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Stability and Hopf bifurcation periodic orbits in delay coupled Lotka-Volterra ring system

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Abstract: In this paper, we consider a class of delay coupled Lotka-Volterra ring systems. Based on the symmetric bifurcation theory of delay differential equations and representation theory of standard dihedral groups, properties of phase locked periodic solutions are given. Moreover, the direction and the stability of the Hopf bifurcation periodic orbits are obtained by using normal form and center manifold theory. Finally, the research results are verified by numerical simulation.

Keywords: Lotka-Volterra; Hopf bifurcation; Periodic solution; Equivariant; Normal form

MSC: 37-XX

1 Introduction

Mathematical models are typically used in ecology to illustrate the basic processes and dynamic mechanisms of ecosystems [1–3]. By the description and analysis of dynamics models, the essential characteristics of life processes can be understood more impressively. Mathematically, we usually describe the ecological mathematical model depending on the theory of functional differential equation. Among them, Lotka-Volterra system described by the theory of functional differential equation is one of the most famous and important ecological population dynamic models [4, 5].

$$\frac{dx_i}{dt} = g_i x_i(1 - \sum_{j=1}^{N} a_{ij} x_j)$$

where g_i represents the linear growth rate of species i, a_{ij} represents the interaction between species i and j, $A = (a_{ij})$ represents the interaction matrix with a_{ij}. It is easy to see that the model is bidirectional,that is, the growth of the species i depends on self-feedback and feedback from the species $i + 1$ and $i - 1$. Among these feedbacks, the interaction between species is not necessarily symmetrical. So in general $a_{ij} \neq a_{ji}$. Without loss of generality, we assume that all $g_i = a_i = 1$, which is equivalent to the carrying capacity of each population x_i in the absence of other species and the unit time of the reverse growth rate of each species [5].

Golubitsky and his collaborators [6] proved that some phase relations can be modeled by coupled systems with observing the gait of animals. Under some conditions, coupled systems produce vibration, while uncoupled systems do not produce vibration. Therefore, the rich dynamic characteristics of the coupled oscillator can be understood by discussing the coupled system. Based on the practical significance of coupled

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Lotka-Volterra ring system, it has been widely used in natural science and ecology. There are many researchers who have conducted deeply studies on the dynamic characteristics of Lotka-Volterra system such as stable, unstable and oscillatory behavior [7–19]. In [20, 21], various continuity theorems play an important role in studying the existence of periodic solutions for Lotka-Volterra systems.

Recently, symmetry has grown up to be an important subject in the study of nonlinear dynamic systems. Generally speaking, symmetry reflects some spatial invariants of dynamical systems. When the system is symmetric, it can exchange with the action of a compact Lie group Γ in Euclidean space [6]. Although symmetry makes the analysis of system more complicated, but it also imposes many special restrictions on the system. Bifurcation can occur to the system with specific symmetry and smaller size, and in some cases, even if the system is symmetric, bifurcation will not occur. Bifurcation phenomenon refers to the qualitative change of some attributes of the object of study. In nature, bifurcation phenomenon is ubiquitous. Therefore, whether in mathematical theory or in practical application, bifurcation theory research has its great significance, especially in symmetric system. So the symmetric Lotka-Volterra system will produce more interesting bifurcation phenomenon, which have been studied initially by some scholars [23, 24]

We mainly study a class of Lotka-Volterra ring systems with coupling [25].

$$\frac{dx_i}{dt} = x_i(1 - a_{-2}x_{i-2} - a_{-1}x_{i-1} - x_i - a_1x_{i+1} - a_2x_{i+2}) \tag{1.1}$$

where i represents the number of species from 1 to N, and assume $x_{N+1} = x_1$ for create periodic boundary conditions. The system is a closed coupled Lotka-Volterra system consisting of N identical species. Each species competes with two of the four neighbouring species for limited resources. Splott [25] studied this coupled ring system deeply and found that the system exhibits spatiotemporal chaos in a spatial dimension, and its quasi-periodic paths are chaos, bifurcation, spontaneous symmetry destruction and spatial pattern formation, however, due to its impossible connectivity, it is not a very realistic model, because the author neglected the growth cycle of species and did not consider the effect of time delay on the model (1.1). Because time delay is an important controls parameter, it is imperative to introduce species growth time. Various scholars also incorporate time delays into the symmetric model [26–34]. Wu [27–29] used bifurcation theory to study local and global Hopf bifurcations of symmetric functional differential equations. Zheng and Zhang [30–32] obtained some results of the symmetric neural network model with delay. Hu [34] and Guo [35] discussed Hopf bifurcation periodic orbits and the spatial patterns of periodic orbits respectively in neuron ring systems with delay.

Based on the original model, the dynamic behavior of three identical species connected into a ring system is considered, a new three-dimensional coupled Lotka-volterra ring system with delays is constructed by adding appropriate delay τ. We just introduce the time delay into adjacent species, assuming that any species is coupled with the nearest species and symmetrical. In the model (1.1), a simpler choice of parameters are $a_{-2} = a_2 = 0$ and $a_{-1} = a_1 = b$.

$$\begin{cases} \dot{x_1} = x_1(1 - x_1 - bx_3(t-\tau) - bx_2(t-\tau)) \\ \dot{x_2} = x_2(1 - x_2 - bx_1(t-\tau) - bx_3(t-\tau)) \\ \dot{x_3} = x_3(1 - x_3 - bx_2(t-\tau) - bx_1(t-\tau)) \end{cases} \tag{1.2}$$

where $b > 0$ and $\tau > 0$. Because the system is symmetry, then the characteristic equation corresponding to the linearization of the Eq. (1.2) has multiple pure imaginary roots at specific parameter values, so the classical Hopf bifurcation theory can not be applied.

In this paper we mainly consider the dynamical properties of Eq. (1.2). The remainder of this paper is organized as follows. In section 2, we proved that a series of Hopf bifurcations will occur when the delay τ increases. In section 3, we obtained the existence and spatial pattern of multiple periodic solutions of Eq. (1.2). In section 4, the detailed calculations of the normal form on center manifold of Eq. (1.2) near to Equivariant-Hopf bifurcation points are determined. We also analysis the direction and the stability conditions of bifurcation nonsynchronous periodic solutions. In section 5, some numerical simulation are given to illustrate the results. Finally, we provides a brief conclusion of our results. The Appendix contains some detailed calculation procedures of coefficients h.

2 Hopf bifurcation

Consider the complex delay Eq. (1.2). In order to study the effect of τ on the stability of equilibrium point, we need to analyze the distribution of roots of eigenvalue equation corresponding to the linear part of the system. It is clear that $(\frac{1}{1+2b}, \frac{1}{1+2b}, \frac{1}{1+2b})$ is an unique positive equilibrium point of Eq.(1.2). Let $c = \frac{1}{1+2b}$, then make an equilibrium transformation of Eq. (1.2), we have

$$\begin{cases} \dot{x_1} = (x_1 + c)\big(1 - (x_1 + c) - b(x_2(t - \tau) + c) - b(x_3(t - \tau) + c)\big) \\ \dot{x_2} = (x_2 + c)\big(1 - (x_2 + c) - b(x_1(t - \tau) + c) - b(x_3(t - \tau) + c)\big) \\ \dot{x_3} = (x_3 + c)\big(1 - (x_3 + c) - b(x_1(t - \tau) + c) - b(x_2(t - \tau) + c)\big) \end{cases} \tag{2.1}$$

The linearization of Eq. (2.1) at origin as follows:

$$\begin{cases} \dot{x_1} = -cx_1 - bcx_2(t - \tau) - bcx_3(t - \tau) \\ \dot{x_2} = -cx_2 - bcx_1(t - \tau) - bcx_3(t - \tau) \\ \dot{x_3} = -cx_3 - bcx_1(t - \tau) - bcx_2(t - \tau) \end{cases} \tag{2.2}$$

regarding τ as the bifurcating parameter, the associated characteristic equation of Eq. (2.2) takes the form

$$\Delta(\lambda) = \Delta_1(\lambda)\Delta_2(\lambda) = (\lambda + c + 2bce^{-\lambda\tau})(\lambda + c - bce^{-\lambda\tau})^2 = 0 \tag{2.3}$$

then we have $\Delta_1(\lambda) = 0$ or $\Delta_2(\lambda) = 0$.

Lemma 2.1. *Consider $\Delta_1(\lambda) = 0$, under the condition of $b > 0$ and $c > 0$, then*
1. In case $\tau = 0$, the root of $\Delta_1(\lambda)$ has negative real part.
2. In case $\tau > 0$ and $b > \frac{1}{2}$, $\Delta_1(\lambda) = 0$ has a pair of purely imaginary roots $\pm i\omega_1$ if and only if

$$\begin{cases} \omega_1 = c\sqrt{4b^2 - 1} \\ \tau = \tau_j^{(1)} = \frac{1}{\omega_1}\{\pi - arcsin\frac{\omega_1}{2bc} + 2j\pi,\} \;\; j = 0, 1, 2, 3\cdots \end{cases} \tag{2.4}$$

Proof 1. When $\tau = 0$, $\Delta_1(\lambda)$ simplified as $\lambda = -1$.
2. When $\tau > 0$, if $i\omega_1(\omega_1 > 0)$ is the purely imaginary root of $\Delta_1(\lambda)$ with

$$\begin{cases} \omega_1 - 2bc\ sin\omega_1\tau = 0 \\ c + 2bc\ cos\omega_1\tau = 0 \end{cases} \tag{2.5}$$

ω_1 and $\tau_j^{(1)}$ are defined by (2.5).

