}, \tag{3.26}$$

where $\mathbf{U}_i \in \mathbb{R}^{k_i \times n}$ *are arbitrary,* $i = 1, 2$. *The corresponding* $g_i(\widehat{\mathbf{K}}_i)$ *and* $g_i(\mathbf{K}_i)$ *in* (3.17) *are given by*

$$\begin{aligned} g_i(\widehat{\mathbf{K}}_i) &= \mathrm{Cov}[\,\mathrm{BLUP}_{\mathscr{L}}(\boldsymbol{\psi}_i) - \boldsymbol{\psi}_i\,] \\ &= \left([\,\mathbf{G}_i\mathbf{S}_i,\ \mathbf{J}_i\mathbf{X}^{\perp}\,][\,\mathbf{X},\ \boldsymbol{\Sigma}\mathbf{X}^{\perp}\,]^{+} - \mathbf{H}_i\mathbf{T}_i\right)\boldsymbol{\Sigma}\left([\,\mathbf{G}_i\mathbf{S}_i,\ \mathbf{J}_i\mathbf{X}^{\perp}\,][\,\mathbf{X},\ \boldsymbol{\Sigma}\mathbf{X}^{\perp}\,]^{+} - \mathbf{H}_i\mathbf{T}_i\right)', \end{aligned} \tag{3.27}$$

$$\begin{aligned} g_i(\mathbf{K}_i) &= g_i(\widehat{\mathbf{K}}_i) + (\mathbf{K}_i\boldsymbol{\Sigma} - \mathbf{H}_i\mathbf{T}_i\boldsymbol{\Sigma})\left(\mathbf{X}^{\perp}\boldsymbol{\Sigma}\mathbf{X}^{\perp}\right)^{+}(\mathbf{K}_i\boldsymbol{\Sigma} - \mathbf{H}_i\mathbf{T}_i\boldsymbol{\Sigma})' \\ &= g_i(\widehat{\mathbf{K}}_i) + (\mathbf{K}_i\boldsymbol{\Sigma} - \mathbf{J}_i)\left(\mathbf{X}^{\perp}\boldsymbol{\Sigma}\mathbf{X}^{\perp}\right)^{+}(\mathbf{K}_i\boldsymbol{\Sigma} - \mathbf{J}_i)' \end{aligned} \tag{3.28}$$

for $i = 1, 2$. *Further, the following results hold.*

(a) $r[\,\mathbf{X},\ \boldsymbol{\Sigma}\mathbf{X}^{\perp}\,] = r[\,\mathbf{X},\ \boldsymbol{\Sigma}\,]$, $\mathscr{R}[\,\mathbf{X},\ \boldsymbol{\Sigma}\mathbf{X}^{\perp}\,] = \mathscr{R}[\,\mathbf{X},\ \boldsymbol{\Sigma}\,]$, *and* $\mathscr{R}(\mathbf{X}) \cap \mathscr{R}(\boldsymbol{\Sigma}\mathbf{X}^{\perp}) = \{\mathbf{0}\}$.

(b) $\widehat{\mathbf{K}}$ *is unique if and only if* $r[\,\mathbf{X},\ \boldsymbol{\Sigma}\,] = n$.

(c) $\mathrm{BLUP}_{\mathscr{L}}(\boldsymbol{\psi})$ *is unique with probability* 1 *if and only if* $\mathbf{y} \in \mathscr{R}[\,\mathbf{X},\ \boldsymbol{\Sigma}\,]$ *holds with probability* 1.

(d) *The following covariance matrix formulas*

$$\begin{aligned} &\mathrm{Cov}[\mathrm{BLUP}_{\mathscr{L}}(\boldsymbol{\psi})] = \left([\,\mathbf{G},\ \mathbf{J}\mathbf{X}^{\perp}\,][\,\mathbf{X},\ \boldsymbol{\Sigma}\mathbf{X}^{\perp}\,]^{+}\right)\boldsymbol{\Sigma}\left([\,\mathbf{G},\ \mathbf{J}\mathbf{X}^{\perp}\,][\,\mathbf{X},\ \boldsymbol{\Sigma}\mathbf{X}^{\perp}\,]^{+}\right)', \\ &\mathrm{Cov}\{\mathrm{BLUP}_{\mathscr{L}}(\boldsymbol{\psi}),\ \boldsymbol{\psi}\} = [\,\mathbf{G},\ \mathbf{J}\mathbf{X}^{\perp}\,][\,\mathbf{X},\ \boldsymbol{\Sigma}\mathbf{X}^{\perp}\,]^{+}\mathbf{J}', \\ &\mathrm{Cov}(\boldsymbol{\psi}) - \mathrm{Cov}[\mathrm{BLUP}_{\mathscr{L}}(\boldsymbol{\psi})] = \mathbf{H}\boldsymbol{\Sigma}\mathbf{H}' - \left([\,\mathbf{G},\ \mathbf{J}\mathbf{X}^{\perp}\,][\,\mathbf{X},\ \boldsymbol{\Sigma}\mathbf{X}^{\perp}\,]^{+}\right)\boldsymbol{\Sigma}\left([\,\mathbf{G},\ \mathbf{J}\mathbf{X}^{\perp}\,][\,\mathbf{X},\ \boldsymbol{\Sigma}\mathbf{X}^{\perp}\,]^{+}\right)', \end{aligned}$$

