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Manuel Norman*

A new way to represent functions as series

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Abstract: In this paper we will show a new way to represent functions as infinite series, finding some conditions under which a function is expandable with this method, and showing how it allows us to find the values of many interesting series. At the end, we will prove one of the main results of the paper, a Representation Theorem.

Keywords: Function series, Lagrange's mean value theorem, values of infinite series

MSC: Primary 26A06, 40A30; Secondary 41A58

1 Introduction

It has always been an interesting problem to find the sums of infinite series. There are many examples of series whose values are known: Taylor Series, Fourier Series, ... (see, for example, the list of references at the end of this manuscript).

Here our aim is to show a new method to expand functions as series. In the first sections, we present how a particular iteration of Lagrange's mean value theorem leads us to a new series expansion. Then, we find some conditions of expandability for some common functions. In the last section, we develop a more general theory, which gives us the values of other series.

2 A comparison with other well known series

Before introducing this new kind of expansion, we will analyse the conditions of expandability of Taylor series and Fourier series, comparing them with the new results presented here.

2.1 Taylor series

This kind of series allows us to represent a function f as an "infinite polynomial". More precisely, we call power series a series of this type:

$$\sum_{k=0}^{\infty} a_k (x - x_0)^k,$$

where $\{a_k\}_{k=0}^{\infty} \subset \mathbb{R}$, and x_0 is a given real value.

Let

$$L := \limsup_{n \to +\infty} |a_n|^{1/n}.$$

***Corresponding Author: Manuel Norman:** Scientific High School, 26837 Mulazzano, Italy,
E-mail: manuel.norman02@gmail.com

It is apparent that $L \geq 0$; the convergence radius of the power series is defined as $R := \frac{1}{L}$, using the convention $1/0 = \infty$ and $1/\infty = 0$ if either $L = 0$, or $L = \infty$. Then the power series converges $\forall x \in (x_0 - R, x_0 + R)$, whereas at the extrema of the interval, the convergence depends on the particular series considered.

We say that a function f is analytic in $I = (a, b)$ if $\forall x_0 \in I$ it can be written as a power series with center x_0 and a proper convergence radius $R > 0$. Such a power series is usually known as Taylor series. If a function is analytic, then it turns out that the coefficients a_k are directly linked to the derivative of order k of f evaluated at x_0, namely,

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k;$$

moreover, the series converges pointwise to $f(x)$ when x is in a suitable neighborhood of x_0 (a Taylor series with $x_0 = 0$ is also called a McLaurin series). It is also clear that, if f is analytic in I, then $f \in C^\infty(I)$.

2.2 Fourier series

Let $f : \mathbb{R} \to \mathbb{R}$ be a periodic function with period 2π; in such a case, under mild conditions on f, Fourier series allows us to associate an "infinite trigonometrical polynomial" to f. We can write

$$f(x) \sim \frac{1}{2}a_0 + \sum_{k=1}^{\infty}(a_k \cos(kx) + b_k \sin(kx)), \tag{2.1}$$

where at this stage $\sim$ simply means that the series is associated to the function f, since

$$a_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(kx)dx, \qquad \forall k \geq 0, \tag{2.2}$$

$$b_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(kx)dx, \qquad \forall k \geq 1. \tag{2.3}$$

Although we assumed that the period is 2π, the theory can be generalized to any period $T > 0$; see, for example, [9].

When dealing with Fourier series, we can consider different notions of convergence. Here we will consider only two of them. The first one is the norm convergence: let $L^2 = L^2(-\pi, \pi)$ denote the Hilbert Space of square-integrable functions over the interval $(-\pi, \pi)$. If $f \in L^2$, then

$$\|f - S_n f\|_2 = \Big(\int_0^{2\pi}(f(x) - S_n f(x))^2 dx\Big)^{\frac{1}{2}} \to 0 \tag{2.4}$$

when $n \to +\infty$, where $S_n f = \frac{1}{2}a_0 + \sum_{k=1}^{n}(a_k \cos(kx) + b_k \sin(kx))$.

On the other hand, we can consider the pointwise convergence of Fourier Series: suppose that f is a bounded function in $[0, 2\pi]$ and that this interval can be decomposed as a finite number of subintervals, such that in each of them f is continuous and differentiable. Suppose also that the limits of f and f' at the extrema of these subintervals are finite. Then, if we let $f(x_0^\pm)$ denote the right/left limit of the function at x_0, we have

$$\frac{1}{2}a_0 + \sum_{k=1}^{\infty}(a_k \cos(kx_0) + b_k \sin(kx_0)) = \frac{f(x_0^+) + f(x_0^-)}{2} \tag{2.5}$$

2.3 A new kind of expansion

The new kind of expansion we develop here requires two fundamental conditions: $f \in C^\infty([x_0 - 1, x_0 + 1])$ and the one in Proposition 4.1. We will analyse some special cases of this Proposition, which will lead us to many interesting results. Using this method, the function f can be represented with a series of products depending on the inverses of the derivatives $f^{(i)}$. Eventually, in Section 8, we develop a more general method, that relaxes the previous assumptions on f, and does not require the differentiability of f; as a matter of fact, in that case we can find a general representation for any real number x.

We can conclude that
- Taylor series require strong conditions on f, but ensure good convergence properties;
- Fourier series require weak conditions on f, but convergence needs to be understood in a proper sense;
- The first kind of series developed here requires strong conditions on f, namely $f \in C^\infty$, whereas the second one requires weaker conditions.

3 A particular case

We will start with an example related to a general smooth function in the interval $[2, 4]$. First of all, recall that a function f is smooth when it has derivatives of all orders. If a function is smooth in an interval $[a, b]$ we write $f \in C^\infty([a, b])$.
Let $f \in C^\infty([2, 4])$. Consider the interval (2,3); since f is continuous (also at the extrema 2 and 3) and differentiable in it, for Lagrange's mean value theorem we have:

$$f(3) - f(2) = f'(c_1) \tag{3.1}$$

for a point $c_1 \in (2, 3)$. For the same reason,

$$f(4) - f(3) = f'(c_2) \tag{3.2}$$

for a point $c_2 \in (3, 4)$. Lagrange's mean value theorem also says what these values are; in fact, c_1 and c_2 are the extreme point of

$$\omega_1(y) = f(y) - (f(2) + (f(3) - f(2))(y - 2),$$
$$\omega_2(y) = f(y) - (f(3) + (f(4) - f(3))(y - 3)).$$

For what we said, we now know that $f'(c_1) + f'(c_2) = f(4) - f(2)$. Now we want to find the difference between $f'(c_1)$ and $f'(c_2)$, so that, with a system, we can find their values. To do this, we use again Lagrange's theorem

$$\exists c_3 \in (c_1, c_2) : f'(c_2) - f'(c_1) = (c_2 - c_1)f''(c_3). \tag{3.3}$$