Lemma 2.2. *The transversality conditions is*

$$(\frac{dRe\lambda}{d\tau})^{-1}|_{\tau=\tau_j^{(1)}} > 0$$

Proof By substituting $\lambda(\tau)$ into $\Delta_1(\lambda)$ and taking the derivative with the respect τ from it, we get

$$\frac{d\lambda}{d\tau} = \frac{2bc\lambda e^{-\lambda\tau}}{1 - 2bc\tau e^{-\lambda\tau}}$$

then

$$(\frac{d\lambda}{d\tau})^{-1} = \frac{1}{2bc\lambda e^{-\lambda\tau}} - \frac{\tau}{\lambda}$$

we deduce that

$$(\frac{dRe\lambda}{d\tau})^{-1}|_{\tau=\tau_j^{(1)}} = \frac{1}{\omega_1^2 + c^2} > 0$$

Lemma 2.3. *Consider $\Delta_2(\lambda) = 0$, under the condition of $b > 0$ and $c > 0$, then*
1. In case $\tau = 0$ and $b < 1$, all the roots of $\Delta_2(\lambda) = 0$ has negative real parts.

2. In case $\tau > 0$ and $b > 1$, $\Delta_2(\lambda) = 0$ has two pairs of purely imaginary roots $\pm i\omega_2$ if and only if

$$\begin{cases} \omega_2 = c\sqrt{b^2 - 1}, \\ \tau = \tau_j^{(2)} = \frac{1}{\omega_2}\{2\pi - arccos\frac{1}{b} + 2j\pi, \} \;\; j = 0, 1, 2, 3\cdots \end{cases} \tag{2.6}$$

Proof 1. When $\tau = 0$, and $b < 1$, $\Delta_2(\lambda)$ simplified as $\lambda = c(b - 1)$.
2. When $\tau > 0$, and $b > 1$, if $i\omega_2(\omega_2 > 0)$ is the purely imaginary root of $\Delta_2(\lambda)$ with

$$\begin{cases} \omega_2 + bc\; sin\omega_2\tau = 0 \\ c - bc\; cos\omega_2\tau = 0 \end{cases} \tag{2.7}$$

ω_2 and $\tau_j^{(2)}$ are defined by (2.7).

Lemma 2.4. *The transversality conditions is*

$$(\frac{dRe\lambda}{d\tau})^{-1}|_{\tau=\tau_j^{(2)}} > 0$$

Proof By substituting $\lambda(\tau)$ into $\Delta_2(\lambda)$ and taking the derivative with the respect τ from it, we get

$$\frac{d\lambda}{d\tau} = \frac{-bc\lambda e^{-\lambda\tau}}{1 + bc\tau e^{-\lambda\tau}}$$

then

$$(\frac{d\lambda}{d\tau})^{-1} = \frac{1}{-bc\lambda e^{-\lambda\tau}} - \frac{\tau}{\lambda}$$

we deduce that

$$(\frac{dRe\lambda}{d\tau})^{-1}|_{\tau=\tau_j^{(1)}} = \frac{1}{\omega_2^2 + c^2} > 0$$

Based on the above analysis, we have the following theorem.

Theorem 2.1. *Let $b > 0$ and $c > 0$, and define $(\tau_j^{(1)}, \omega_1)$ and $(\tau_j^{(2)}, \omega_2)$ as in (2.4)(2.6) respectively. Then*
1. If $\tau \in \left[0, min\{\frac{1}{\omega_1}(\pi - arcsin\frac{\omega_1}{2bc} + 2j\pi), \frac{1}{\omega_2}(2\pi - arccos\frac{1}{b} + 2j\pi)\}\right]$, then the equilibrium solution of Eq. (2.1) is asymptotically stable.
2. If $\tau = \tau_j^{(1)}$, $j = 0, 1, 2, 3\cdots$, then Eq. (2.1) shows the general-Hopf bifurcation.
3. If $\tau = \tau_j^{(2)}$, $j = 0, 1, 2, 3\cdots$, then Eq. (2.1) shows the equivariant-Hopf bifurcation.

3 Multiple periodic solutions

Next, we consider the symmetric characteristic of the Eq. (1.2). we know that the Eq. (1.2) is D_3-equivariant with

$$(\rho x)_r = x_{r+1}(mod\; 3)$$
$$(\kappa x)_r = x_{5-r}(mod\; 3)$$

the dihedral group D_3 is generated by the cyclic subgroup subgroup of Z_3 acts with the generator ρ and the flip of κ.

Let $T = \frac{2\pi}{\omega_2}$, P_T represents the set of all continuous T-periodic function $x(t) : R \to R^3$. According to maximum norm, P_T is a Banach space. Apply the action of $D_3 \times S^1$ on P_T with

$$(r, \theta)x(t) = rx(t + \theta) \qquad (r, \theta) \in D_3 \times S^1, x \in P_T$$

Let SP_T be a subspace of P_T, which consist of all T-periodic solution of Eq. (1.2) when parameter $\tau = \tau_j^{(2)}$, then for each closed subgroup of $D_3 \times S^1$ is called the isotropy group Σ, and $\Sigma = \{(r, \theta) \in D_3 \times S^1; (r, \theta)x(t) =$

$x(t)\}$. Under usual non-resonance and transversality conditions, the Σ-fixed-point subspace of SP_T as follows.

$$Fix(\Sigma, SP_T) = \{x \in SP_T,\ (r,\theta)x = x,\ for\ all\ (r,\theta) \in \Sigma\}$$

See [6], symmetric delay differential equations have a bifurcation of periodic solution whose spatiotemporal symmetry can be completely characterized by Σ. We consider the following subgroups of $D_3 \times S^1$ to describe the symmetry of periodic solution of system Eq. (1.2). $\Sigma_1 = \{(\kappa, 1)\}$, $\Sigma_2 = \{(\kappa, -1)\}$, $\Sigma_3 = \{(\rho, e^{i\frac{2\pi}{3}})\}$, $\Sigma_4 = \{(\rho, e^{-i\frac{2\pi}{3}})\}$. Since the Eq. (1.2) is cyclic, we choose

$$v(\theta) = (0, e^{i\frac{2\pi}{3}}, e^{-i\frac{2\pi}{3}})^T e^{i\omega_2\theta}$$

We know $v(\theta)$ is the corresponding eigenvectors of $\Delta_2(\lambda)$ with $\pm i\omega_2$. Generalized eigenspace $U_{\pm i\omega_2}$ is

$$\begin{aligned}
U_{\pm i\omega_2} &= \left[\{\textstyle\sum_{j=1}^{4} y_j\varepsilon_j,\ y_j \in R,\ j = 1,2,3,4\}\right]\\
\varepsilon_1 &= cos(\omega_2 t)Re\{v(0)\} - sin(\omega_2 t)Im\{v(0)\}\\
\varepsilon_2 &= sin(\omega_2 t)Re\{v(0)\} + cos(\omega_2 t)Im\{v(0)\}\\
\varepsilon_3 &= cos(\omega_2 t)Re\{v(0)\} + sin(\omega_2 t)Im\{v(0)\}\\
\varepsilon_4 &= sin(\omega_2 t)Re\{v(0)\} - cos(\omega_2 t)Im\{v(0)\}
\end{aligned}$$

From [27], we have the following theorem.

Theorem 3.1. *Near $\tau = \tau_j^{(2)}$, the Eq.(2.1) has three kinds of small-amplitude periodic solutions from the trivial solution $x = 0$, and the period T is close to $\frac{2\pi}{\omega_2}$.*
1. Mirror-reflecting waves: $x_i(t) = x_j(t) \neq x_k(t)$, for $t \in R$, $(i,j,k) \in (1,2,3)$,
2. Discrete waves: $x_i(t) = x_j(t \pm \frac{T}{3})$, for $i, j \in (1,2,3)$, $i \neq j$,
3. Standing waves: $x_i(t) = x_j(t + \frac{T}{2})$, for $(i,j) \in (1,2,3)$, $i \neq j$.

4 Normal form for equivariant-Hopf bifurcation

In this part, we only consider the case that the Eq. (2.3) has double characteristic values $\pm i\omega_2$, where the equivariant-Hopf bifurcation occurs. Center manifold theory and normal form method [36, 37] are used to study Hopf bifurcation. Firstly, rescale the time by $t \to \frac{t}{\tau}$, Eq. (2.1) can be written as

$$\dot{x}_t = F(x_t, \tau) \tag{4.1}$$

where

$$(F(x_t,\tau))_i = \tau\big(-cx_i(t) - bcx_{i+1}(t-1) - bcx_{i-1}(t-1) - x_i^2(t) - bx_i(t)x_{i+1}(t-1) - bx_i(t)x_{i-1}(t-1)\big)$$

Suppose that the Eq. (4.1) undergoes Equivariant-Hopf bifurcation at $\tau = \tau_j^2 = \tau_\star$. Choosing the phase space $C = C([-1,0]; R^3)$, where for $\phi = (\phi_1, \phi_2, \phi_3)^T \in C$. Then Eq. (4.1) can be written as

$$(F(\phi_t,\tau))_i = -\tau(\phi_i(0) + c)[1 - (\phi_i(0) + c) - b(\phi_{i+1}(-1) + c) - b(\phi_{i-1}(-1) + c)]$$

with $i = mod(3)$.

