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Research Article

Jun Zhao, Emmanuel Lépinette, and Peibiao Zhao*

Pricing under dynamic risk measures

Abstract: In this paper, we study the discrete-time super-replication problem of contingent claims with respect to an acceptable terminal discounted cash flow. Based on the concept of Immediate Profit, i.e., a negative price which super-replicates the zero contingent claim, we establish a weak version of the fundamental theorem of asset pricing. Moreover, time consistency is discussed and we obtain a representation formula for the minimal super-hedging prices of bounded contingent claims.

Keywords: Super-hedging, Dynamic risk measures, Time consistency, Absence of immediate profit, Pricing

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1 Introduction

In mathematical finance, it is very classical to solve the problem of super-replicating a contingent claim under a no-arbitrage condition (NA). In particular, in frictionless markets, the so-called fundamental theorem of asset pricing (FTAP) characterising NA condition has been studied by numerous authors, see [1–3] in discrete time and [4, 5] in continuous time. It states that NA condition holds if and only if there exist equivalent martingale measures (EMM). In complete markets, such a martingale measure $Q \sim P$ is unique and the (replicating) price of a derivative is uniquely computed as the expectation of the discounted payoff under Q. However, in incomplete markets, there exists an infinite number of EMM and the (minimal) super-hedging price is difficult to compute in practice. Indeed, this is a supremum of the expected discounted payoff over all probability measures (see [6] and [7, Theorem 2.1.11]).

A new pricing technique called No Good Deal (NGD) pricing has been proposed in [8, 9]. A good deal is a trade with an unusually high profit/loss or Sharpe ratio. Cherny [10] introduced the concept of good deal with respect to a risk measure as a trade with negative risk. Contrarily to the classical approach where super-replication holds almost surely, Cherny assumes that the agent seller accepts some non null risk for its portfolio not to super-hedge the payoff. In the setting of coherent risk measures, Cherny [10] provides a version of the FTAP under absence of NGD.

Risk measures are more studied and known on the space L^∞, i.e. the space of essentially bounded random variables. And the space L^p, $p \in [1, \infty)$ is a natural extension, see [11, 12]. Actually, working on the restricted subspaces of L^0, such as L^∞ and L^p, is mainly motivated by the robust representation of risk measures. However, the space L^0, equipped with the topology of convergence in probability, is more adapted for some classical financial and actuarial problems such as hedging, pricing, portfolio choice, equilibrium and optimal reinsurance with respect to risk measures.

Jun Zhao: Department of Applied Mathematics, Nanjing University of Science and Technology, Nanjing 210094, Jiangsu, P.R. China; E-mail: zhaojun.njust@hotmail.com
Emmanuel Lépinette: Ceremade, UMR CNRS 7534, Paris Dauphine University, PSL National Research, Place du Maréchal De Lattre De Tassigny, 75775 Paris cedex 16, France; E-mail: emmanuel.lepinette@ceremade.dauphine.fr
*Corresponding Author: Peibiao Zhao: Department of Applied Mathematics, Nanjing University of Science and Technology, Nanjing 210094, Jiangsu, P.R. China; E-mail: pbzhao@njust.edu.cn

Delbaen in [13, 14] extends the coherent risk measure to the space L^0 by enlarging its range to $\mathbf{R} \cup \{+\infty\}$ as there is no real-valued coherent risk measure on L^0 when the probability space $(\Omega, \mathcal{F}, \mathrm{P})$ is atomless [14, Theorem 5.1]. A robust representation with respect to a set of probability measures is then given [14, Theorem 5.4]. As the space L^0 contains non integrable random variables, Delbaen in [14] truncates the random variables from above, i.e. only considers possible future wealth up to some threshold. It is then possible to compute the risk measures as in L^∞ and then make n tend to infinity [14, Definition 5.3]). Therefore, the robust representation on L^∞ appears to be the key point to extend coherent risk measure to L^0, see ([10, Definition 2.2] and [15]), which allow to formulate a FTAP with respect to NGD and solve super-replication problems. In this approach, coherent risk measures remain characterised through families of probability measures which are not necessarily easy to handle in practice, see e.g. the explicit representation of this family for the Weighted VaR risk measure [16, 17].