Putting in a system this equation with the one with the sum of f' at $c_{1,2}$ and solving, we obtain

$$f'(c_1) = \frac{1}{2}f(4) - \frac{1}{2}f(2) - \frac{c_2 - c_1}{2}f''(c_3) \tag{3.4}$$

and

$$f'(c_2) = \frac{1}{2}f(4) - \frac{1}{2}f(2) + \frac{c_2 - c_1}{2}f''(c_3). \tag{3.5}$$

Since $f(3) = f(2) + f'(c_1)$, we have that

$$f(3) = \frac{1}{2}f(4) + \frac{1}{2}f(2) - \frac{c_2 - c_1}{2}f''(c_3). \tag{3.6}$$

Now, always for the mean value theorem, we have that ($\forall h \geq 3$)

$$\exists c_h \in (c_1, c_{h-1}) : f^{(h-2)}(c_{h-1}) - f^{(h-2)}(c_1) = (c_{h-1} - c_1)f^{(h-1)}(c_h). \tag{3.7}$$

Knowing that $f''(c_3) - f''(c_1) = (c_3 - c_1)f^{(3)}(c_4)$ (obtained from the above formula with $h = 4$), we can put this value into (3.6) and get

$$f(3) = \frac{1}{2}f(2) + \frac{1}{2}f(4) - \frac{(c_3 - c_1)(c_2 - c_1)}{2}f^{(3)}(c_4) - \frac{c_2 - c_1}{2}f''(c_1). \tag{3.8}$$

Iterating this process, it is easy to prove the following.

Theorem 3.1. *Let c_1 and c_2 defined as above. Then, $\forall h \geq 3$, $\exists c_h \in (c_1, c_{h-1})$ such that*

$$f(3) = \frac{1}{2}f(2) + \frac{1}{2}f(4) - \frac{1}{2}f^{(h-1)}(c_h)\prod_{a=2}^{h-1}(c_a - c_1) - \frac{1}{2}\sum_{i=2}^{h-2}(f^{(i)}(c_1)\prod_{b=2}^{i}(c_b - c_1)).$$

If $h = 3$, we obtain (3.6).

Proof. The proof is by induction. We know that for $h = 3$ the formula holds true. Now, suppose that it holds true for a number h. We have to prove that it also holds for $h + 1$. Then, by induction, we can conclude that the formula holds true $\forall h \geq 3$. So, suppose that

$$f(3) = \frac{1}{2}f(2) + \frac{1}{2}f(4) - \frac{1}{2}f^{(h-1)}(c_h)\prod_{a=2}^{h-1}(c_a - c_1) - \frac{1}{2}\sum_{i=2}^{h-2}(f^{(i)}(c_1)\prod_{b=2}^{i}(c_b - c_1)).$$

Consider now this formula for $h + 1$:

$$f(3) = \frac{1}{2}f(2) + \frac{1}{2}f(4) - \frac{1}{2}f^{(h)}(c_{h+1})\prod_{a=2}^{h}(c_a - c_1) - \frac{1}{2}\sum_{i=2}^{h-1}(f^{(i)}(c_1)\prod_{b=2}^{i}(c_b - c_1)).$$

We know that $\forall h \geq 2$

$$\frac{f^{(h-1)}(c_h) - f^{(h-1)}(c_1)}{c_h - c_1} = f^{(h)}(c_{h+1}).$$

This can be written as

$$\begin{aligned}
&f^{(h)}(c_{h+1})\prod_{a=2}^{h}(c_a - c_1) + \sum_{i=2}^{h-2}(f^{(i)}(c_1)\prod_{b=2}^{i}(c_b - c_1)) + f^{(h-1)}(c_1)\prod_{b=2}^{h-1}(c_b - c_1)\\
&= f^{(h-1)}(c_h)\prod_{a=2}^{h-1}(c_a - c_1) + \sum_{i=2}^{h-2}(f^{(i)}(c_1)\prod_{b=2}^{i}(c_b - c_1))
\end{aligned}$$

and it implies that

$$\begin{aligned}
&f^{(h)}(c_{h+1})\prod_{a=2}^{h}(c_a - c_1) + \sum_{i=2}^{h-1}(f^{(i)}(c_1)\prod_{b=2}^{i}(c_b - c_1))\\
&= f^{(h-1)}(c_h)\prod_{a=2}^{h-1}(c_a - c_1) + \sum_{i=2}^{h-2}(f^{(i)}(c_1)\prod_{b=2}^{i}(c_b - c_1)).
\end{aligned}$$

Hence, we have that

$$\begin{aligned}
&\frac{1}{2}f(2) + \frac{1}{2}f(4) - \frac{1}{2}f^{(h-1)}(c_h)\prod_{a=2}^{h-1}(c_a - c_1) - \frac{1}{2}\sum_{i=2}^{h-2}(f^{(i)}(c_1)\prod_{b=2}^{i}(c_b - c_1))\\
&= \frac{1}{2}f(2) + \frac{1}{2}f(4) - \frac{1}{2}f^{(h)}(c_{h+1})\prod_{a=2}^{h}(c_a - c_1) - \frac{1}{2}\sum_{i=2}^{h-1}(f^{(i)}(c_1)\prod_{b=2}^{i}(c_b - c_1)).
\end{aligned}$$

Since the first expression equals $f(3)$, also the second is equal to $f(3)$, and therefore, we proved the theorem. □

Instead of considering the intervals $(2, 3)$ and $(3, 4)$, we can take $(x_0 - 1, x_0)$ and $(x_0, x_0 + 1)$ for any $x_0 \in \mathbb{R}$. In the same way as in Theorem 3.1, we then get the following result.

Theorem 3.2. *Let $f \in C^\infty([x_0-1, x_0+1])$. Let $c_1 \in (x_0-1, x_0)$, $c_2 \in (x_0, x_0+1)$ such that $f(x_0)-f(x_0-1) = f'(c_1)$, $f(x_0 + 1) - f(x_0) = f'(c_2)$. Then, $\forall h \geq 3$, $\exists c_h \in (c_1, c_{h-1})$ such that*

$$\begin{aligned}
f(x_0) =& \frac{1}{2}f(x_0 - 1) + \frac{1}{2}f(x_0 + 1)\\
&- \frac{1}{2}f^{(h-1)}(c_h)\prod_{a=2}^{h-1}(c_a - c_1) - \frac{1}{2}\sum_{i=2}^{h-2}(f^{(i)}(c_1)\prod_{b=2}^{i}(c_b - c_1)).
\end{aligned}$$

Proof. Analogous to the one of the previous Theorem. □

Notice that to find the values of all the c_j, we have to solve

$$f'(c_1) = f(x_0) - f(x_0 - 1),$$
$$f'(c_2) = f(x_0 + 1) - f(x_0),$$
$$\frac{f^{(h-2)}(c_{h-1}) - f^{(h-2)}(c_1)}{c_{h-1} - c_1} = f^{(h-1)}(c_h), \quad \forall h \geq 3.$$

4 Series associated to a function

To show the next results, we will use the following notation:

Definition 4.1. *Let c_j be the points defined before. For $j \geq 2$ we will let*

$$l(x_0, j) := c_j - c_1 \ \ (l \text{ depends on } j \text{ and } x_0 \text{ in general}),$$
$$n(x_0, j) := \prod_{b=2}^{j} l(x_0, b).$$

Now our aim is to take the limit for $h \to +\infty$ of the expansion in Theorem 3.2. More precisely, we want to understand under which conditions, we can say that

$$f(x_0) = \frac{1}{2}f(x_0 - 1) + \frac{1}{2}f(x_0 + 1) - \frac{1}{2}\sum_{i=2}^{+\infty}\left(f^{(i)}(c_1)\prod_{b=2}^{i}(c_b - c_1)\right). \tag{4.1}$$

In the following, when we say that a function is expandable, we mean that can be written as a series above.