The linearized equation of Eq. (4.1) at zero as follows

$$\dot{x}_t = \mathcal{L}(\tau)x_t \tag{4.2}$$

where

$$\mathcal{L}(\tau)(\phi) = \mathbb{A}\phi(0) + \mathbb{B}\phi(-1) = -c\tau\phi(0) - bc\tau\delta\phi(-1)$$

with $\delta = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. The characteristic equation of Eq. (4.2) at origin is $det\Delta_2(0, \frac{\lambda}{\tau}) = 0$, where $det\Delta_2(0, \lambda) = 0$ is the characteristic equation of the linearizaion of Eq. (2.1). Since $\Delta_2(0, i\omega_2)v_j = 0, j = 1, 2$, the center space at $\tau = \tau_\star$ and in complex coordinates is $P = span(\varphi_1, \varphi_2, \varphi_3, \varphi_4)$, where

$$\begin{aligned} \varphi_1 &= e^{i\tau_\star\omega_2\theta}v_1 \\ \varphi_2 &= e^{-i\tau_\star\omega_2\theta}\bar{v}_1 \\ \varphi_3 &= e^{i\tau_\star\omega_2\theta}\bar{v}_1 \\ \varphi_4 &= e^{-i\tau_\star\omega_2\theta}v_1 \qquad \theta \in [-1, 0] \end{aligned}$$

and

$$v_1 = v(0) = (1, e^{\frac{2i\pi}{3}}, e^{-\frac{2i\pi}{3}})^T$$

Let $\Phi = (\varphi_1, \varphi_2, \varphi_3, \varphi_4)$ and $\bar{v}_1 = v_2$. Note that $v_j^T v_i = 3, i \neq j \in 1, 2$ and $v_i^T v_i = 0, i \in (1, 2)$. It is easy to check that a basis of the adjoint space of $P^\star$ is

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} \bar{a}^{-1}e^{-i\omega_2\tau_\star s}\bar{v}_1^T \\ a^{-1}e^{i\omega_2\tau_\star s}v_1^T \\ \bar{a}^{-1}e^{-i\omega_2\tau_\star s}v_1^T \\ a^{-1}e^{i\omega_2\tau_\star s}\bar{v}_1^T \end{pmatrix}$$

with $\langle\Psi, \Phi\rangle = I_{4\times4}$ for the adjoint bilinear form on $C^\star \times C$ define by

$$\langle\psi, \varphi\rangle = \psi(0)\varphi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta} \psi(\xi - \theta)d\eta(\theta)\varphi(\xi)d\xi$$

with $\varphi \in C, \psi \in C^\star$, and

$$a = 1 + c\tau_\star - i\omega_2\tau_\star$$

Introducing new parameter variables $\mu = \tau - \tau_\star$, we can rewrite Eq. (4.1) as

$$\dot{z}(t) = \mathcal{L}(\tau_\star)z_t + G(z_t, \mu) \tag{4.3}$$

where

$$G(z_t, \mu) = \mathcal{L}(\mu)z_t + (\tau_\star + \mu)\begin{pmatrix} -z_1^2 - bz_1z_3(t-1) - bz_1z_2(t-1) \\ -z_2^2 - bz_2z_1(t-1) - bz_2z_3(t-1) \\ -z_3^2 - bz_3z_1(t-1) - bz_3z_2(t-1) \end{pmatrix} + h.o.t$$

Let $B = (i\omega_2\tau_\star, -i\omega_2\tau_\star, i\omega_2\tau_\star, -i\omega_2\tau_\star)$ and P is the generalized eigenspace associated with B, $P^\star$ is the adjoint space of P. Then C can be decomposed as $C = P \oplus Q$ where $Q = (\varphi \in C :< \psi, \varphi >= 0, for\ all\ \psi \in P^\star)$. Using the decomposition $z_t = \Phi x(t) + y(t)$, then we have

$$\begin{aligned} z_1(0) &= c_4 + y_1(0),\ z_1(-1) = c_2 + c_3 + y_1(-1) \\ z_2(0) &= c_1 + y_2(0),\ z_2(-1) = e^{-\frac{2}{3}i\pi}c_2 + e^{\frac{2}{3}i\pi}c_3 + y_2(-1) \\ z_3(0) &= \bar{c}_1 + y_3(0),\ z_3(-1) = e^{\frac{2}{3}i\pi}c_2 + e^{-\frac{2}{3}i\pi}c_3 + y_3(-1) \end{aligned}$$

We can decompose Eq. (4.3) as

$$\begin{cases} \dot{x} = Bx + \Psi(0)G(\Phi x + y, \mu) \\ \dot{y} = \mathcal{A}_{Q^1 y} + (I - \pi)X_0G(\Phi x + y, \mu) \end{cases} \tag{4.4}$$

with $x \in C^4, y \in Q^1$. We will write the Taylor expansion

$$\Psi(0)G(\Phi x + y, \mu) = \sum_{j\geq2}\frac{1}{j!}f_j^1(x, y, \mu) \tag{4.5}$$

and we have

$$\begin{aligned}
\Phi(0)x &= [v_1, \bar{v}_1, \bar{v}_1, v_1] = (x_1 + x_4)v_1 + (x_2 + x_3)\bar{v}_1 \\
\Phi(-1)x &= (e^{-i\omega_2\tau_*}x_1 + e^{i\omega_2\tau_*}x_4)v_1 + (e^{i\omega_2\tau_*}x_2 + e^{-i\omega_2\tau_*}x_3)\bar{v}_1 \\
\delta(\Phi(-1)x) &= -(e^{-i\omega_2\tau_*}x_1 + e^{i\omega_2\tau_*}x_4)v_1 - (e^{i\omega_2\tau_*}x_2 + e^{-i\omega_2\tau_*}x_3)\bar{v}_1 = -\Phi(-1)x
\end{aligned}$$

Using the idea of Faria [37], we know that Eq.(4.4) can be written as

$$\dot{x} = Bx + \sum_{j\geq 2} \frac{1}{j!} f_j^1(x, y, \mu) \tag{4.6}$$

where $f_i(x, y, \mu)$ is homogeneous polynomials of degree j about (x, y, μ) with coefficients in C^4. Then the normal form of Eq. (1.2) on the center manifold

$$\dot{x} = Bx + \frac{1}{2} g_2^1(x, 0, \mu) + \frac{1}{3!} g_3^1(x, 0, \mu) + h.o.t$$

where g_2^1, g_3^1 will be calculated in the following part of this section.

Fist of all, we get

$$\begin{aligned}
\tfrac{1}{2} f_2^1(x, 0, \mu) &= \Psi(0)\mathcal{L}(\mu)(z_t) + \tau_*\Psi(0) \begin{pmatrix} -z_1^2 - bz_1z_3(t-1) - bz_1z_2(t-1) \\ -z_2^2 - bz_2z_1(t-1) - bz_2z_3(t-1) \\ -z_3^2 - bz_3z_1(t-1) - bz_3z_2(t-1) \end{pmatrix} \\
&= \mu \begin{pmatrix} \bar{a}^{-1} i\omega_2(x_1 - x_4) \\ a^{-1} i\omega_2(x_3 - x_2) \\ \bar{a}^{-1} i\omega_2(x_3 - x_2) \\ a^{-1} i\omega_2(x_1 - x_4) \end{pmatrix} + \tau_*\Psi(0) \begin{pmatrix} -z_1^2 - bz_1z_3(t-1) - bz_1z_2(t-1) \\ -z_2^2 - bz_2z_1(t-1) - bz_2z_3(t-1) \\ -z_3^2 - bz_3z_1(t-1) - bz_3z_2(t-1) \end{pmatrix}
\end{aligned}$$

These are the second-order terms of (μ, x) in Eq. (4.6). From Faria and Hal[36–39], we have the second-order terms of (μ, x) in the normal form on center manifold as follows:

$$\frac{1}{2} g_2^1(x, 0, \mu) = Proj_{Ker(M_2^1)} \frac{1}{2} f_2^1(x, 0, \mu)$$

Here, define M_j to be the operator in $V_j^5(C^4 \times Ker_\pi)$ with the range in the same space by

$$M_j(p, h) = (M_j^1(p), M_j^2(h))$$

where

$$\begin{cases} M_j^1(p) = D_x p(x, \mu)Bx - Bp(x, \mu) \\ M_j^1(h) = D_x h(x, \mu)Bx - A_{Q^1} h(x, \mu) \end{cases} \tag{4.7}$$

In particular,

$$M_j^1(\mu x^q e^k) = i\mu\tau_*(q_1 - q_2 + q_3 - q_4 + (-1)^k)x^q e_k \quad |q| = j - 1$$

where $j \geq 2$, $1 \leq k \leq 4$, and $\{e_1, e_2, e_3, e_4\}$ is the canonical basis for C^4. Therefore, if $|q| = 1$, then

$$\begin{aligned}
&(Ker(M_2^1)) \\
&\quad = \{\mu x_3 e_1, \mu x_1 e_1, \mu x_4 e_2, \mu x_2 e_2, \mu x_3 e_3, \mu x_1 e_3, \mu x_4 e_4, \mu x_2 e_4\} \\
&(Ker(M_3^1)) \\
&\quad = \{\mu^2 x_1 e_1, \mu^2 x_3 e_1, x_1x_2x_3 e_1, x_1x_3x_4 e_1, x_1^2x_2 e_1, x_1^2x_4 e_1, x_3^2x_2 e_1, x_3^2x_4 e_1 \\
&\qquad \mu^2 x_2 e_2, \mu^2 x_4 e_2, x_1x_2x_4 e_2, x_2x_3x_4 e_2 x_2^2x_1 e_2, x_2^2x_3 e_2, x_4^2x_1 e_2, x_4^2x_3 e_2 \\
&\qquad \mu^2 x_1 e_3, \mu^2 x_3 e_3, x_1x_2x_3 e_3, x_1x_3x_4 e_3 x_1^2x_2 e_3, x_1^2x_4 e_3, x_3^2x_2 e_3, x_3^2x_4 e_3 \\
&\qquad \mu^2 x_2 e_4, \mu^2 x_4 e_4, x_1x_2x_4 e_4, x_2x_3x_4 e_4 x_2^2x_1 e_4, x_2^2x_3 e_4, x_4^2x_1 e_4, x_4^2x_3 e_4\}
\end{aligned}$$

and

$$\frac{1}{2}g_2^1(x,0,\mu) = \mu \begin{pmatrix} i\omega_2\bar{a}^{-1}x_1 \\ -i\omega_2 a^{-1}x_2 \\ i\omega_2\bar{a}^{-1}x_3 \\ -i\omega_2 a^{-1}x_4 \end{pmatrix}$$