and

$$\begin{aligned} \mathrm{Cov}[\mathrm{BLUP}_{\mathscr{L}}(\boldsymbol{\psi}_i)] &= \left([\,\mathbf{G}_i\mathbf{S}_i,\ \mathbf{J}_i\mathbf{X}^{\perp}\,][\,\mathbf{X},\ \boldsymbol{\Sigma}\mathbf{X}^{\perp}\,]^{+}\right)\boldsymbol{\Sigma}\left([\,\mathbf{G}_i\mathbf{S}_i,\ \mathbf{J}_i\mathbf{X}^{\perp}\,][\,\mathbf{X},\ \boldsymbol{\Sigma}\mathbf{X}^{\perp}\,]^{+}\right)', \\ \mathrm{Cov}\{\mathrm{BLUP}_{\mathscr{L}}(\boldsymbol{\psi}_i),\ \boldsymbol{\psi}_i\} &= [\,\mathbf{G}_i\mathbf{S}_i,\ \mathbf{J}_i\mathbf{X}^{\perp}\,][\,\mathbf{X},\ \boldsymbol{\Sigma}\mathbf{X}^{\perp}\,]^{+}\mathbf{J}_i', \\ \mathrm{Cov}(\boldsymbol{\psi}_i) - \mathrm{Cov}[\mathrm{BLUP}_{\mathscr{L}}(\boldsymbol{\psi}_i)] &= \mathbf{H}_i\mathbf{T}_i\boldsymbol{\Sigma}(\mathbf{H}_i\mathbf{T}_i)' \\ &\quad - \left([\,\mathbf{G}_i\mathbf{S}_i,\ \mathbf{J}_i\mathbf{X}^{\perp}\,][\,\mathbf{X},\ \boldsymbol{\Sigma}\mathbf{X}^{\perp}\,]^{+}\right)\boldsymbol{\Sigma}\left([\,\mathbf{G}_i\mathbf{S}_i,\ \mathbf{J}_i\mathbf{X}^{\perp}\,][\,\mathbf{X},\ \boldsymbol{\Sigma}\mathbf{X}^{\perp}\,]^{+}\right)' \end{aligned}$$

hold for $i = 1, 2$.

(e) *The BLUPs of* $\boldsymbol{\psi}$ *and* $\boldsymbol{\psi}_i$ *satisfy the following identities*

$$\begin{aligned} &\mathrm{BLUP}_{\mathscr{L}}(\boldsymbol{\psi}) = \mathrm{BLUE}_{\mathscr{L}}(\mathbf{G}\boldsymbol{\beta}) + \mathrm{BLUP}_{\mathscr{L}}(\mathbf{H}\boldsymbol{\varepsilon}), \\ &\mathrm{BLUP}_{\mathscr{L}}(\boldsymbol{\psi}_i) = \mathrm{BLUE}_{\mathscr{L}}(\mathbf{G}_i\boldsymbol{\beta}_i) + \mathrm{BLUP}_{\mathscr{L}}(\mathbf{H}_i\boldsymbol{\varepsilon}_i), \quad i = 1, 2. \end{aligned}$$

(f) $\mathbf{T}\boldsymbol{\psi}$ *is predictable under* $\mathscr{L}$, *then* $\mathrm{BLUP}_{\mathscr{L}}(\mathbf{T}\boldsymbol{\psi}) = \mathbf{T}\mathrm{BLUP}_{\mathscr{L}}(\boldsymbol{\psi})$ *holds for all matrices* $\mathbf{T} \in \mathbb{R}^{t \times k}$.