In this paper, we define risk measures on the space L^0 with values in $\overline{\mathbf{R}} = [-\infty, +\infty]$. They are naturally defined through the concept of acceptable set, i.e. a risk measure is seen as the minimal capital requirement added to the position for it to be acceptable. Under some natural assumptions satisfied by the acceptable set, we show that a risk measure is lower semi-continuous. This allows to compute ω-wise risk measure using similar new results on conditional essential supremum [18]. Inspired by [10], the aim of this paper is to reconsider the super-replication problem in discrete-time with respect to a risk measure without using a dual representation. The minimal super-hedging prices of a contingent claim are recursively defined in the spirit of [18].

Based on the concept of immediate profit, introduced in [18], we establish a weak version of FTAP to equivalently characterise the condition of absence of immediate profit (AIP). Moreover, we show that for bounded non-negative contingent claims, the minimal super-hedging price may be computed through a conditional (dynamic) coherent risk measure derived from the underlying risk measure. At last, we discuss the time consistency, i.e. coherent evaluations of risk in time, since it is a very important concept developed in the literatures for dynamic risk measures, see [19, 20].

The paper is organized as follows. Section 2 gives the definition of risk measures and some important properties for these risk measures are showed. Section 3 introduces the model of super-replication with respect to acceptable sets. We simplify the problem of minimal super-hedging price involving the essential infimum into a classic minimization problem just with infimum. In Section 4, a weak version of fundamental theorem of asset pricing is proved. Section 5 gives a price representation for the bounded non-negative contingent claims.

2 Dynamic risk measure

Notations:

$L^0(\mathbf{R}, \mathcal{F})$ is the metric space of all $\mathbf{R}$-valued random variables which are $\mathcal{F}$-measurable;

$L^p(\mathbf{R}, \mathcal{F}, \mathrm{P})$, $p \in [1, \infty)$ (resp. $p = \infty$), is the normed space of all $\mathbf{R}$-valued random variables which are $\mathcal{F}$-measurable and admit a moment of order p under the probability P (resp. bounded). Without any confusions, we omit the notation P and just denote $L^p(\mathbf{R}, \mathcal{F})$;

$L^p(\mathbf{R}_+, \mathcal{F}) := \{X \in L^p(\mathbf{R}, \mathcal{F}) | X \geq 0\}$, $L^p(\mathbf{R}_{++}, \mathcal{F}) := \{X \in L^p(\mathbf{R}, \mathcal{F}) | X > 0\}$ and $L^p(\mathbf{R}_-, \mathcal{F}) := \{X \in L^p(\mathbf{R}, \mathcal{F}) | X \leq 0\}$;

In the following, we consider a complete discrete-time stochastic basis $(\Omega, \mathcal{F} := (\mathcal{F}_t)_{t=0,\cdots,T}, \mathrm{P})$ where $\mathcal{F}_t$ represents the available information of the market at time t;

$\mathbb{E}_\mathrm{P}$ and $\mathbb{E}_\mathrm{Q}$ are the expectations of any integrable random variable with respect to the probability measure P and Q. In general, we denote $\mathbb{E}_\mathrm{P}$ as $\mathbb{E}$ without of any confusions. All equalities and inequalities of random variables are understood up to a negligible set.

The dynamic risk measure $X \mapsto (\rho_t(X))_{t=0,\cdots,T}$ we consider is defined on L^0. It is constructed from its acceptance sets defined as follows:

Definition 2.1. *A dynamic acceptable set is a family $(\mathcal{A}_t)_{t=0,\cdots,T}$ of subsets of $L^0(\mathbf{R}, \mathcal{F}_T)$ satisfying the following conditions:*

(1) $X + Y \in \mathcal{A}_t$ for all $X, Y \in \mathcal{A}_t$;

(2) $Y \in \mathcal{A}_t$ whenever $Y \geq X$ for some $X \in \mathcal{A}_t$;

(3) $\mathcal{A}_t \cap L^0(\mathbf{R}, \mathcal{F}_t) = L^0(\mathbf{R}_+, \mathcal{F}_t)$;

(4) $k_t X \in \mathcal{A}_t$ for any $X \in \mathcal{A}_t$ and $k_t \in L^0(\mathbf{R}_+, \mathcal{F}_t)$.