Our aim is to find out when a function can be written as in (4.1), and we also want to determine the expansions of some of the principal functions of Analysis.

We will start with the following.

Proposition 4.1. *Let f be a function with the properties in Theorem 3.2. Then f is expandable if and only if $f^{(h-1)}(c_h)n(x_0, h-1) \to 0$ when $h \to +\infty$.*

Proof. From Theorem 3.2, we know that

$$\begin{aligned} f(x_0) = &\frac{1}{2}f(x_0 - 1) + \frac{1}{2}f(x_0 + 1) - \frac{1}{2}f^{(h-1)}(c_h)n(x_0, h-1) \\ &- \frac{1}{2}\sum_{i=2}^{h-2}\left(f^{(i)}(c_1)n(x_0, i)\right). \end{aligned}$$

If f is expandable, since for $h \to +\infty$ we have

$$\sum_{i=2}^{h-2}\left(f^{(i)}(c_1)n(x_0, i)\right) \to \sum_{i=2}^{+\infty}\left(f^{(i)}(c_1)n(x_0, i)\right),$$

in order to obtain (4.1), we must have $f^{(h-1)}(c_h)n(x_0, h-1) \to 0$. □

Equation (4.1) can be expressed also in this way.

$$\sum_{i=2}^{+\infty}\left(f^{(i)}(c_1)n(x_0, i)\right) = f(x_0 - 1) + f(x_0 + 1) - 2f(x_0). \tag{4.2}$$

This can be represented in another interesting way

Proposition 4.2. $\forall \sigma \geq 2, \sigma \in \mathbb{N}$,

$$\sum_{i=\sigma}^{+\infty}(f^{(i)}(c_1)n(x_0,i)) = n(x_0,\sigma)f^{(\sigma)}(c_{\sigma+1}).$$

Proof. First of all, notice that, since $f(x_0) - f(x_0-1) = f'(c_1)$ and $f(x_0+1) - f(x_0) = f'(c_2)$, we have that

$$f(x_0-1) + f(x_0+1) - 2f(x_0) = f'(c_2) - f'(c_1).$$

Furthermore,

$$n(x_0,2)f''(c_3) = (c_2-c_1)f''(c_3) = f'(c_2) - f'(c_1).$$

Hence

$$\sum_{i=2}^{+\infty}(f^{(i)}(c_1)n(x_0,i)) = n(x_0,2)f''(c_3).$$

Since

$$\sum_{i=2}^{+\infty}(f^{(i)}(c_1)n(x_0,i)) = f''(c_1)n(x_0,2) + \sum_{i=3}^{+\infty}(f^{(i)}(c_1)n(x_0,i)) = n(x_0,2)f''(c_3),$$

we can write

$$\sum_{i=3}^{+\infty}(f^{(i)}(c_1)n(x_0,i)) = (f''(c_3) - f''(c_1))n(x_0,2).$$

From the formulae that give c_j, we know that $f''(c_3) - f''(c_1) = f'''(c_4)l(x_0,3)$. Therefore,

$$\begin{aligned}\sum_{i=3}^{+\infty}(f^{(i)}(c_1)n(x_0,i)) &= (f''(c_3) - f''(c_1))n(x_0,2)\\ &= f'''(c_4)l(x_0,3)n(x_0,2)\\ &= f'''(c_4)n(x_0,3).\end{aligned}$$

Iterating this, we obtain the above theorem. □

5 Expandability of a function

We now want to find some theorems that can be easily used to determine when a given function f can be expanded in series. To prove the first one, we need the following:

Lemma 5.1. *If* $l(x_0,i_0) \leq 1$ *for a certain* $i_0 \in \mathbb{N}$, $i_0 \geq 2$, *then* $l(x_0,i_0) \leq 1\ \forall i \geq i_0$.

Proof. First of all, notice that $l(x_0,i) > 0$ because $c_i > c_1$ $(i > 1)$. Furthermore: $l(x_0,i+1) < l(x_0,i)\ \forall i \geq 2$, since $c_{i+1} < c_i$. Generalizing this, we can say that $c_{i+j} < c_i$, $\forall i \geq 2$, $\forall j \geq 1$. So, if $l(x_0,i_0) \leq 1$, this inequality holds true $\forall i \geq i_0$. □

Theorem 5.1. *Let* $f \in C^\infty([x_0-1,x_0+1])$. *Then, if there exists an index* $i_0 \geq 2$ *such that* $l(x_0,i_0) \leq 1$ *and,* $\forall x \in (x_0-1,x_0+1)$, $|f^{(i)}(x)| \to 0$ *when* $i \to +\infty$, *the function is expandable.*

Proof. We have to show that $f^{(h-1)}(c_h)n(x_0,h-1) \to 0$ when $h \to +\infty$. This is equivalent to prove that $|f^{(h-1)}(c_h)n(x_0,h-1)| \to 0$.

Since $|f^{(i)}(x)| \to 0\ \forall x \in (x_0-1,x_0+1)$, and $c_h \in (x_0-1,x_0+1)$, we obtain $|f^{(h-1)}(c_h)| \to 0$; we have to show that $n(x_0,h-1) \nrightarrow \infty$, so that the product goes to 0.

Since $l(x_0,i_0) \leq 1$, by Lemma 5.1 $l(x_0,i) \leq 1\ \forall i \geq i_0$. Hence,

$$n(x_0,h-1) = \prod_{b=2}^{h-1} l(x_0,b) = \prod_{b=2}^{i_0-1} l(x_0,b)\prod_{b=i_0}^{h-1} l(x_0,b) \leq \prod_{b=2}^{i_0-1} l(x_0,b) \nrightarrow \infty$$

because it is constant. If $i_0 = 2$, consider $\prod_{b=2}^{i_0-1} l(x_0, b) = 1$. □

Theorem 5.2. *Let $f \in C^\infty([x_0 - 1, x_0 + 1])$. If $|f^{(i)}(x)| \le M \in \mathbb{R}_+$ $\forall x \in (x_0 - 1, x_0 + 1)$ and $\forall i \in \mathbb{N}$, and if $n(x_0, i) \to 0$ when $i \to +\infty$, then f is expandable.*

Proof. We have to show that $f^{(h-1)}(c_h)n(x_0, h-1) \to 0$ when $h \to +\infty$. Since $|f^{(h-1)}(c_h)n(x_0, h-1)| \le Mn(x_0, h-1)$ by hypothesis, and $n(x_0, i) \to 0$, the product goes to 0 and we have finished. □

We would like to have also some conditions under which $n(x_0, i) \to 0$, since in general we cannot write a closed formula for this quantity. In order to do this, we need a theorem about infinite products.