Proof. It is obvious from (3.15) that

$$\begin{aligned} &\mathrm{E}\,(\mathbf{K}\mathbf{y} - \boldsymbol{\psi}) = \mathbf{0} \Leftrightarrow \mathbf{K}\mathbf{X}\boldsymbol{\beta} - \mathbf{G}\boldsymbol{\beta} = \mathbf{0} \text{ for all } \boldsymbol{\beta} \Leftrightarrow \mathbf{K}\mathbf{X} = \mathbf{G}, \\ &\mathrm{E}\left(\mathbf{K}_i\mathbf{y} - \boldsymbol{\psi}_i\right) = \mathbf{0} \Leftrightarrow \mathbf{K}_i\mathbf{X}\boldsymbol{\beta} - \mathbf{G}_i\mathbf{S}_i\boldsymbol{\beta} = \mathbf{0} \text{ for all } \boldsymbol{\beta} \Leftrightarrow \mathbf{K}_i\mathbf{X} = \mathbf{G}_i\mathbf{S}_i, \ i = 1, 2. \end{aligned}$$

From Lemma 2.2, the matrix equations are solvable respectively if and only if (3.19) and (3.24) hold, respectively. In these cases, we see from Lemma 2.2 that the first parts of (3.20) and (3.25) are equivalent to finding solutions $\widehat{\mathbf{K}}$ of the solvable matrix equations $\widehat{\mathbf{K}}\mathbf{X} = \mathbf{G}$ and $\widehat{\mathbf{K}}_i$ of the solvable matrix equations $\widehat{\mathbf{K}}_i\mathbf{X} = \mathbf{G}_i\mathbf{S}_i$ such that

$$g(\mathbf{K}) \succcurlyeq g(\widehat{\mathbf{K}}) \ \text{ s.t. } \ \mathbf{K}\mathbf{X} = \mathbf{G}, \tag{3.29}$$

$$g_i(\mathbf{K}_i) \succcurlyeq g_i(\widehat{\mathbf{K}}_i) \ \text{ s.t. } \ \mathbf{K}_i\mathbf{X} = \mathbf{G}_i\mathbf{S}_i, \ \ i = 1, 2, \tag{3.30}$$

hold, respectively, in the Löwner partial ordering. Further from Lemma 2.3, there always exist solutions $\widehat{\mathbf{K}}$ of $\widehat{\mathbf{K}}\mathbf{X} = \mathbf{G}$ and $\widehat{\mathbf{K}}_i$ of $\widehat{\mathbf{K}}_i\mathbf{X} = \mathbf{G}_i\mathbf{S}_i$ such that (3.29) and (3.30) hold, respectively. Applying Lemma 2.3 to (3.29) and (3.30) leads to the conclusions in the theorem. □

4 How to establish decomposition identities between BLUPs under SURMs

The exact formulas of BLUPs and their analytical properties presented in Section 3 enable us to conduct many new and valuable statistical inference for SURMs via various matrix analysis tools. Especially through comparing the formulas of the BLUPs of the same unknown parameters under two different models, people can propose various types of equality between the BLUPs. Some previous and recent work on the equivalence of BLUPs under linear regression models can be found in [27–30]. In this section, we derive necessary and sufficient conditions for (1.13)–(1.14) to hold, and present some of their direct consequences.

Theorem 4.1. *Assume that* $\boldsymbol{\psi}_i$ *in* (1.5) *are predictable under* $\mathscr{L}_1$ *and* $\mathscr{L}_2$, *i.e.*, (3.5) *holds*, $i = 1, 2$. *Then, they are predictable under* $\mathscr{L}$ *as well. Also let* $\mathrm{BLUP}_{\mathscr{L}_i}(\boldsymbol{\psi}_i)$ *and* $\mathrm{BLUP}_{\mathscr{L}}(\boldsymbol{\psi}_i)$ *be as given in* (3.7) *and* (3.26), *respectively,* $i = 1, 2$. *Then, the following statements are equivalent*:

(a) $\mathrm{BLUP}_{\mathscr{L}}(\boldsymbol{\psi}_i) = \mathrm{BLUP}_{\mathscr{L}_i}(\boldsymbol{\psi}_i)$, $i = 1, 2$.

(b) $r\begin{bmatrix} \mathbf{X}_i & \mathrm{Cov}\{\mathbf{y}_i, \mathbf{y}\} \\ \mathbf{0} & \mathbf{X}' \\ \mathbf{G}_i & \mathrm{Cov}\{\boldsymbol{\psi}_i, \mathbf{y}\} \end{bmatrix} = r\begin{bmatrix} \mathbf{X}_i & \mathrm{Cov}\{\mathbf{y}_i, \mathbf{y}\} \\ \mathbf{0} & \mathbf{X}' \end{bmatrix}$, $i = 1, 2$.