To compute $g_3^1(x,0,\mu)$, we first note that from Eq. (4.7), it follows that

$$\begin{aligned} \frac{1}{3!}g_3^1(x,0,\mu) &= Proj_{(kerM_3^1)}\frac{1}{3!}\tilde{f}_3^1(x,0,\mu) \\ &= Proj_{ker(M_3^1)}\frac{1}{3!}\tilde{f}_3^1(x,0,0) + O(|\mu|^2|x|) + O(|\mu||x|^2) \\ &= Proj_{ker(M_3^1)}\frac{1}{3!}f_3^1(x,0,0) \\ &\quad + Proj_{ker(M_3^1)}[\frac{1}{4}(D_x f_2^1)(x,0,0)U_2^1(x.0) + \frac{1}{4}(D_y f_2^1)(x,0,0)U_2^2(x,0)] \end{aligned}$$

since $\mu x^q e_j \notin Ker(M_3^1)$, for $|q| = 2, j = 1,2,3,4$. For $\mu = 0, f_2^1(x,0,0) = g_2^1(x,0,0) = 0$, a simplified formula is given.

$$\begin{aligned} c_1 &= e^{\frac{2}{3}i\pi}(x_1 + x_4) + e^{-\frac{2}{3}i\pi}(x_2 + x_3) \\ c_2 &= e^{i\omega_2\tau_\star}x_2 + e^{-i\omega_2\tau_\star}x_3 \\ c_3 &= e^{i\omega_2\tau_\star}x_4 + e^{-i\omega_2\tau_\star}x_1 \\ c_4 &= x_1 + x_2 + x_3 + x_4 \\ r_1 &= (1 + c\tau_\star) \\ r_2 &= \omega_2\tau_\star \\ p &= \bar{a}^{-1}a^{-1} = ((1 + c\tau_\star)^2 + \omega_2^2\tau_\star^2)^{-1} \\ q &= p^2(r_1^2 - r_2^2) - 2ip^2 r_1 r_2 \end{aligned}$$

Case I: Compute $Proj_{ker(M_3^1)}\frac{1}{3!}f_3^1(x,0,0) = 0$.
Case II: Compute $Proj_{ker(M_3^1)}[\frac{1}{4}(D_x f_2^1)(x,0,0)U_2^1(x.0)]$.
In fact, from the $e^{(\pm i\omega_2\tau_\star)} = \frac{c\mp i\omega_2}{bc}$, we have

$$\frac{1}{2}f_2^1(x,0,0) = -\tau_\star \begin{pmatrix} \bar{a}^{-1}[2(x_2+x_3)^2 + i\frac{\omega_2}{c}(x_3^2 - x_2^2)] \\ a^{-1}[2(x_1+x_4)^2 + i\frac{\omega_2}{c}(x_1^2 - x_4^2)] \\ \bar{a}^{-1}[2(x_1+x_4)^2 + i\frac{\omega_2}{c}(x_1^2 - x_4^2)] \\ a^{-1}[2(x_2+x_3)^2 + i\frac{\omega_2}{c}(x_3^2 - x_2^2)] \end{pmatrix}$$

since

$$\begin{aligned} U_2^1(x,0) &= U_2^1(x,\mu)|_{\mu=0} = (M_2^1)^{-1}Proj_{Im(M_2^1)}f_2^1(x,0,0) \\ &= -\tau_\star(M_2^1)^{-1} \begin{pmatrix} \bar{a}^{-1}[2(x_2+x_3)^2 + i\frac{\omega_2}{c}(x_3^2 - x_2^2)] \\ a^{-1}[2(x_1+x_4)^2 + i\frac{\omega_2}{c}(x_1^2 - x_4^2)] \\ \bar{a}^{-1}[2(x_1+x_4)^2 + i\frac{\omega_2}{c}(x_1^2 - x_4^2)] \\ a^{-1}[2(x_2+x_3)^2 + i\frac{\omega_2}{c}(x_3^2 - x_2^2)] \end{pmatrix} \\ &= -\frac{\tau_\star}{i\omega_2} \begin{pmatrix} \bar{a}^{-1}[-\frac{2}{3}x_2^2 - 4x_2x_3 + 2x_3^2 + i\frac{\omega_2}{c}(x_3^2 + \frac{1}{3}x_2^2)] \\ a^{-1}[\frac{2}{3}x_1^2 + 4x_1x_4 - 2x_4^2 + i\frac{\omega_2}{c}(\frac{1}{3}x_1^2 + x_4^2)] \\ \bar{a}^{-1}[2x_1^2 - 4x_1x_4 - \frac{2}{3}x_4^2 + i\frac{\omega_2}{c}(x_1^2 + \frac{1}{3}x_4^2)] \\ a^{-1}[-2x_2^2 + 4x_2x_3 + \frac{2}{3}x_3^2 + i\frac{\omega_2}{c}(\frac{1}{3}x_3^2 + x_2^2)] \end{pmatrix} \end{aligned}$$

we have

$$\begin{aligned} Proj_{ker(M_3^1)}[\tfrac{1}{4}(D_x f_2^1)(x,0,0)U_2^1(x.0)] &= \tfrac{1}{2}Proj_{ker(M_3^1)}[\tfrac{1}{2}(D_x f_2^1)(x,0,0)U_2^1(x.0)] \\ &= \begin{pmatrix} b_{11}x_1x_3x_4 + b_{12}x_1^2x_2 \\ \bar{b}_{11}x_2x_3x_4 + \bar{b}_{12}x_2^2x_1 \\ b_{11}x_1x_2x_3 + b_{12}x_3^2x_4 \\ \bar{b}_{11}x_1x_2x_4 + \bar{b}_{12}x_4^2x_3 \end{pmatrix} \end{aligned}$$

where

$$b_{11} = -\frac{\tau_\star^2 K_2}{2\omega_2} + i\frac{\tau_\star^2(K_1 - 16p)}{2\omega_2}$$
$$b_{12} = \frac{\tau_\star^2}{2\omega_2}K_5 - i\frac{\tau_\star^2(K_3 + K_4)}{2\omega_2}$$
$$K_1 = 16p^2(r_1^2 - r_2^2) - \frac{2\omega_2}{c}p^2 r_1 r_2$$
$$K_2 = -\frac{\omega_2}{c}p^2(r_1^2 - r_2^2) - 32p^2 r_1 r_2$$
$$K_3 = \frac{p(8c^2 + 2\omega_2^2)}{3c^2}$$
$$K_4 = 8p^2(r_1^2 - r_2^2) + \frac{8\omega_2}{c}p^2 r_1 r_2$$
$$K_5 = -\frac{4\omega_2}{c}p^2(r_1^2 - r_2^2) - 16p^2 r_1 r_2$$

Case III: Compute $\frac{1}{4}Proj_{ker(M_3^1)}(D_y f_2^1)(x,0,0)U_2^2(x,0)]$.
Define $h = h(x)(\theta) = U_2^2(x,0)$, and write

$$h(\theta) = \begin{pmatrix} h^{(1)}(\theta) \\ h^{(2)}(\theta) \\ h^{(3)}(\theta) \end{pmatrix} = h_{2000}x_1^2 + h_{0200}x_2^2 + h_{0020}x_3^2 + h_{0002}x_4^2 + h_{1100}x_1x_2$$
$$+ h_{1010}x_1x_3 + h_{1001}x_1x_4 + h_{0110}x_2x_3 + h_{0101}x_2x_4 + h_{0011}x_3x_4$$

where $h_{2000}, h_{0200}, h_{0020}, h_{0002}, h_{1100}, h_{1010}, h_{1001}, h_{0110}, h_{0101}, h_{0011} \in Q^1$. The coefficients of h are determined by $M_2^2 h(x) = f_2^2(x,0,0)$, which is equivalent to

$$D_x hBx - A_{Q^1}(h) = (I - \pi)X_0 F_2(\Phi x, 0)$$

Applying the definition of A_{Q^1} and π, we obtain

$$\dot{h} - D_x hBx = \Phi(\theta)\Psi(0)F_2(\Phi x, 0)$$
$$\dot{h}(0) - \mathcal{L}h = F_2(\Phi x, 0)$$

where $\dot{h}$ denotes the derivative of $h(\theta)$ relative to θ. Let

$$F_2(\Phi x, 0) = A_{2000}x_1^2 + A_{0200}x_2^2 + A_{0020}x_3^2 + A_{0002}x_4^2 + A_{1100}x_1x_2$$
$$+ A_{1010}x_1x_3 + A_{1001}x_1x_4 + A_{0110}x_2x_3 + A_{0101}x_2x_4 + A_{0011}x_3x_4$$

where $A_{ijmn} \in C^2$, $0 \le i,j,m,n \le 2$ and $i + j + m + n = 2$. Comparing the coefficients of $x_1^2, x_2^2, x_3^2, x_4^2, x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4$ and x_3x_4, we have $\bar{h}_{2000} = h_{0200} = h_{0020} = \bar{h}_{0002}, h_{1010} = \bar{h}_{0101}$ and $h_{1100} = h_{0011} = h_{1001} = h_{0110} = 0$, and h_{2000}, h_{1010} satisfy the following differential equations respectively,

$$\begin{cases} \dot{h_{2000}} - 2i\omega_2\tau_\star h_{2000} = \Phi(\theta)\Psi(0)A_{2000} \\ \dot{h_{2000}}(0) - \mathcal{L}(h_{2000}) = A_{2000} \end{cases} \tag{4.8}$$

$$\begin{cases} \dot{h_{1010}} - 2i\omega_2\tau_\star h_{1010} = \Phi(\theta)\Psi(0)A_{1010} \\ \dot{h_{1010}}(0) - \mathcal{L}(h_{1010}) = A_{1010} \end{cases} \tag{4.9}$$