Theorem 5.3. $\sum_{n=1}^{+\infty} \log(a_n) = -\infty \Leftrightarrow \prod_{n=1}^{+\infty} a_n = 0$, *where $a_n > 0$ $\forall n > 0$.*

Proof. For details, see [3].

$$\log\Big(\prod_{n=1}^{+\infty} a_n\Big) = \sum_{n=1}^{+\infty} \log(a_n).$$

If the sum is $-\infty$, $\log(\prod_{n=1}^{+\infty} a_n) \to -\infty$ and this happens when the argument $\prod_{n=1}^{+\infty} a_n \to 0$. Vice versa, if $\prod_{n=1}^{+\infty} a_n \to 0$, we have that $\log(\prod_{n=1}^{+\infty} a_n) \to -\infty$ and so the sum tends to $-\infty$. □

Theorem 5.4. *$n(x_0, i) \to 0$ when $i \to +\infty \Leftrightarrow \sum_{n=2}^{+\infty} \log(l(x_0, n)) = -\infty$.*

Proof. Let $a_n = l(x_0, n + 1)$. Applying Theorem 5.3, we have that

$$\prod_{n=1}^{+\infty} l(x_0, n + 1) = 0 \ \ \text{(that is the limit of } n(x_0, i)\text{)},$$

if and only if

$$\sum_{n=1}^{+\infty} \log(l(x_0, n + 1)) = -\infty.$$

Writing

$$\sum_{n=1}^{+\infty} \log(l(x_0, n + 1)) = \sum_{n=2}^{+\infty} \log(l(x_0, n)),$$

we have proved the theorem. □

Theorem 5.5. *If there exists $n_0 \ge 2$ such that $l(x_0, n_0) < \frac{1}{e}$, then $n(x_0, i) \to 0$ when $i \to +\infty$.*

Proof. If $l(x_0, n_0) < \frac{1}{e}$, $l(x_0, n) < \frac{1}{e}$ $\forall n \ge n_0$ (for the same reason why Lemma 5.1 holds true). Hence, $\log(l(x_0, n)) < \log(\frac{1}{e}) = -1$. Thus,

$$\begin{aligned}\sum_{n=2}^{+\infty} \log(l(x_0, n)) &= \sum_{n=2}^{n_0-1} \log(l(x_0, n)) + \sum_{n=n_0}^{+\infty} \log(l(x_0, n)) \\ &< \sum_{n=2}^{n_0-1} \log(l(x_0, n)) + \sum_{n=n_0}^{+\infty} (-1) = -\infty.\end{aligned}$$

Therefore, $\sum_{n=2}^{+\infty} \log(l(x_0, n)) = -\infty$. Applying Theorem 5.4, we conclude. □

We end with this important theorem.

Theorem 5.6. *Let $f \in C^\infty([x_0 - 1, x_0 + 1])$ such that $\forall x \in (x_0 - 1, x_0 + 1)$ $|f^{(i)}(x)| \le M \in \mathbb{R}_+$. Then, if there exists a $j \ge 2$ such that $l(x_0, j) < \frac{1}{e}$, f is expandable.*

Proof. We have to prove that $f^{(h-1)}(c_h)n(x_0, h-1) \to 0$. We have

$$|f^{(h-1)}(c_h)n(x_0, h-1)| \leq Mn(x_0, h-1) \to 0,$$

because under the said hypothesis $n(x_0, h-1) \to 0$ by Theorem 5.5. □

6 Expansions of some important functions

In this section we will show the expansions of some functions.

Example 6.1. Let $f(x) = e^{ax}$, $a \in \mathbb{R}$, $a \in (-1, 1)$, $a \neq 0$. $|f^{(i)}(x)| = |a|^i e^{ax} \to 0$. We want to find some of the c_h. To do this, we have to solve

$$\begin{aligned} & f'(c_1) = f(x_0) - f(x_0 - 1) \\ \Rightarrow\ & ae^{ac_1} = e^{a(x_0)} - e^{a(x_0-1)} \\ \Rightarrow\ & ac_1 = \ln\left(\frac{e^{a(x_0)} - e^{a(x_0-1)}}{a}\right) \\ \Rightarrow\ & c_1 = \frac{1}{a}\ln\left(\frac{e^{a(x_0)} - e^{a(x_0-1)}}{a}\right); \end{aligned}$$

$$\begin{aligned} & f'(c_2) = f(x_0 + 1) - f(x_0) \\ \Rightarrow\ & ae^{ac_2} = e^{a(x_0+1)} - e^{a(x_0)} \\ \Rightarrow\ & ac_2 = \ln\left(\frac{e^{a(x_0+1)} - e^{a(x_0)}}{a}\right) \\ \Rightarrow\ & c_2 = \frac{1}{a}\ln\left(\frac{e^{a(x_0+1)} - e^{a(x_0)}}{a}\right); \end{aligned}$$

$$\begin{aligned} & f''(c_3) = \frac{f'(c_2) - f'(c_1)}{c_2 - c_1} = \frac{f(x_0 + 1) - 2f(x_0) + f(x_0 - 1)}{c_2 - c_1} \\ \Rightarrow\ & a^2 e^{ac_3} = e^{a(x_0-1)}(e^2 - 2e + 1) \\ \Rightarrow\ & c_3 = \frac{1}{a}\ln\left(\frac{e^{a(x_0-1)}(e-1)^2}{a^2}\right) \ \text{(noticing that } l(x_0, 2) = 1\text{)} \end{aligned}$$

$$f'''(c_4) = a^3 e^{ac_4} = \frac{f''(c_3) - f''(c_1)}{c_3 - c_1}$$

$$\text{(after some calculations ...)} \ \Rightarrow c_4 = \frac{1}{a}\ln\left(\frac{e^{a(x_0-1)}(e^a - 1)(e^a - 1 - a)}{a^2 \ln(\frac{e^a-1}{a})}\right).$$

As noticed before, $l(x_0, 2) = 1$ and $|f^{(i)}(x)| \to 0\ \forall a \in (-1, 1)$, $a \neq 0$, so by Theorem 5.1, f is expandable for these values of a. To write its series, notice that

$$l(x_0, j-1) = \frac{f^{(j-2)}(c_{j-1}) - f^{(j-2)}(c_1)}{f^{(j-1)}(c_j)} = \frac{1}{a}(e^{a(c_{j-1}-c_j)} - e^{a(c_1-c_j)}),$$

and so

$$n(x_0, i) = \prod_{j=2}^{i} l(x_0, j) = \frac{1}{a^{i-1}}\prod_{j=2}^{i}(e^{a(c_{j-1}-c_j)} - e^{a(c_1-c_j)}).$$