(c) $r\begin{bmatrix} \mathbf{X}_i & \mathrm{Cov}\{\mathbf{y}_i, \mathbf{X}^{\perp}\mathbf{y}\} \\ \mathbf{G}_i & \mathrm{Cov}\{\boldsymbol{\psi}_i, \mathbf{X}^{\perp}\mathbf{y}\} \end{bmatrix} = r[\mathbf{X}_i, \mathrm{Cov}\{\mathbf{y}_i, \mathbf{X}^{\perp}\mathbf{y}\}]$, $i = 1, 2$.

(d) $\mathscr{R}\left([\mathbf{G}_i, \mathrm{Cov}\{\boldsymbol{\psi}_i, \mathbf{X}^{\perp}\mathbf{y}\}]'\right) \subseteq \mathscr{R}\left([\mathbf{X}_i, \mathrm{Cov}\{\mathbf{y}_i, \mathbf{X}^{\perp}\mathbf{y}\}]'\right)$, $i = 1, 2$.

Proof. If (a) holds, the coefficient matrices of $\mathrm{BLUP}_{\mathscr{L}_i}(\boldsymbol{\psi}_i)$ and $\mathrm{BLUP}_{\mathscr{L}}(\boldsymbol{\psi}_i)$ are the same, i.e., the coefficient matrices of $\mathrm{BLUP}_{\mathscr{L}_i}(\boldsymbol{\psi}_i)$ satisfy (3.25)

$$\left([\mathbf{G}_i, \mathbf{C}_i\mathbf{X}_i^{\perp}][\mathbf{X}_i, \boldsymbol{\Sigma}_{ii}\mathbf{X}_i^{\perp}]^{+} + \mathbf{U}_i[\mathbf{X}_i, \boldsymbol{\Sigma}_{ii}\mathbf{X}_i^{\perp}]^{\perp}\right)\mathbf{T}_i[\mathbf{X}, \boldsymbol{\Sigma}\mathbf{X}^{\perp}] = [\mathbf{G}_i\mathbf{S}_i, \mathbf{J}_i\mathbf{X}^{\perp}], \quad i = 1, 2. \tag{4.1}$$

Simplifying both sides by (2.1) and elementary block matrix operations, we obtain

$$\begin{aligned} r\left([\mathbf{X}_i, \boldsymbol{\Sigma}_{ii}\mathbf{X}_i^{\perp}]^{\perp}\mathbf{T}_i[\mathbf{X}, \boldsymbol{\Sigma}\mathbf{X}^{\perp}]\right) &= r\left([\mathbf{X}_i, \boldsymbol{\Sigma}_{ii}]^{\perp}\mathbf{T}_i[\mathbf{X}, \boldsymbol{\Sigma}]\right) \\ &= r[\mathbf{T}_i[\mathbf{X}, \boldsymbol{\Sigma}], \mathbf{X}_i, \boldsymbol{\Sigma}_{ii}] - r[\mathbf{X}_i, \boldsymbol{\Sigma}_{ii}] \\ &= r[[\mathbf{X}_i, \mathbf{0}], \mathbf{T}_i\boldsymbol{\Sigma}, \mathbf{X}_i, \boldsymbol{\Sigma}_{ii}] - r[\mathbf{X}_i, \boldsymbol{\Sigma}_{ii}] \\ &= r[\mathbf{X}_i, \boldsymbol{\Sigma}_{ii}] - r[\mathbf{X}_i, \boldsymbol{\Sigma}_{ii}] \\ &= 0, \end{aligned}$$

that is, $\mathscr{R}(\mathbf{T}_i[\mathbf{X}, \boldsymbol{\Sigma}\mathbf{X}^{\perp}]) \subseteq \mathscr{R}[\mathbf{X}_i, \boldsymbol{\Sigma}_{ii}\mathbf{X}_i^{\perp}]$. In this case, we obtain by (2.3) that