Since

$$F_2(z_t, 0) = -\tau_\star \begin{pmatrix} z_1^2(0) \\ z_2^2(0) \\ z_3^2(0) \end{pmatrix} - b\tau_\star \begin{pmatrix} z_1(0)(z_2(-1) + z_3(-1)) \\ z_2(0)(z_1(-1) + z_3(-1)) \\ z_3(0)(z_1(-1) + z_2(-1)) \end{pmatrix}$$

and

$$\frac{1}{2}f_2^1(x,y,0)=\Psi(0)F_2(\Phi x+y,0)=-\frac{\tau_*}{3}\begin{pmatrix}\bar{a}^{-1}B_1\\ a^{-1}B_2\\ \bar{a}^{-1}B_2\\ a^{-1}B_1\end{pmatrix}$$

where

$$\begin{aligned}B_1&=(c_4+y_1(0))^2+b(c_4+y_1(0))(-c_2-c_3+y_2(-1)+y_3(-1))\\&\quad+e^{-\frac{2}{3}i\pi}[(c_1+y_2(0))^2+b(c_1+y_2(0))(-c_2e^{-\frac{2}{3}i\pi}-c_3e^{\frac{2}{3}i\pi}+y_1(-1)+y_3(-1))]\\&\quad+e^{\frac{2}{3}i\pi}[(\bar{c}_1+y_3(0))^2+b(\bar{c}_1+y_3(0))(-c_2e^{\frac{2}{3}i\pi}-c_3e^{-\frac{2}{3}i\pi}+y_1(-1)+y_2(-1))],\end{aligned}$$

$$\begin{aligned}B_2&=(c_4+y_1(0))^2+b(c_4+y_1(0))(-c_2-c_3+y_2(-1)+y_3(-1))\\&\quad+e^{\frac{2}{3}i\pi}[(c_1+y_2(0))^2+b(c_1+y_2(0))(-c_2e^{-\frac{2}{3}i\pi}-c_3e^{\frac{2}{3}i\pi}+y_1(-1)+y_3(-1))]\\&\quad+e^{-\frac{2}{3}i\pi}[(\bar{c}_1+y_3(0))^2+b(\bar{c}_1+y_3(0))(-c_2e^{\frac{2}{3}i\pi}-c_3e^{-\frac{2}{3}i\pi}+y_1(-1)+y_2(-1))],\end{aligned}$$

Thus

$$\frac{1}{2}(D_yf_2^1)(x,y,0)(h)=-\frac{\tau_*}{3}\begin{pmatrix}\bar{a}^{-1}\frac{\partial B_1}{\partial y_1} & \bar{a}^{-1}\frac{\partial B_1}{\partial y_2} & \bar{a}^{-1}\frac{\partial B_1}{\partial y_3}\\ a^{-1}\frac{\partial B_2}{\partial y_1} & a^{-1}\frac{\partial B_2}{\partial y_2} & a^{-1}\frac{\partial B_2}{\partial y_3}\\ \bar{a}^{-1}\frac{\partial B_2}{\partial y_1} & \bar{a}^{-1}\frac{\partial B_2}{\partial y_2} & \bar{a}^{-1}\frac{\partial B_2}{\partial y_3}\\ a^{-1}\frac{\partial B_1}{\partial y_1} & a^{-1}\frac{\partial B_1}{\partial y_2} & a^{-1}\frac{\partial B_1}{\partial y_3}\end{pmatrix}\begin{pmatrix}h^1(\theta)\\ h^2(\theta)\\ h^3(\theta)\end{pmatrix}$$

where

$$\begin{aligned}\frac{\partial B_1}{\partial y_1}&=2(c_4+y_1(0))+b(-c_2-c_3+y_2(-1)+y_3(-1))+be^{-\frac{2}{3}i\pi}(c_1+y_2(0))\\&\quad+be^{\frac{2}{3}i\pi}(\bar{c}_1+y_3(0))\\ \frac{\partial B_1}{\partial y_2}&=e^{-\frac{2}{3}i\pi}[2(c_1+y_2(0))+b(-c_2e^{-\frac{2}{3}i\pi}-c_3e^{\frac{2}{3}i\pi}+y_1(-1)+y_3(-1))]\\&\quad+b(c_4+y_1(0))+be^{\frac{2}{3}i\pi}(\bar{c}_1+y_3(0))\\ \frac{\partial B_1}{\partial y_3}&=e^{\frac{2}{3}i\pi}[2(\bar{c}_1+y_3(0))+b(-c_2e^{\frac{2}{3}i\pi}-c_3e^{-\frac{2}{3}i\pi}+y_1(-1)+y_2(-1))]\\&\quad b(c_4+y_1(0))+be^{-\frac{2}{3}i\pi}(c_1+y_2(0))\\ \frac{\partial B_2}{\partial y_1}&=2(c_4+y_1(0))+b(-c_2-c_3+y_2(-1)+y_3(-1))+be^{\frac{2}{3}i\pi}(c_1+y_2(0))\\&\quad+be^{-\frac{2}{3}i\pi}(\bar{c}_1+y_3(0))\\ \frac{\partial B_2}{\partial y_2}&=e^{\frac{2}{3}i\pi}[2(c_1+y_2(0))+b(-c_2e^{-\frac{2}{3}i\pi}-c_3e^{\frac{2}{3}i\pi}+y_1(-1)+y_3(-1))]\\&\quad+b(c_4+y_1(0))+be^{-\frac{2}{3}i\pi}(\bar{c}_1+y_3(0))\\ \frac{\partial B_2}{\partial y_3}&=e^{-\frac{2}{3}i\pi}[2(\bar{c}_1+y_3(0))+b(-c_2e^{\frac{2}{3}i\pi}-c_3e^{-\frac{2}{3}i\pi}+y_1(-1)+y_2(-1))]\\&\quad+b(c_4+y_1(0))+be^{\frac{2}{3}i\pi}(c_1+y_2(0))\end{aligned}$$

Thus

$$\frac{1}{4}Proj_{KerM_3^1}(D_yf_2^1)(x,y,0)U_2^2=\begin{pmatrix}d_{11}x_1x_3x_4+d_{12}x_1^2x_2\\ \bar{d}_{11}x_2x_3x_4+\bar{d}_{12}x_2^2x_1\\ d_{11}x_1x_2x_3+d_{12}x_3^2x_4\\ \bar{d}_{11}x_1x_2x_4+\bar{d}_{12}x_4^2x_3\end{pmatrix}$$

where

$$
\begin{aligned}
d_{11} &= -\frac{\tau_\star}{2}p[r_1K_6 + r_2K_7 - i(r_2K_6 - r_1K_7)]\\
d_{12} &= -\frac{\tau_\star}{2}p[r_1K_8 + r_2K_9 - i(r_2K_8 - r_1K_9)]\\
K_6 &= (2+2b)Re(h^{(1)}_{1010}(0)) - Re(h^{(1)}_{1010}(-1)) - \frac{\omega_2}{c}Im(h^{(1)}_{1010}(-1))\\
K_7 &= (2+2b)Im(h^{(1)}_{1010}(0)) - Im(h^{(1)}_{1010}(-1)) + \frac{\omega_2}{c}Re(h^{(1)}_{1010}(-1))\\
K_8 &= (2-b)Re(h^{(1)}_{2000}(0)) - Re(h^{(1)}_{2000}(-1)) - \frac{\omega_2}{c}Im(h^{(1)}_{2000}(-1))\\
K_9 &= (2-b)Im(h^{(1)}_{2000}(0)) - Im(h^{(1)}_{2000}(-1)) + \frac{\omega_2}{c}Re(h^{(1)}_{2000}(-1))
\end{aligned}
$$

and h_{ijmn} will be calculated in Appendix. From Cases I, II, III, we get

$$
\frac{1}{6}g^1_3(x,0,\mu) = \begin{pmatrix}(b_{11}+d_{11})x_1x_3x_4 + (b_{12}+d_{12})x_1^2x_2\\ (\bar{b}_{11}+\bar{d}_{11})x_2x_3x_4 + (\bar{b}_{12}+\bar{d}_{12})x_2^2x_1\\ (b_{11}+d_{11})x_1x_2x_3 + (b_{12}+d_{12})x_3^2x_4\\ (\bar{b}_{11}+\bar{d}_{11})x_1x_2x_4 + (\bar{b}_{12}+\bar{d}_{12})x_4^2x_3\end{pmatrix}
$$