Hence, we have that

$$\sum_{i=2}^{+\infty} f^{(i)}(c_1)n(x_0, i) = f(x_0 - 1) + f(x_0 + 1) - 2f(x_0),$$

which implies

$$\sum_{i=2}^{+\infty}\left[a^i e^{ac_1}\frac{1}{a^{i-1}}\prod_{j=2}^{i}(e^{a(c_{j-1}-c_j)}-e^{a(c_1-c_j)})\right]=e^{a(x_0-1)}(e^{2a}-2e^a+1),$$

and this in turn yields

$$\sum_{i=2}^{+\infty}\left(e^{a(x_0-1)}(e^a-1)\prod_{j=2}^{i}(e^{a(c_{j-1}-c_j)}-e^{a(c_1-c_j)})\right)=e^{a(x_0-1)}(e^a-1)^2,$$

since $ae^{ac_1}=e^{a(x_0-1)}(e^a-1)$. This becomes

$$e^a=1+\sum_{i=2}^{+\infty}\left(\prod_{j=2}^{i}\frac{e^{ac_{j-1}}-e^{ac_1}}{e^{ac_j}}\right). \tag{6.1}$$

Writing some terms, we have

$$e^a=1+a+a\ln\left(\frac{e^a-1}{a}\right)+a\ln\left(\frac{e^a-1}{a}\right)\ln\left(\frac{e^a-1-a}{a\ln(\frac{e^a-1}{a})}\right)+\dots \tag{6.2}$$

Now, notice that when $a=\pm1$, $|f^{(i)}(x)|=e^{\pm x}\le M\in\mathbb{R}_+$ since it is independent of i. We wonder, whether the function is expandable also for these values, or not. We know that the derivatives are limited, so we just have to verify that $\exists j: l(x_0,j)<\frac{1}{e}$ in both cases. Take, for instance, $j=4$; we have

$$l(x_0,4)=\frac{1}{a}\ln\left(\frac{e^a-1-a}{a\ln(\frac{e^a-1}{a})}\right)\approx\begin{cases}0.2828424, & \text{if } a=1\\ 0.2205869, & \text{if } a=-1\end{cases}.$$

Since both these values are $<\frac{1}{e}$, we can expand f. We get a new interesting representation of e, i.e.

$$\begin{aligned}e&=1+\sum_{i=2}^{+\infty}\left(\prod_{j=2}^{i}\frac{e^{c_{j-1}}-e^{c_1}}{e^{c_j}}\right)\\&=2+\ln(e-1)+\ln(e-1)\ln\left(\frac{e-2}{\ln(e-1)}\right)+\dots\end{aligned} \tag{6.3}$$

□

Example 6.2. Let $f(x)=\sin(\omega x)$, $\omega\in(0,1)$. In this interval, we have that $|f^{(i)}(x)|\le|\omega|^i\to0$ when $i\to+\infty$. To know when f is expandable and to write its series, we evaluate some points c_j; notice that we can choose any x_0; here, for the sake of simplicity, we will just consider the particular case $x_0=1$. We have

$$\begin{aligned}&f'(c_1)=\omega\cos(\omega c_1)=f(1)-f(0)=\sin\omega\\ \Rightarrow\ & c_1=\frac{1}{\omega}(2p_1\pi\pm\arccos(\frac{\sin\omega}{\omega}));\\ &f'(c_2)=\omega\cos(\omega c_2)=f(2)-f(1)=\sin(2\omega)-\sin\omega\\ \Rightarrow\ & c_2=\frac{1}{\omega}(2p_2\pi\pm\arccos(\frac{\sin(2\omega)-\sin\omega}{\omega}))\end{aligned}$$

where $p_1,p_2\in\mathbb{Z}$ since $\cos y=t$ for $0<t\le1$ yields $y=2p\pi\pm\arccos t$ for some $p\in\mathbb{Z}$, and $c_1\in(0,1)$, $c_2\in(1,2)$ because of Lagrange's Theorem. So we want to find $p_1,p_2\in\mathbb{Z}$ such that $c_1\in(0,1)$ and $c_2\in(1,2)$. We claim that they are both equal to 0, and that we have to choose the sign +. To prove this, we have to verify that

$$\begin{aligned}0&<\frac{1}{\omega}\arccos(\frac{\sin\omega}{\omega})<1,\\ 1&<\frac{1}{\omega}\arccos(\frac{\sin(2\omega)-\sin\omega}{\omega})<2.\end{aligned}$$

First of all, notice that the function $h(\omega) := \sin\omega - \omega\cos\omega$ is strictly increasing in $(0, 1)$, and $h(0) = 0$; therefore, $\sin\omega - \omega\cos\omega > 0$ in $(0, 1)$ and $\frac{\sin\omega}{\omega} > \cos\omega$. Since arccos is a decreasing function in its domain of definition, we have $\arccos(\frac{\sin\omega}{\omega}) < \omega$. Noting also that $\arccos y = 0$ only when $y = 1$, and that $\frac{\sin\omega}{\omega} \neq 1$ when $\omega \in (0, 1)$, we can conclude that $0 < \frac{1}{\omega}\arccos(\frac{\sin\omega}{\omega}) < 1$. In a similar way we can prove the other inequality.

We can actually verify that when $\omega \in (0, 1)$

$$\arccos(\sin 1) < c_1 < \frac{1}{\sqrt{3}}$$

and

$$\arccos(\sin 2 - \sin 1) < c_2 < \sqrt{\frac{7}{3}}.$$

Indeed, both $g_1(\omega) := \frac{1}{\omega}\arccos(\frac{\sin\omega}{\omega})$ and $g_2(\omega) := \frac{1}{\omega}\arccos(\frac{\sin(2\omega)-\sin\omega}{\omega})$ are decreasing functions in $(0, 1)$, and therefore,

$$\inf_{(0,1)} c_1 = \arccos(\sin 1), \quad \sup_{(0,1)} c_1 = \lim_{\omega\to 0^+} c_1 = \frac{1}{\sqrt{3}},$$

$$\inf_{(0,1)} c_2 = \arccos(\sin 2 - \sin 1), \quad \sup_{(0,1)} c_2 = \lim_{\omega\to 0^+} c_2 = \sqrt{\frac{7}{3}}.$$

Since $\sup_{(0,1)} c_2 - \inf_{(0,1)} c_1 \approx 0.9567289\ldots < 1$, we certainly have $l(1, 2) = c_2 - c_1 < 1$ for $\omega \in (0, 1)$, and we conclude that $\forall\omega \in (0, 1)$ we can expand f.