$$\begin{aligned} r\left\{[\mathbf{G}_i\mathbf{S}_i, \mathbf{J}_i\mathbf{X}^{\perp}] - [\mathbf{G}_i, \mathbf{C}_i\mathbf{X}_i^{\perp}][\mathbf{X}_i, \boldsymbol{\Sigma}_{ii}\mathbf{X}_i^{\perp}]^{+}\mathbf{T}_i[\mathbf{X}, \boldsymbol{\Sigma}\mathbf{X}^{\perp}]\right\} &= r\begin{bmatrix} [\mathbf{G}_i\mathbf{S}_i, \mathbf{J}_i\mathbf{X}^{\perp}] & [\mathbf{G}_i, \mathbf{C}_i\mathbf{X}_i^{\perp}] \\ [\mathbf{T}_i\mathbf{X}, \mathbf{T}_i\boldsymbol{\Sigma}\mathbf{X}^{\perp}] & [\mathbf{X}_i, \boldsymbol{\Sigma}_{ii}\mathbf{X}_i^{\perp}] \end{bmatrix} - r[\mathbf{X}_i, \boldsymbol{\Sigma}_{ii}] \\ &= r\begin{bmatrix} \mathbf{G}_i & \mathbf{J}_i\mathbf{X}^{\perp} & \mathbf{G}_i & \mathbf{C}_i\mathbf{X}_i^{\perp} \\ \mathbf{X}_i & \mathbf{T}_i\boldsymbol{\Sigma}\mathbf{X}^{\perp} & \mathbf{X}_i & \boldsymbol{\Sigma}_{ii}\mathbf{X}_i^{\perp} \end{bmatrix} - r[\mathbf{X}_i, \boldsymbol{\Sigma}_{ii}] \\ &= r\begin{bmatrix} \mathbf{G}_i & \mathbf{J}_i & \mathbf{C}_i \\ \mathbf{X}_i & \mathbf{T}_i\boldsymbol{\Sigma} & \boldsymbol{\Sigma}_{ii} \\ \mathbf{0} & \mathbf{X}' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{X}_i' \end{bmatrix} - r(\mathbf{X}) - r(\mathbf{X}_i) - r[\mathbf{X}_i, \boldsymbol{\Sigma}_{ii}] \ \text{(by (2.2))} \end{aligned}$$

$$
\begin{aligned}
&= r\begin{bmatrix} \mathbf{G}_i & \mathbf{H}_i\mathbf{T}_i\boldsymbol{\Sigma} & \mathbf{H}_i\mathbf{T}_i\boldsymbol{\Sigma}\mathbf{T}_i' \\ \mathbf{X}_i & \mathbf{T}_i\boldsymbol{\Sigma} & \mathbf{T}_i\boldsymbol{\Sigma}\mathbf{T}_i' \\ \mathbf{0} & \mathbf{X}' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{X}_i' \end{bmatrix} - r(\mathbf{X}) - r(\mathbf{X}_i) - r[\,\mathbf{X}_i,\ \boldsymbol{\Sigma}_{ii}\,] \\
&= r\begin{bmatrix} \mathbf{G}_i & \mathbf{H}_i\mathbf{T}_i\boldsymbol{\Sigma} & \mathbf{0} \\ \mathbf{X}_i & \mathbf{T}_i\boldsymbol{\Sigma} & \mathbf{0} \\ \mathbf{0} & \mathbf{X}' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{X}_i' \end{bmatrix} - r(\mathbf{X}) - r(\mathbf{X}_i) - r[\mathbf{X}_i,\ \boldsymbol{\Sigma}_{ii}] \\
&= r\begin{bmatrix} \mathbf{G}_i & \mathrm{Cov}\{\boldsymbol{\psi}_i, \mathbf{y}\} \\ \mathbf{X}_i & \mathrm{Cov}\{\mathbf{y}_i, \mathbf{y}\} \\ \mathbf{0} & \mathbf{X}' \end{bmatrix} - r(\mathbf{X}) - r[\,\mathbf{X}_i,\ \boldsymbol{\Sigma}_{ii}\,] \\
&= r\begin{bmatrix} \mathbf{G}_i & \mathrm{Cov}\{\boldsymbol{\psi}_i, \mathbf{X}^{\perp}\mathbf{y}\} \\ \mathbf{X}_i & \mathrm{Cov}\{\mathbf{y}_i, \mathbf{X}^{\perp}\mathbf{y}\} \end{bmatrix} - r[\,\mathbf{X}_i,\ \boldsymbol{\Sigma}_{ii}\,] \ \text{(by (2.2))}.
\end{aligned}
$$

Combining this equality with (4.1) leads to the equivalence of (a)–(c). The equivalence of (c) and (d) follows from Lemma 2.1 (b). □

The following results are direct consequences of Theorem 4.1.