So, we can express Eq. (4.1) as

$$
\begin{cases}
\dot{x_1} = i\omega_2\tau_\star x_1 + \mu\bar{a}^{-1}i\omega_2x_1 + (b_{11}+d_{11})x_1x_3x_4 + (b_{12}+d_{12})x_1^2x_2\\
\dot{x_2} = -i\omega_2\tau_\star x_2 - \mu a^{-1}i\omega_2x_2 + (\bar{b}_{11}+\bar{d}_{11})x_2x_3x_4 + (\bar{b}_{12}+\bar{d}_{12})x_2^2x_1\\
\dot{x_3} = i\omega_2\tau_\star x_3 + \mu\bar{a}^{-1}i\omega_2x_3 + (b_{11}+d_{11})x_1x_2x_3 + (b_{12}+d_{12})x_3^2x_4\\
\dot{x_4} = -i\omega_2\tau_\star x_4 - \mu a^{-1}i\omega_2x_4 + (\bar{b}_{11}+\bar{d}_{11})x_1x_2x_4 + (\bar{b}_{12}+\bar{d}_{12})x_4^2x_3
\end{cases}
\tag{4.10}
$$

Since $x_1 = \bar{x}_2, x_3 = \bar{x}_4$, through the change of variables $x_1 = \alpha_1 - i\alpha_2, x_2 = \alpha_1 + i\alpha_2, x_3 = \alpha_3 - i\alpha_4, x_4 = \alpha_3 + i\alpha_4$, we obtain

$$
\begin{aligned}
\begin{pmatrix}\dot{\alpha}_1\\ \dot{\alpha}_2\end{pmatrix} &= \omega_2\tau_\star\begin{pmatrix}\alpha_2\\ -\alpha_1\end{pmatrix} + \mu\omega_2\begin{pmatrix}-Im[\bar{a}^{-1}]\alpha_1 + Re[\bar{a}^{-1}]\alpha_2\\ -Re[\bar{a}^{-1}]\alpha_1 - Im[\bar{a}^{-1}]\alpha_2\end{pmatrix}\\
&+\begin{pmatrix}\alpha_1\big(Re[b_{11}+d_{11}]\rho_2^2 + Re[b_{12}+d_{12}]\rho_1^2\big) + \alpha_2\big(Im[b_{11}+d_{11}]\rho_2^2 + Im[b_{12}+d_{12}]\rho_1^2\big)\\ -\alpha_1\big(Im[b_{11}+d_{11}]\rho_2^2 + Im[b_{12}+d_{12}]\rho_1^2\big) + \alpha_2\big(Re[b_{11}+d_{11}]\rho_2^2 + Re[b_{12}+d_{12}]\rho_1^2\big)\end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\begin{pmatrix}\dot{\alpha}_3\\ \dot{\alpha}_4\end{pmatrix} &= \omega_2\tau_\star\begin{pmatrix}\alpha_4\\ -\alpha_3\end{pmatrix} + \mu\omega_2\begin{pmatrix}-Im[\bar{a}^{-1}]\alpha_3 + Re[\bar{a}^{-1}]\alpha_4\\ -Re[\bar{a}^{-1}]\alpha_3 - Im[\bar{a}^{-1}]\alpha_4\end{pmatrix}\\
&+\begin{pmatrix}\alpha_3\big(Re[b_{11}+d_{11}]\rho_1^2 + Re[b_{12}+d_{12}]\rho_2^2\big) + \alpha_4\big(Im[b_{11}+d_{11}]\rho_1^2 + Im[b_{12}+d_{12}]\rho_2^2\big)\\ -\alpha_3\big(Im[b_{11}+d_{11}]\rho_1^2 + Im[b_{12}+d_{12}]\rho_2^2\big) + \alpha_4\big(Re[b_{11}+d_{11}]\rho_1^2 + Re[b_{12}+d_{12}]\rho_2^2\big)\end{pmatrix}
\end{aligned}
$$

If we use double polar coordinates $\alpha_1 = \rho_1 cos\chi_1, \alpha_2 = \rho_1 sin\chi_1$, and $\alpha_3 = \rho_2 cos\chi_2, \alpha_4 = \rho_2 sin\chi_2$, then we get

$$
\begin{cases}
\dot{\rho}_1 = \rho_1\big(-\mu\omega_2Im[\bar{a}^{-1}] + Re[b_{11}+d_{11}]\rho_2^2 + Re[b_{12}+d_{12}]\rho_1^2\big)\\
\qquad +o(|\mu|^2|,(\rho_1,\rho_2)|) + o(|(\rho_1,\rho_2)|^4)\\
\dot{\rho}_2 = \rho_2\big(-\mu\omega_2Im[\bar{a}^{-1}] + Re[b_{11}+d_{11}]\rho_1^2 + Re[b_{12}+d_{12}]\rho_2^2\big)\\
\qquad +o(|\mu|^2|,(\rho_1,\rho_2)|) + o(|(\rho_1,\rho_2)|^4)\\
\dot{\chi_1} = -\omega_2\tau_\star - \mu\omega_2Re[\bar{a}^{-1}] - Im[b_{11}+d_{11}]\rho_2^2 - Im[b_{12}+d_{12}]\rho_1^2\\
\qquad +o(|\mu|^2|,(\rho_1,\rho_2)|) + o(|(\rho_1,\rho_2)|^4)\\
\dot{\chi_2} = -\omega_2\tau_\star - \mu\omega_2Re[\bar{a}^{-1}] - Im[b_{11}+d_{11}]\rho_1^2 - Im[b_{12}+d_{12}]\rho_2^2\\
\qquad +o(|\mu|^2|,(\rho_1,\rho_2)|) + o(|(\rho_1,\rho_2)|^4)
\end{cases}
\tag{4.11}
$$

Introducing periodic variable parameters ς, and

$$z_1(t) = \alpha_1(s) + i\alpha_2(s), z_2(t) = \alpha_3(s) + i\alpha_4(s), s = \frac{t}{(1+\varsigma)\omega_2\tau_\star}$$

we obtain

$$\begin{aligned}
(1+\varsigma)\dot{z}_1(t) &= \alpha_2(s) - i\alpha_1(s) \\
&+ \frac{\mu}{\tau_\star}[-Im(\bar{a}^{-1})\alpha_1(s) + Re(\bar{a}^{-1})\alpha_2(s) - iRe(\bar{a}^{-1})\alpha_1(s) - iIm(\bar{a}^{-1})\alpha_2(s)] \\
&+ \frac{1}{\omega_2\tau_\star}\{(Re(b_{11}+d_{11})|z_2|^2 + Re[b_{12}+d_{12}]|z_1|^2)\alpha_1(s) \\
&+ i(Re(b_{11}+d_{11})|z_2|^2 + Re(b_{12}+d_{12})|z_1|^2)\alpha_2(s) \\
&+ (Im(b_{11}+d_{11})|z_2|^2 + Im(b_{12}+d_{12})|z_1|^2)\alpha_2(s) \\
&- i(Im(b_{11}+d_{11})|z_2|^2 + Im(b_{12}+d_{12})|z_1|^2)\alpha_1(s)\} + o(\mu^2|z|^2) + o(|z|^4) \\
&= -iz_1(t) - i\frac{\mu}{\tau_0}a^{-1}z_1(t) + \frac{1}{\omega_2\tau_\star}\overline{(b_{11}+d_{11})}|z_2(t)|^2 z_1(t) \\
&+ \frac{1}{\omega_2\tau_\star}\overline{(b_{12}+d_{12})}|z_1(t)|^2 z_1(t) + o(\mu^2|z|^2) + o(|z|^4)
\end{aligned}$$

Similarly, we get an equation for $z_2(t)$. Thus, ignoring the terms $o(\mu^2|z|^2) + o(|z|^4)$, we get the normal form

$$\begin{cases}
(1+\varsigma)\dot{z}_1(t) = -iz_1(t) - i\frac{\mu}{\tau_\star}a^{-1}z_1(t) + \frac{1}{\omega_2\tau_\star}\overline{(b_{11}+d_{11})}|z_2(t)|^2 z_1(t) \\
+\frac{1}{\omega_2\tau_\star}\overline{(b_{12}+d_{12})}|z_1(t)|^2 z_1(t) \\
(1+\varsigma)\dot{z}_2(t) = -iz_2(t) - i\frac{\mu}{\tau_\star}a^{-1}z_2(t) + \frac{1}{\omega_2\tau_\star}\overline{(b_{11}+d_{11})}|z_1(t)|^2 z_2(t) \\
+\frac{1}{\omega_2\tau_\star}\overline{(b_{12}+d_{12})}|z_2(t)|^2 z_2(t)
\end{cases} \tag{4.12}$$

Let $g : C \oplus C \oplus R \to C \oplus C$ be given so that $-g(z_1, z_2, \mu)$ is the right-hand side of Eq.(4.12), then Eq.(4.12) can be written as

$$(1+\varsigma)\dot{z} + g(z, \mu) = 0 \tag{4.13}$$

Note that

$$D_z g(0,0)(z_1, z_2) = i(z_1, z_2) \quad z = (z_1, z_2) \in C \oplus C$$

Also note that $g(., \mu) : C \oplus C \to C \oplus C$ is $D_3 \times S^1$–equivariant with respect to the following $D_3 \times S^1$–action on $C \oplus C$:

$$\begin{aligned}
&\gamma(z_1, z_2) = (e^{i\frac{2\pi}{3}}z_1, e^{-i\frac{2\pi}{3}}z_2) && Z_3 = \langle\gamma\rangle \le D_3 \\
&\kappa(z_1, z_2) = (z_2, z_1) && Z_2 = \langle\kappa\rangle \le D_3 \\
&e^{i\theta}(z_1, z_2) = (e^{i\theta}z_1, e^{i\theta}z_2) && e^{i\theta} \in S^1
\end{aligned}$$