By (4.1), we can write the series

$$\sin\omega = \frac{1}{2}\sin(2\omega) - \frac{1}{2}\sum_{i=2}^{+\infty}\left(f^{(i)}\left(\frac{1}{\omega}\arccos\left(\frac{\sin\omega}{\omega}\right)\right)\prod_{b=2}^{i}(c_b - c_1)\right). \tag{6.4}$$

We can write some terms, namely

$$\begin{aligned}\sin\omega =& \frac{1}{2}\sin(2\omega)\\ &-\frac{1}{2}(-\omega\sin(\arccos(\frac{\sin\omega}{\omega}))\left(\arccos(\frac{\sin(2\omega)-\sin\omega}{\omega})\right.\\ &\left.-\arccos(\frac{\sin\omega}{\omega})\right) + \ldots\end{aligned} \tag{6.5}$$

We now wonder, whether or not f is expandable for $\omega = 1$. To answer this question, we have to find a $l(1, j) < \frac{1}{e}$. We can easily verify that, for this value of ω, $l(1, 4) \approx 0.21119234 < \frac{1}{e}$, so we can expand f $\forall\omega \in (0, 1]$. If we want to expand f for an argument in $[-1, 0)$, we can just remark that

$$\sin(-\omega) = -\frac{1}{2}\sin(2\omega) + \frac{1}{2}\sum_{i=2}^{+\infty}\left(f^{(i)}\left(\frac{1}{\omega}\arccos\left(\frac{\sin\omega}{\omega}\right)\right)\prod_{b=2}^{i}(c_b - c_1)\right), \tag{6.6}$$

since $\sin\omega = -\sin(-\omega)$ for any ω.

Furthermore, the following fact is also interesting: consider $\omega = \frac{\pi}{4} \in (0, 1]$; we have $\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$. Hence, we can write

$$\frac{1}{\sqrt{2}} = \frac{1}{2}\sin\frac{\pi}{2} - \frac{1}{2}\sum_{i=2}^{+\infty}\left(f^{(i)}\left(\frac{4}{\pi}\arccos\left(\frac{\frac{1}{\sqrt{2}}}{\frac{\pi}{4}}\right)\right)\prod_{b=2}^{i}(c_b - c_1)\right).$$

After some algebraic manipulations, we eventually obtain

$$\sqrt{2} = 1 - \sum_{i=2}^{+\infty}\left(f^{(i)}\left(\frac{4}{\pi}\arccos\left(\frac{2\sqrt{2}}{\pi}\right)\right)\prod_{b=2}^{i}(c_b - c_1)\right). \tag{6.7}$$

□

The next example is about the expansion of composite functions. We can expand these functions in the same way we did before, but we would then obtain for the c_j equations that are not solvable with "standard" methods, and so we would have to use approximation methods to find these values. Hence, it is sometimes better to expand these functions as shown below.

Example 6.3. Let $f(x) = e^{\sin x}$. We could expand f as usual, but we would not have the exact values of the c_j. In this case it can be better to consider $a = \sin(x)$ and expand first e^a. Since, by Example 6.1, $\forall a \in [-1, 1]$, $a \neq 0$,

$$e^a = 1 + \sum_{i=2}^{+\infty} \left(\prod_{j=2}^{i} \frac{e^{ac_{j-1}} - e^{ac_1}}{e^{ac_j}} \right),$$

we just have to put $a = \sin x \in [-1, 1]$, $\sin x \neq 0$. Since $\sin x \in [-1, 1]$ holds $\forall x \in \mathbb{R}$, we just have to exclude those values of x for which $\sin x = 0$. This means that we can expand f $\forall x \neq t\pi$, $t \in \mathbb{Z}$. The expansion is obtained by putting $\sin x$ instead of a in the expansion of e^a.

$$\begin{aligned} e^{\sin x} &= 1 + \sum_{i=2}^{+\infty} \left(\prod_{j=2}^{i} \frac{e^{c_{j-1}\sin x} - e^{c_1 \sin x}}{e^{c_j \sin x}} \right) \\ &= 1 + \sin x + \sin x \ln\left(\frac{e^{\sin x} - 1}{\sin x} \right) \\ &\quad + \sin x \ln\left(\frac{e^{\sin x} - 1}{\sin x} \right) \ln\left(\frac{e^{\sin x} - 1 - \sin x}{\sin x \ln\left(\frac{e^{\sin x} - 1}{\sin x} \right)} \right) + \ldots \end{aligned}$$

We could also expand all the terms with $\sin x$ in this equation with the series of Example 6.2, if we wanted. □

7 Approximation of a function with a finite sum and error term

We can now expand a function f according to a new type of series. An important question is: if we consider a finite sum instead of the series, what is the error due to the approximation? To answer this question, we can use the formula

$$\begin{aligned} f(x_0) =& \frac{1}{2} f(x_0 - 1) + \frac{1}{2} f(x_0 + 1) \\ & - \frac{1}{2} f^{(h-1)}(c_h) \prod_{a=2}^{h-1} (c_a - c_1) - \frac{1}{2} \sum_{i=2}^{h-2} (f^{(i)}(c_1) \prod_{b=2}^{i} (c_b - c_1)). \end{aligned}$$

What we want to do is to write

$$f(x_0) = \frac{1}{2} f(x_0 - 1) + \frac{1}{2} f(x_0 + 1) - \frac{1}{2} \sum_{i=2}^{h-2} (f^{(i)}(c_1) \prod_{b=2}^{i} (c_b - c_1)) + E_r(x_0, h).$$

It is easy to see, looking at these two formulae, that

$$E_r(x_0, h) = -\frac{1}{2} f^{(h-1)}(c_h) \prod_{a=2}^{h-1} (c_a - c_1).$$

If $E_r(x_0, h) \to 0$ when $h \to +\infty$, then we can write the series expansion of the function. Notice that, if

$$\begin{aligned} & |f^{(i)}(x)| \leq M_1(x, i) \ \forall x \in (x_0 - 1, x_0 + 1), \ \forall i \geq 1, \\ \wedge \ & |n(x_0, i)| \leq M_2(x_0, i) \ \forall i \geq 2, \end{aligned} \tag{7.1}$$

we have that

$$|E_r(x_0, i)| \leq M_1(x, i - 1) M_2(x_0, i - 1). \tag{7.2}$$

If $M_1(x, i) M_2(x_0, i) \to 0$ when $i \to +\infty$, the function can certainly be expanded as a series (the vice versa is not necessarily true, because we have inequalities) as said before.