Corollary 4.2. *Let* $\mathrm{BLUP}_{\mathscr{L}_i}(\boldsymbol{\psi}_i)$ *and* $\mathrm{BLUP}_{\mathscr{L}}(\boldsymbol{\psi}_i)$ *be as given in* (3.7) *and* (3.26), *respectively,* $i = 1, 2$. *Then, the following statements are equivalent*:

(a) $\mathrm{BLUE}_{\mathscr{L}}(\mathbf{X}_i\boldsymbol{\beta}_i) = \mathrm{BLUE}_{\mathscr{L}_i}(\mathbf{X}_i\boldsymbol{\beta}_i)$, $i = 1, 2$.

(b) $\mathrm{BLUP}_{\mathscr{L}}(\boldsymbol{\varepsilon}_i) = \mathrm{BLUP}_{\mathscr{L}_i}(\boldsymbol{\varepsilon}_i)$, $i = 1, 2$.

(c) $r\begin{bmatrix} \mathbf{X}_i & \mathrm{Cov}\{\mathbf{y}_i, \mathbf{y}\} \\ \mathbf{0} & \mathbf{X}' \\ \mathbf{0} & \mathrm{Cov}\{\boldsymbol{\varepsilon}_i, \mathbf{y}\} \end{bmatrix} = r\begin{bmatrix} \mathbf{X}_i & \mathrm{Cov}\{\mathbf{y}_i, \mathbf{y}\} \\ \mathbf{0} & \mathbf{X}' \end{bmatrix}$, $i = 1, 2$.

(d) $r\begin{bmatrix} \mathbf{X}_i & \mathrm{Cov}\{\mathbf{y}_i, \mathbf{X}^{\perp}\mathbf{y}\} \\ \mathbf{0} & \mathrm{Cov}\{\boldsymbol{\varepsilon}_i, \mathbf{X}^{\perp}\mathbf{y}\} \end{bmatrix} = r[\mathbf{X}_i, \mathrm{Cov}\{\mathbf{y}_i, \mathbf{X}^{\perp}\mathbf{y}\}]$, $i = 1, 2$.

(e) $\mathscr{R}\left([\mathbf{0}, \mathrm{Cov}\{\boldsymbol{\varepsilon}_i, \mathbf{X}^{\perp}\mathbf{y}\}]'\right) \subseteq \mathscr{R}\left([\mathbf{X}_i, \mathrm{Cov}\{\mathbf{y}_i, \mathbf{X}^{\perp}\mathbf{y}\}]'\right)$, $i = 1, 2$.

(f) $\mathscr{R}\left([\mathrm{Cov}\{\boldsymbol{\varepsilon}_i, \mathbf{X}^{\perp}\mathbf{y}\}]'\right) \subseteq \mathscr{R}\left([\mathrm{Cov}\{\mathbf{X}_i^{\perp}\mathbf{y}_i, \mathbf{X}^{\perp}\mathbf{y}\}]'\right)$, $i = 1, 2$.

Corollary 4.3. *The following statistical facts are equivalent*:

(a) $\mathrm{BLUP}_{\mathscr{L}}(\boldsymbol{\psi}) = \begin{bmatrix} \mathrm{BLUP}_{\mathscr{L}_1}(\boldsymbol{\psi}_1) \\ \mathrm{BLUP}_{\mathscr{L}_2}(\boldsymbol{\psi}_2) \end{bmatrix}$.

(b) $\mathrm{BLUP}_{\mathscr{L}}(\boldsymbol{\psi}_1) = \mathrm{BLUP}_{\mathscr{L}_1}(\boldsymbol{\psi}_1)$ *and* $\mathrm{BLUP}_{\mathscr{L}}(\boldsymbol{\psi}_2) = \mathrm{BLUP}_{\mathscr{L}_2}(\boldsymbol{\psi}_2)$.

Corollary 4.4. *The following statistical facts are equivalent*:

(a) $\mathrm{BLUE}_{\mathscr{L}}(\mathbf{X}\boldsymbol{\beta}) = \begin{bmatrix} \mathrm{BLUE}_{\mathscr{L}_1}(\mathbf{X}_1\boldsymbol{\beta}_1) \\ \mathrm{BLUE}_{\mathscr{L}_2}(\mathbf{X}_2\boldsymbol{\beta}_2) \end{bmatrix}$.