According to [28] and [35], the bifurcations of small-amplitude periodic solutions of Eq.(4.13) are completely determined by the signs of three eigenvalues of

$$-i(1+\varsigma)z + g(z, \mu) = 0 \tag{4.14}$$

and their orbital stability is determined by the signs of three eigenvalues of

$$Dg(z, 0) - i(1+\varsigma)Id \tag{4.15}$$

that are not forced to zero by the group action. To be more precise, we note that Eq. (4.13) is equivalent to

$$\begin{aligned}
-i\varsigma z_1 + i\frac{\mu}{\tau_\star}a^{-1}z_1 - \frac{1}{\omega_2\tau_\star}\overline{(b_{11}+d_{11})}|z_2(t)|^2 z_1(t) - \frac{1}{\omega_2\tau_\star}\overline{(b_{12}+d_{12})}|z_1(t)|^2 z_1(t) = 0 \\
-i\varsigma z_2 + i\frac{\mu}{\tau_\star}a^{-1}z_2 - \frac{1}{\omega_2\tau_\star}\overline{(b_{11}+d_{11})}|z_1(t)|^2 z_2(t) - \frac{1}{\omega_2\tau_\star}\overline{(b_{12}+d_{12})}|z_2(t)|^2 z_2(t) = 0
\end{aligned} \tag{4.16}$$

It is known that Eq. (4.16) can be written as

$$A\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + B\begin{pmatrix} z_1^2\bar{z}_1 \\ z_2^2\bar{z}_2 \end{pmatrix}$$

with

$$A = A_0 + A_N(|z_1|^2 + |z_2|^2)$$
$$B = B_0$$

for some complex numbers A_0, A_N, B_0 given by

$$A_0 = i\frac{\mu}{\tau_\star}a^{-1} - i\varsigma$$
$$A_N = -\frac{1}{\omega_2\tau_\star}\overline{(b_{11}+d_{11})}$$
$$B_0 = \frac{1}{\omega_2\tau_\star}\left(\overline{(b_{11}+d_{11})} - \overline{(b_{12}+d_{12})}\right)$$

By the results of [6, 28] and [35], $Re(A_n + b_0) > 0$ or $Re(A_n + B_0) < 0$ determines whether the bifurcation of the phase-locked oscillation occurring in the system is supercritical or subcritical. When $Re(A_n + B_0) > 0$ and $Re(B_0) < 0$ these are orbitally asymptotically stable. In addition, $Re(2A_N + B_0) > 0$ or $Re(2A_N + B_0) < 0$ determines whether the bifurcation of mirror-reflecting waves and standing waves are supercritical or subcritical. When $Re(2A_N + B_0) > 0$ and $Re(B_0) > 0$ these are orbitally asymptotically stable.

Note that

$$\begin{aligned}
H_1 &= Re(B_0 + A_N) = Re(-\frac{1}{\omega_2\tau_\star}\overline{(b_{12}+d_{12})}) \\
&= \frac{1}{2\omega_2^2}[\omega_2 p(r_1K_8 + r_2K_9) - \tau_\star K_5] \\
H_2 &= Re(2A_N + B_0) = Re\Big(-\frac{1}{\omega_2\tau_\star}(\overline{(b_{11}+d_{11})} + \overline{(b_{12}+d_{12})})\Big) \\
&= \frac{1}{2\omega_2^2}[\omega_2 p(r_1(K_6 + K_8) + r_2(K_7 + K_9)) + \tau_\star(K_2 - K_5)] \\
H_3 &= Re(B_0) = Re\Big(\frac{1}{\omega_2\tau_\star}\big(\overline{(b_{11}+d_{11})} - \overline{(b_{12}+d_{12})}\big)\Big) \\
&= \frac{1}{2\omega_2^2}[\omega_2 p(r_1(K_8 - K_6) + r_2(K_9 - K_7)) - \tau_\star(K_2 + K_5)]
\end{aligned}$$

Theorem 4.1. *Assume $b > 1$ and define $(\tau_\star, \omega_2)$ as in (2.6), near the critical value $\tau = \tau_\star$. Eight asynchronous periodic solutions of Eq. (1.2) are branched from the trivial solution $x = 0$, and the periodic T is close to $(\frac{2\pi}{\omega_2})$. These waves are*
1. If $H_1 < 0$, there exists two supercritical phase-locked oscillation bifurcations: $x_i(t) = x_{i-1}(t \pm \frac{T}{3})$, for $i(mod3), t \in R$, and bifurcated periodic solution exists at $\tau > \tau_\star$, otherwise it's subcritical, and these are orbitally asymptotically stable if and only if $H_1 > 0, H_3 < 0$.
2. If $H_2 < 0$, there exists three mirror-reflecting waves: $x_i(t) = x_j(t) \neq x_k(t)$, for $t \in R$ and for some distinct (i, j, k) in $(1, 2, 3)$. Three standing waves: $x_i(t) = x_{ij}(t + \frac{T}{2})$,for $t \in R$ and for some pair of distinct elements (i, j) in $(1, 2, 3)$, and bifurcated periodic solution exists at $\tau > \tau_\star$, otherwise it's subcritical, and these are orbitally asymptotically stable if and only if $H_2 > 0, H_3 > 0$.

5 Numerical simulations

In this part, we use Matlab to simulate the research results of Eq.(1.2).

As shown in fig.1, take $b = 0.8795, \tau = 3.59$, Eq.(1.2) has an asymptotic stable equilibrium point $(0.362, 0.362, 0.362)$. That means the growth of three identical species gradually tends to a balanced state.

As shown in fig.2, take $b = 1.0009, \tau = 16.9$, Eq.(1.2) occurs the synchronous bifurcating periodic solutions. That means three identical species will change synchronously and periodically over a period of time, eventually reach an equilibrium state.

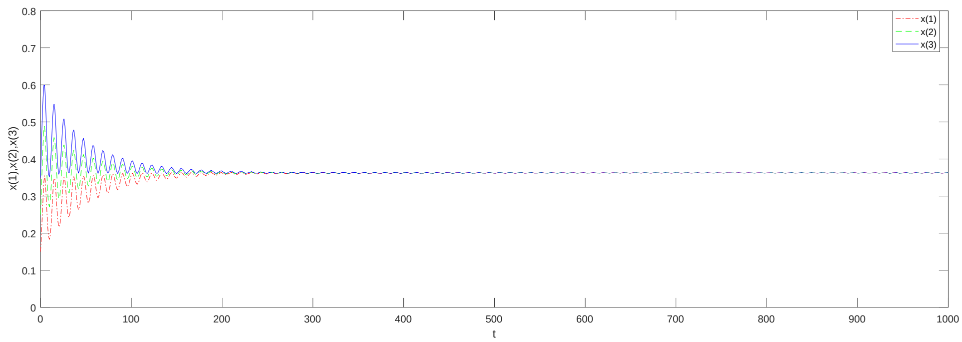

Figure 1: Equilibrium point $(0.362, 0.362, 0.362)$ is asymptotic stability with $b = 0.8795$, $\tau = 3.59$.

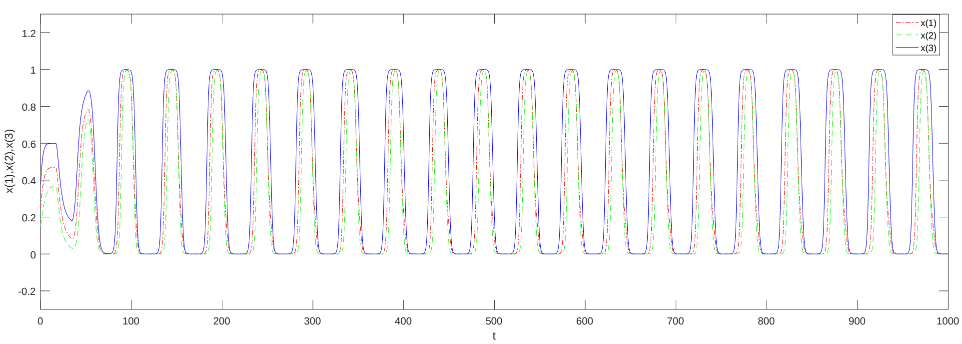

Figure 2: Trajectories $x(1)$, $x(2)$ and $x(3)$ of system (1.1) with $b = 1.0009$, $\tau = 16.9$.

6 Conclusion

This paper introduces a typical biological model: Lotka-Volterra, which is mainly used in space ecology, disease transmission and species evolution. In recent years, many mathematicians have made gratifying achievements in the study of this model. However, due to the necessary of considering the ′evolution′ problem, delay differential equation can often describe a real development system more objectively than ordinary differential equations. In this paper, we introduce time-delay into the Lotka-Volterra model based on biological background, considering the introduction of time-delay into two adjacent species and making the coefficient of influence between adjacent species is $b(b > 0)$. under such a circumstance and based on the theory of Equivariant Hopf bifurcation, we discuss that the Lotka-Volterra ring system composed by three species can produce some singular and interesting bifurcation phenomena.