8 A more general method of expansion

We now consider a more general case. Let

$$\frac{f(x_0)-f(x_0-1)}{g(x_0)-g(x_0-1)}=\frac{s_1(c_1)}{p_1(c_1)}, \tag{8.1}$$

$$\frac{f(x_0+1)-f(x_0)}{g(x_0+1)-g(x_0)}=\frac{s_1(c_2)}{p_1(c_2)}, \tag{8.2}$$

$$\forall j\geq 3, \qquad \frac{s_{j-2}(c_{j-1})-s_{j-2}(c_1)}{p_{j-2}(c_{j-1})-p_{j-2}(c_1)}=\frac{s_{j-1}(c_j)}{p_{j-1}(c_j)}, \tag{8.3}$$

where $f,\ g\in C^0(I)$, $\forall j\geq 1$, s_j, p_j are continuous functions on $I\subseteq\mathbb{R}$, a closed interval containing all the points we need, and

$$\begin{aligned} g(x_0)-g(x_0-1)&\neq 0,\\ g(x_0+1)-g(x_0)&\neq 0,\\ \forall j\geq 3, \qquad p_{j-2}(c_{j-1})-p_{j-2}(c_1)&\neq 0. \end{aligned}$$

Working in the same way as in the proof of Theorem 3.2, by induction on h we get

Theorem 8.1. *Let $x_0\in\mathbb{R}$ and $I\subseteq\mathbb{R}$ a closed interval such that $[x_0-1,x_0+1]\subseteq I$, assume that $f,\ g\in C^0(I)$, and $\forall j\geq 1$ let s_j, p_j be continuous functions on I too. Suppose that all the operations below are possible, namely that the quantities at the denominator never vanish. Then, $\forall h\geq 3$*

$$\begin{aligned} f(x_0)=&\frac{\Delta_{1,1}}{\Delta_{1,1}+\Delta_{1,2}\frac{p_1(c_1)}{p_1(c_2)}}f(x_0+1)+\Big(1-\frac{\Delta_{1,1}}{\Delta_{1,1}+\Delta_{1,2}\frac{p_1(c_1)}{p_1(c_2)}}\Big)f(x_0-1)\\ &-\frac{s_{h-1}(c_h)\Delta_{1,1}\Delta_{1,2}}{\Delta_{1,1}p_1(c_2)+\Delta_{1,2}p_1(c_1)}n(x_0,h-1)\\ &-\frac{\Delta_{1,1}\Delta_{1,2}}{\Delta_{1,1}p_1(c_2)+\Delta_{1,2}p_1(c_1)}\sum_{i=2}^{h-2}(s_i(c_1)n(x_0,i)), \end{aligned}$$

where $n(x_0,i)=\prod_{b=2}^{i}l(x_0,b)$, $l(x_0,b)=\frac{\Delta_b}{p_b(c_{b+1})}$, $\Delta_b=p_{b-1}(c_b)-p_{b-1}(c_1)$ $\forall b\geq 2$, $\Delta_{1,1}=g(x_0)-g(x_0-1)$, $\Delta_{1,2}=g(x_0+1)-g(x_0)$.

We will say that f is expandable with respect to g, s_j and p_j if

$$\begin{aligned} f(x_0)=&\frac{\Delta_{1,1}}{\Delta_{1,1}+\Delta_{1,2}\frac{p_1(c_1)}{p_1(c_2)}}f(x_0+1)+\Big(1-\frac{\Delta_{1,1}}{\Delta_{1,1}+\Delta_{1,2}\frac{p_1(c_1)}{p_1(c_2)}}\Big)f(x_0-1)\\ &-\frac{\Delta_{1,1}\Delta_{1,2}}{\Delta_{1,1}p_1(c_2)+\Delta_{1,2}p_1(c_1)}\sum_{i=2}^{+\infty}(s_i(c_1)n(x_0,i)) \end{aligned} \tag{8.4}$$

Theorem 8.2. *Under the same assumptions of Theorem 8.1, if $|s_j(x)|\to 0$ when $j\to+\infty$ $\forall x\in I$ and $|l(x_0,j)|\leq 1$ for j enough large, then f is expandable with respect to s_j, p_j and g.*

Example 8.1. Let $g(x)=x$, $s_j(x)=\frac{x}{j}$, $p_j(x)=x^2$. For any x in a bounded and closed interval, $|s_j(x)|\to 0$ when $j\to+\infty$, so it suffices to verify, for example, that $|l(x_0,j)|\leq 1$ $\forall j\geq j_0$, to conclude that a function f is expandable. First of all, notice that $\forall j\geq 3$

$$\begin{aligned} c_1&=\frac{1}{f(x_0)-f(x_0-1)},\\ c_2&=\frac{1}{f(x_0+1)-f(x_0)}, \end{aligned}$$

$$\vdots$$

$$c_j = \frac{j-2}{j-1}(c_{j-1} - c_1).$$

From this formula, we can easily prove by induction that $c_j = \frac{j-2}{2}c_1 + \frac{1}{j-1}c_2$. Now,

$$|l(x_0, j)| = \left|\frac{c_j^2 - c_1^2}{c_{j+1}^2}\right| = \left|\frac{(\frac{j}{j-1}c_{j+1} - c_1)^2 - c_1^2}{c_{j+1}^2}\right| = \left|\frac{j^2}{(j-1)^2} - \frac{2jc_1}{(j-1)c_{j+1}}\right|.$$

Suppose $1 < c_1 \le 2$ and $0 < c_2 \le 1$. Under these conditions, we have

$$0 < c_{j+1} = \frac{j-1}{2}c_1 + \frac{1}{j}c_2 \le j - 1 + \frac{1}{j};$$

hence

$$\frac{c_1}{c_{j+1}} = \frac{c_1}{\frac{j-1}{2}c_1 + \frac{1}{j}c_2} \ge \frac{c_1}{j - 1 + \frac{1}{j}}.$$

This implies

$$\frac{j^2}{(j-1)^2} - \frac{2jc_1}{(j-1)c_{j+1}} \le \frac{j^2}{(j-1)^2} - \frac{2jc_1}{(j-1)(j-1+\frac{1}{j})}.$$

Moreover, when $1 < c_1 \le 2$, we have

$$\frac{j^2}{(j-1)^2} - \frac{2jc_1}{(j-1)(j-1+\frac{1}{j})} \le 1$$

for j enough large. We now have to verify that this value is ≥ -1 for a large j. Since

$$l(x_0, j) = \frac{c_j^2 - c_1^2}{c_{j+1}^2} = \frac{\frac{(j-1)^2}{4}c_1^2 + \frac{c_2^2}{(j-1)^2} + c_1c_2 - c_1^2}{\frac{j^2}{4}c_1^2 + \frac{c_2^2}{j^2} + c_1c_2} \to 1$$

when $j \to +\infty$, it is certainly ≥ -1 for a large j. Thus, we have proved

$$-1 \le l(x_0, j) \le 1$$

for large j, which is equivalent to $|l(x_0, j)| \le 1$. Therefore, we can expand f when the said conditions are satisfied.

For example, let $f(x) = e^{\frac{1}{2}x}$, $x_0 = \frac{3}{2}$, and $I = [-3, 3]$ (although the precise definition of I plays no role here). It is easy to verify that the conditions above are satisfied. Therefore, we can write the series, which is obtained simplifying $e^{\frac{1}{4}}$,

$$\sqrt{e} = \frac{2e}{e+1} - \frac{e(\sqrt{e}-1)}{e+1}\sum_{i=2}^{+\infty}\left[\frac{1}{i}\prod_{j=2}^{i}\left(\frac{j^2}{(j-1)^2} - \frac{2j}{(j-1)(\frac{j-1}{2} + \frac{1}{j\sqrt{e}})}\right)\right]. \tag{8.5}$$

This can also be written as

$$\sum_{i=2}^{+\infty}\left[\frac{1}{i}\prod_{j=2}^{i}\left(\frac{j^2}{(j-1)^2} - \frac{2j}{(j-1)(\frac{j-1}{2} + \frac{1}{j\sqrt{e}})}\right)\right] = \frac{1}{\sqrt{e}} - 1. \tag{8.6}$$

We can actually verify that $\forall x \in [-1, 0)$ we have

$$e^x = 1 + \sum_{i=2}^{+\infty}\left[\frac{1}{i}\prod_{j=2}^{i}\left(\frac{j^2}{(j-1)^2} - \frac{2j}{(j-1)(\frac{j-1}{2} + \frac{e^x}{j})}\right)\right]. \tag{8.7}$$

□

The following theorem holds true.