(b) $\mathrm{BLUP}_{\mathscr{L}}(\boldsymbol{\varepsilon}) = \begin{bmatrix} \mathrm{BLUP}_{\mathscr{L}_1}(\boldsymbol{\varepsilon}_1) \\ \mathrm{BLUP}_{\mathscr{L}_2}(\boldsymbol{\varepsilon}_2) \end{bmatrix}$.

(c) $\mathrm{BLUE}_{\mathscr{L}}(\mathbf{X}_1\boldsymbol{\beta}_1) = \mathrm{BLUE}_{\mathscr{L}_1}(\mathbf{X}_1\boldsymbol{\beta}_1)$ *and* $\mathrm{BLUE}_{\mathscr{L}}(\mathbf{X}_2\boldsymbol{\beta}_2) = \mathrm{BLUE}_{\mathscr{L}_2}(\mathbf{X}_2\boldsymbol{\beta}_2)$.

(d) $\mathrm{BLUP}_{\mathscr{L}}(\boldsymbol{\varepsilon}_1) = \mathrm{BLUP}_{\mathscr{L}_1}(\boldsymbol{\varepsilon}_1)$ *and* $\mathrm{BLUP}_{\mathscr{L}}(\boldsymbol{\varepsilon}_2) = \mathrm{BLUP}_{\mathscr{L}_2}(\boldsymbol{\varepsilon}_2)$.

Finally, we present a group of consequences for the covariance matrix $\boldsymbol{\Sigma}$ in (1.3) given by $\boldsymbol{\Sigma} = \mathrm{diag}(\sigma_1^2\mathbf{I}_{n_1},\ \sigma_2^2\mathbf{I}_{n_2})$, where σ_1^2 and σ_2^2 are unknown positive numbers. In this situation, (1.8) and (1.9) reduce to

$$\mathrm{Cov}(\boldsymbol{\psi}_i) = \sigma_i^2\mathbf{H}_i\mathbf{H}_i',\ \ \mathrm{Cov}\{\boldsymbol{\psi}_i,\ \mathbf{y}_i\} = \sigma_i^2\mathbf{H}_i,\ \ i = 1, 2,$$

$$\mathrm{Cov}(\boldsymbol{\psi}) = \begin{bmatrix}\sigma_1^2\mathbf{H}_1\mathbf{H}_1' & \mathbf{0}\\ \mathbf{0} & \sigma_2^2\mathbf{H}_2\mathbf{H}_2'\end{bmatrix},\ \ \mathrm{Cov}\{\boldsymbol{\psi},\ \mathbf{y}\} = \begin{bmatrix}\sigma_1^2\mathbf{H}_1 & \mathbf{0}\\ \mathbf{0} & \sigma_2^2\mathbf{H}_2\end{bmatrix}.$$

Corollary 4.5. *Let* $\mathscr{L}_1$ *and* $\mathscr{L}_2$ *be as given in* (1.1) *and* (1.2), *respectively, and assume that the parameter vectors* $\boldsymbol{\psi}_i$ *in* (1.5) *are predictable by* $\mathbf{y}_i$ *in* (1.1) *and* (1.2), *respectively. Then*

$$\mathrm{BLUP}_{\mathscr{L}_i}(\boldsymbol{\psi}_i) = [\,\mathbf{G}_i,\ \sigma_i^2\mathbf{H}_i\mathbf{X}_i^{\perp}\,][\,\mathbf{X}_i,\ \sigma_i^2\mathbf{X}_i^{\perp}\,]^{+}\mathbf{y}_i = (\mathbf{G}_i\mathbf{X}_i^{+} + \mathbf{H}_i\mathbf{X}_i^{\perp})\mathbf{y}_i,\ \ i = 1, 2.$$

Further, the following results hold.

(a) *The covariance matrices of* $\mathrm{BLUP}_{\mathscr{L}_i}(\boldsymbol{\psi}_i)$, *as well as the covariance matrices between* $\mathrm{BLUP}_{\mathscr{L}_i}(\boldsymbol{\psi}_i)$ *and* $\boldsymbol{\psi}_i$ *are unique, and satisfy the equalities*