According to the stability theory of symmetric periodic solutions of Golubitsky [6], many scholars have made classical academic researches. Based on the three dimensional ring neural network model, Wu [28] extended the symmetric local Hopf bifurcation theory to delay differential equation, and gave a feasible method to solve the case of non-single pairs of purely imaginary eigenvalues. Guo [35] and Fan [39] discussed the bifurcation of the n-dimensional ring neural network model. There is no second-order term in the normal form given by the former, and there is a simple second-order term in the normal form given by the latter. We have studied the Equivariant Hopf bifurcation of the symmetric Lotka-Volterra ring system with delay on the basic of predecessors' theory. When the eigenvalues have multiple pure imaginary roots, the classical Hopf bifurcation theorem of delay differential equation have lost its validity. Since we introduce the concept of Lie group, we know that the system that we consider now is D_3-Equivariant. This enables us to use the Hopf bifurcation existence theorem of the delay differential equation with symmetric structure of Wu [28] to obtain the bifurcation periodic solutions of the system. In this model, the normal form contains such cumbersome second and third order terms, and different oscillation periodic solutions are obtained by simplification. It is summarized that when the time delay varies and passes through some critical values, eight asynchronous periodic solutions can be derived at the zero equilibrium point of the system in some specific subdomains. Among them, there are two stable phase-locked periodic solutions, three unstable mirror reflections and three unstable standing waves.

A Appendix

In this Appendix, we well compute $h_{2000}(0)$, $h_{2000}(-1)$, $h_{0020}(0)$, $h_{0020}(-1)$, $h_{1100}(0)$, $h_{1100}(-1)$, $h_{0011}(0)$, $h_{0011}(-1)$, $h_{1010}(0)$, $h_{1010}(-1)$, $h_{1001}(0)$, $h_{1001}(-1)$ in expressions of $d_{11}, d_{16}, d_{24}, d_{26}$.

Lemma A.1 Solving Eq.(4.8),we obtain

$$\begin{aligned}
h^{(1)}_{2000}(0) = {} & \frac{1}{3b^3c^4(bc^2+2ibc\omega_2-c^2-2ic\omega_2+\omega_2^2)}\big(3\bar{a}^{-1}b^3c^5+6i\bar{a}^{-1}b^3c^4\omega_2\\
& -3\bar{a}^{-1}b^3c^3\omega^2-3\bar{a}^{-1}b^2c^5-3i\bar{a}^{-1}b^2c^4\omega_2-3\bar{a}^{-1}b^2c^3\omega_2^2-3i\bar{a}^{-1}b^2c^2\omega_2^3+a^{-1}b^3c^5\\
& +2ia^{-1}b^3c^4\omega_2-a^{-1}b^3c^3\omega_2^2+a^{-1}c^5+ia^{-1}c^4\omega_2-2a^{-1}c^3\omega_2^2+2ia^{-1}c^2\omega_2^3-a^{-1}c\omega_2^4\\
& +ia^{-1}\omega_2^5-6ib^4c^5p\tau_\star\omega_2-6ib^4c^4p\omega_2+3ib^4c^4\omega_2\big)
\end{aligned}$$

$$\begin{aligned}
h^{(1)}_{2000}(-1) = {} & \frac{1}{3b^5c^4(bc^2+2ibc\omega_2-c^2-2ic\omega_2+\omega_2^2)}\big(3\bar{a}^{-1}b^4c^5+9i\bar{a}^{-1}b^4c^4\omega_2-6\bar{a}^{-1}b^4c^3\omega_2^2\\
& -3\bar{a}^{-1}b^3c^5-6i\bar{a}^{-1}b^3c^4\omega_2-3\bar{a}^{-1}b^3c^3\omega_2^2 6i\bar{a}^{-1}b^3c^2\omega_2^3+a^{-1}b^4c^5+3ia^{-1}b^4c^4\omega_2\\
& -2a^{-1}b^4c^3\omega_2^2-a^{-1}bc^5-4a^{-1}bc^3\omega_2^2+2ia^{-1}bc^2\omega_2^3-3a^{-1}bc\omega_2^4+2ia^{-1}b\omega_2^5\\
& +2a^{-1}c^5+2ia^{-1}c^4\omega_2-6ib^4c^5p\tau_\star\omega_2+6b^4c^4p\tau_\star\omega_2^2-6ib^4c^4p\omega_2+3ib^4c^4\omega_2\\
& +6b^4c^3p\omega^2-3b^4c^3\omega^2\big)
\end{aligned}$$

Proof.
From the first equation of Eq.(4.8), we have

$$h_{2000}(\theta) = e^{2i\omega_2\tau_\star\theta}\int_0^\theta e^{-2i\omega_2\tau_\star t}\Phi(t)\Psi(0)A_{2000}dt + \tilde{c}e^{2i\omega_2\tau_\star\theta}$$

where $\tilde{c} \in C^3$ is a constant and hence

$$\dot{h}_{2000}(0) = \Phi(0)\Psi(0)A_{2000} + \tilde{c}2i\omega_2\tau_\star$$

and

$$\mathcal{L}(h_{2000}) = \mathcal{L}(e^{2i\omega_2\tau_\star\theta})\tilde{c} + \mathbb{B}e^{-2i\omega_2\tau_\star}\int_0^{-1}\Phi(t)\Psi(0)e^{-2i\omega_2\tau_\star t}A_{2000}dt$$

From the second equation of Eq. (4.8), we have

$$(2i\omega_2\tau_\star I - \mathcal{L}(e^{2i\omega_2\tau_\star\theta})\tilde{c} = (I-\Phi(0)\Psi(0))A_{2000} + \mathbb{B}e^{-2i\omega_2\tau_\star}\int_0^{-1}\Phi(t)\Psi(0)e^{-2i\omega_2\tau_\star t}A_{2000}dt$$

Since $2i\omega_2\tau_\star$ is not an eigenvalue of $\mathcal{L}$, the matrix $(2i\omega_2\tau_\star I - \mathcal{L}(e^{2i\omega_2\tau_\star\theta}))$ is invertible. So we have

$$\tilde{c} = (2i\omega_2\tau_\star I - \mathcal{L}(e^{2i\omega_2\tau_\star\theta})^{-1}(I-\Phi(0)\Psi(0))A_{2000} + \mathbb{B}e^{-2i\omega_2\tau_\star}\int_0^{-1}\Phi(t)\Psi(0)e^{-2i\omega_2\tau_\star t}A_{2000}dt$$

By using the software Mathematica, after easy but long computation, we have the expression of $h_{2000}(0)$ and $h_{2000}(-1)$.

Lemma A.2 Solving Eq.(4.9), we obtain

$$h^{(1)}_{1010}(0) = \frac{2(ib^2c^2\omega_2 - 2b^2c\omega_2^2 + ibc^2\omega_2 + 2bc\omega_2^2 + ib\omega_2^3)}{(bc^2 + 2ibc\omega_2 + 2c^2 + 4ic\omega_2 - 2\omega_2^2)(bc^2 + 2ibc\omega_2 - c^2 - 2ic\omega_2 + \omega_2^2)}$$

$$h^{(1)}_{1010}(-1) = \frac{2(ibc^3\omega_2 - 3bc^2\omega_2^2 - 2ibc\omega_2^3 + ic^3\omega_2 + c^2\omega_2^2 + 3ic\omega_2^3 - \omega_2^4)}{bc(bc^2 + 2ibc\omega_2 + 2c^2 + 4ic\omega_2 - 2\omega_2^2)(bc^2 + 2ibc\omega_2 - c^2 - 2ic\omega_2 + \omega_2^2)}$$

Proof.

From the first equation of Eq.(4.9), we have

$$h_{1010}(\theta) = e^{2i\omega_2\tau_\star\theta}\int_0^\theta e^{-2i\omega_2\tau_\star t}\Phi(t)\Psi(0)A_{1010}dt + \tilde{c}e^{2i\omega_2\tau_\star\theta}$$

where $\tilde{c} \in C^3$ is a constant and hence

$$\dot{h}_{1010}(0) = \Phi(0)\Psi(0)A_{1010} + \tilde{c}2i\omega_2\tau_\star$$

and

$$\mathcal{L}(h_{1010}) = \mathcal{L}(e^{2i\omega_2\tau_\star\theta})\tilde{c} + \mathbb{B}e^{-2i\omega_2\tau_\star}\int_0^{-1}\Phi(t)\Psi(0)e^{-2i\omega_2\tau_\star t}A_{1010}dt$$

From the second equation of Eq.(4.9), we have

$$(2i\omega_2\tau_\star I - \mathcal{L}(e^{2i\omega_2\tau_\star\theta})\tilde{c} = (I - \Phi(0)\Psi(0))A_{1010} + \mathbb{B}e^{-2i\omega_2\tau_\star}\int_0^{-1}\Phi(t)\Psi(0)e^{-2i\omega_2\tau_\star t}A_{1010}dt$$

Since $2i\omega_2\tau_\star$ is not an eigenvalue of $\mathcal{L}$, the matrix $(2i\omega_2\tau_\star I - \mathcal{L}(e^{2i\omega_2\tau_\star\theta}))$ is invertible. So we have

$$\tilde{c} = (2i\omega_2\tau_\star I - \mathcal{L}(e^{2i\omega_2\tau_\star\theta})^{-1}(I - \Phi(0)\Psi(0))A_{1010} + \mathbb{B}e^{-2i\omega_2\tau_\star}\int_0^{-1}\Phi(t)\Psi(0)e^{-2i\omega_2\tau_\star t}A_{1010}dt$$

By using the software Mathematica, after easy but long computation, we have the expression of $h_{1010}(0)$ and $h_{1010}(-1)$.

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