Theorem 8.3. *Let $x_0 \in \mathbb{R}$ and $I \subseteq \mathbb{R}$ a closed interval such that $[x_0 - 1, x_0 + 1] \subseteq I$, assume that $f \in C^0(I)$, and $1 < \frac{1}{f(x_0)-f(x_0-1)} \leq 2$, $0 < \frac{1}{f(x_0+1)-f(x_0)} \leq 1$. Then*

$$\begin{aligned}&\frac{f(x_0) - f(x_0 - 1)}{f(x_0 + 1) - f(x_0)} \\ &= 1 + \sum_{k=2}^{\infty} \frac{1}{k} \prod_{j=2}^{k} \left(\frac{j^2}{(j-1)^2} - \frac{2j}{(j-1)(\frac{j-1}{2} + \frac{1}{j}\frac{f(x_0)-f(x_0-1)}{f(x_0+1)-f(x_0)})} \right).\end{aligned} \tag{8.8}$$

Proof. By Example 8.1, under the above conditions, the function f is expandable with respect to $g(x) = x$, $s_j(x) = \frac{x}{j}$, $p_j(x) = x^2$. Knowing that

$$\begin{aligned}c_1 &= \frac{1}{f(x_0) - f(x_0 - 1)}, \\ c_2 &= \frac{1}{f(x_0 + 1) - f(x_0)}, \\ l(x_0, j) &= \frac{j^2}{(j-1)^2} - \frac{2jc_1}{(j-1)c_{j+1}} = \frac{j^2}{(j-1)^2} - \frac{2jc_1}{(j-1)(\frac{j-1}{2}c_1 + \frac{1}{j}c_2)},\end{aligned}$$

we can use (8.4) to write the expansion of f. After some straightforward algebraic manipulations, we get (8.8). □

Example 8.2. By Theorem 8.3 we get

$$\frac{x-2}{(x-1)^2} = 1 + \sum_{k=2}^{\infty} \frac{1}{k} \prod_{j=2}^{k} \left(\frac{j^2}{(j-1)^2} - \frac{2j}{(j-1)(\frac{j-1}{2} + \frac{1}{j}\frac{x-2}{(x-1)^2})} \right), \tag{8.9}$$

for any $x \in (\rho, 3)$, where $\rho \approx 2.561926\ldots$ is a solution to $\frac{1}{\Gamma(x-1)(x-2)} = 2$ (here $f(x) = \Gamma(x)$).
For example, for $x = \frac{11}{4}$

$$\sum_{k=2}^{\infty} \frac{1}{k} \prod_{j=2}^{k} \left(\frac{j^2}{(j-1)^2} - \frac{2j}{(j-1)(\frac{j-1}{2} + \frac{12}{49j})} \right) = -\frac{37}{49} \tag{8.10}$$

□

Remark 8.1. As a matter of fact, the conditions on c_1 and c_2 in Theorem 8.3 are not so restrictive. Let $s_j(x) = \frac{x^k}{j}$, $k \in \mathbb{N}$, $k \neq 0$, $p_j(x) = x^{2k}$, and $g(x) = x$. The expansion of f is still given by (8.8), but using the same method as in Example 8.1, we can verify that the expansion holds for $\frac{1}{(f(x)-f(x-1))^{1/k}} \in (2^{\frac{nk-1}{n}}, 2^n]$ (n can be any non zero natural number), $\frac{1}{(f(x+1)-f(x))^{1/k}} \in (0, 1]$.

We can now prove the following.

Theorem 8.4. *[Representation Theorem] Let $J = [a, +\infty)$ with $a \leq 0$ and $t : J \to (0, +\infty)$ be such that $\lim_{j\to+\infty} t(j) = +\infty$. Moreover, let $x_0 \in \mathbb{R}$, $I \subseteq \mathbb{R}$ a closed interval such that $[x_0 - 1, x_0 + 1] \subseteq I$, $f \in C^0(I)$, and assume that $f(x_0) - f(x_0 - 1) \neq 0$, $f(x_0 + 1) - f(x_0) \neq 0$. Define*

$$\begin{aligned}c_1 &:= \frac{1}{t(1)(f(x_0) - f(x_0 - 1))}, \\ c_2 &:= \frac{1}{t(1)(f(x_0 + 1) - f(x_0))}, \\ \forall j \geq 3 \qquad c_j &:= \frac{t(1)}{t(j-1)}c_2 + \frac{c_1}{t(j-1)} \sum_{p=1}^{j-2} t(p).\end{aligned}$$

Suppose that $0 < c_j < q(j)$, *where* $q(j)$ *is such that*

$$\frac{t^2(j)}{t^2(j-1)} - \frac{2t(j)c_1}{t(j-1)q(j)} \leq 1$$

for j enough large. Suppose also that

$$\frac{t(j)}{t(j-1)} \to 1, \quad \frac{\sum_{p=1}^{j-1} t(p)}{t(j)} \to +\infty,$$

when $j \to +\infty$. *Then, If* $x \in \mathbb{R}$ *is such that* $x = \frac{f(x_0)-f(x_0-1)}{f(x_0+1)-f(x_0)}$, *we have*

$$x = 1 + \sum_{k=2}^{+\infty} \prod_{j=2}^{k} \left(\frac{t(j)}{t(j-1)} - \frac{2t(j)}{xt(1) + \sum_{p=1}^{j-1} t(p)} \right) \tag{8.11}$$

Proof. Let $s_j(x) = \frac{x}{t(j)}$, $p_j(x) = x^2$, $g(x) = x$. Proceed as in Example 8.1. The conditions $\frac{t(j)}{t(j-1)} \to 1$, $\frac{\sum_{p=1}^{j-1} t(p)}{t(j)} \to +\infty$ assure us that $l(x_0, j) \to 1$ when $j \to +\infty$, and therefore $l(x_0, j) \geq -1$ for j enough large. □

Remark 8.2. For example, we can take $t(j) = j^k$ $(k > 0)$, $t(j) = \ln(j+1)$, $t(j) = \psi_0(j)$ (the digamma function), and work as discussed above.

9 Conclusion

In this article we have shown a new way to expand functions as infinite series. With this new kind of expansion, we can find the values of many interesting numerical series. We think that one of the main results, Theorem 8.4 presented above, can be generalized, obtaining other interesting and significative results.

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