$$\mathrm{Cov}[\mathrm{BLUP}_{\mathscr{L}_i}(\boldsymbol{\psi}_i)] = \sigma_i^2(\mathbf{G}_i\mathbf{X}_i^{+} + \mathbf{H}_i\mathbf{X}_i^{\perp})(\mathbf{G}_i\mathbf{X}_i^{+} + \mathbf{H}_i\mathbf{X}_i^{\perp})',$$
$$\mathrm{Cov}\{\mathrm{BLUP}_{\mathscr{L}_i}(\boldsymbol{\psi}_i),\ \boldsymbol{\psi}_i\} = \sigma_i^2(\mathbf{G}_i\mathbf{X}_i^{+} + \mathbf{H}_i\mathbf{X}_i^{\perp})\mathbf{H}_i',$$
$$\mathrm{Cov}(\boldsymbol{\psi}_i) - \mathrm{Cov}[\mathrm{BLUP}_{\mathscr{L}_i}(\boldsymbol{\psi}_i)] = \sigma_i^2\mathbf{H}_i\mathbf{H}_i' - \sigma_i^2(\mathbf{G}_i\mathbf{X}_i^{+} + \mathbf{H}_i\mathbf{X}_i^{\perp})(\mathbf{G}_i\mathbf{X}_i^{+} + \mathbf{H}_i\mathbf{X}_i^{\perp})'$$

for $i = 1,\ 2$.

(b) *The BLUPs of* $\boldsymbol{\psi}_i$ *can be decomposed as the sums*

$$\mathrm{BLUP}_{\mathscr{L}_i}(\boldsymbol{\psi}_i) = \mathrm{BLUE}_{\mathscr{L}_i}(\mathbf{G}_i\boldsymbol{\beta}_i) + \mathrm{BLUP}_{\mathscr{L}_i}(\mathbf{H}_i\boldsymbol{\varepsilon}_i),\ \ i = 1,\ 2.$$

(c) *If* $\boldsymbol{\psi}_1$ *and* $\boldsymbol{\psi}_2$ *are predictable under* (1.1) *and* (1.2), *respectively, then* $\mathbf{P}_1\boldsymbol{\psi}_1$ *and* $\mathbf{P}_2\boldsymbol{\psi}_2$ *are predictable under* (1.1) *and* (1.2), *respectively, and* $\mathrm{BLUP}_{\mathscr{L}_i}(\mathbf{P}_i\boldsymbol{\psi}_i) = \mathbf{P}_i\mathrm{BLUP}_{\mathscr{L}_i}(\boldsymbol{\psi}_i)$ *hold for any matrices* $\mathbf{P}_i \in \mathbb{R}^{t_i\times k_i}$, $i = 1,\ 2$.

Corollary 4.6. *Let* $\mathscr{L}$ *be as given in* (1.4), *and assume that* $\boldsymbol{\psi}$ *in* (1.6) *is predictable by* $\mathbf{y}$ *in* $\mathscr{L}$. *Then*

$$\mathrm{BLUP}_{\mathscr{L}}(\boldsymbol{\psi}) = \begin{bmatrix}\mathrm{BLUP}_{\mathscr{L}}(\boldsymbol{\psi}_1)\\ \mathrm{BLUP}_{\mathscr{L}}(\boldsymbol{\psi}_2)\end{bmatrix} = \begin{bmatrix}\mathrm{BLUP}_{\mathscr{L}_1}(\boldsymbol{\psi}_1)\\ \mathrm{BLUP}_{\mathscr{L}_2}(\boldsymbol{\psi}_2)\end{bmatrix} = \begin{bmatrix}(\mathbf{G}_1\mathbf{X}_1^{+} + \mathbf{H}_1\mathbf{X}_1^{\perp})\mathbf{y}_1\\ (\mathbf{G}_2\mathbf{X}_2^{+} + \mathbf{H}_2\mathbf{X}_2^{\perp})\mathbf{y}_2\end{bmatrix},$$
$$\mathrm{BLUP}_{\mathscr{L}}(\boldsymbol{\psi}) = \mathrm{BLUE}_{\mathscr{L}}(\mathbf{G}\boldsymbol{\beta}) + \mathrm{BLUP}_{\mathscr{L}}(\mathbf{H}\boldsymbol{\varepsilon}),$$
$$\mathrm{BLUP}_{\mathscr{L}}(\boldsymbol{\psi}_i) = \mathrm{BLUE}_{\mathscr{L}}(\mathbf{G}_i\boldsymbol{\beta}_i) + \mathrm{BLUP}_{\mathscr{L}}(\mathbf{H}_i\boldsymbol{\varepsilon}_i) = \mathrm{BLUE}_{\mathscr{L}_i}(\mathbf{G}_i\boldsymbol{\beta}_i) + \mathrm{BLUP}_{\mathscr{L}_i}(\mathbf{H}_i\boldsymbol{\varepsilon}_i),\ \ i = 1,\ 2.$$

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