Any element of $\mathcal{A}_t$ is said acceptable at time t. For any $X \in L^0(\mathbf{R}, \mathcal{F}_T)$, we denote by $\mathcal{A}_t^X$ the set of all $Y \in L^0(\mathbf{R}, \mathcal{F}_t)$ such that $X + Y \in \mathcal{A}_t$.

Definition 2.2. *Let $(\mathcal{A}_t)_{t=0,\cdots,T}$ be a dynamic acceptance set. The risk measure associated to $(\mathcal{A}_t)_{t=0,\cdots,T}$ is, at time t, the mapping $\rho_t : L^0(\mathbf{R}, \mathcal{F}_T) \to L^0(\mathbf{R} \cup \{-\infty, +\infty\}, \mathcal{F}_t)$ defined as*

$$\rho_t(X) := \text{ess inf } \mathcal{A}_t^X \tag{2.1}$$

up to a negligible set.

Observe that $\rho_t(X)$ is the the minimal capital requirement we add to the position X for it to be acceptable at time t. The effective domain of ρ_t is denoted as

$$\text{dom } \rho_t := \{X \in L^0(\mathbf{R}, \mathcal{F}_T) | \rho_t(X) < +\infty\}.$$

In this paper, we just consider the positions whose risk measures are not infinite at any time t. In other words, we assume that $\rho_t(X) < +\infty$ for any $X \in L^0(\mathbf{R}, \mathcal{F}_T)$.

Lemma 2.3. *For any $X \in L^0(\mathbf{R}, \mathcal{F}_T)$, there exists a sequence $Y_n \in \mathcal{A}_t^X$ such that $\rho_t(X) = \lim_{n \to \infty} \downarrow Y_n$ a.s.*

Proof. We first observe that the set $\mathcal{A}_t^X$ is $\mathcal{F}_t$-decomposable, i.e. if $\Lambda_t \in \mathcal{F}_t$ and $Y_1, Y_2 \in \mathcal{A}_t^X$, then $Y_1 1_{\Lambda_t} + Y_2 1_{\Omega \backslash \Lambda_t} \in \mathcal{A}_t^X$. To see it, we use conditions 1) and 4) of Definition 2.1. We then deduce that $\mathcal{A}_t^X$ is directed downward, i.e. if $Y_1, Y_2 \in \mathcal{A}_t^X$, then $Y_1 \wedge Y_2 \in \mathcal{A}_t^X$. Indeed, $Y_1 \wedge Y_2 = Y_1 1_{\{Y_1 \leq Y_2\}} + Y_2 1_{\{Y_1 > Y_2\}}$ with $\{Y_1 \leq Y_2\} \in \mathcal{F}_t$. Therefore, there exists a sequence $Y_n \in \mathcal{A}_t^X$ such that $\rho_t(X) = \lim_{n \to \infty} \downarrow Y_n$ a.s., see [7, Section 5.3.1.]

The following proposition is straightforward due to the definition. The proofs are showed in the Appendix C.

Proposition 2.4. *The risk measure ρ_t defined as (2.1) satisfies the following properties:*

Normalization: $\rho_t(0) = 0$;

Monotonicity: $X \leq X'$ means $\rho_t(X) \geq \rho_t(X')$;

Cash invariance: for all $m_t \in L^0(\mathbf{R}, \mathcal{F}_t)$, $\rho_t(X + m_t) = \rho_t(X) - m_t$;

Subadditivity: for all $X, X' \in L^0(\mathbf{R}, \mathcal{F}_T)$, $\rho_t(X + X') \leq \rho_t(X) + \rho_t(X')$;

Positive homogeneity: for all $k \in L^0(\mathbf{R}_+, \mathcal{F}_t)$, $\rho_t(kX) = k\rho_t(X)$.

Moreover, if acceptable set $\mathcal{A}_t$ is closed, then ρ_t is lower semi-continuous a.s. with the constraint $\rho_t(X) > -\infty$ a.s. for all $X \in L^0(\mathbf{R}, \mathcal{F}_T)$ and $\mathcal{A}_t$ can be represented by ρ_t:

$$\mathcal{A}_t = \{X \in L^0(\mathbf{R}, \mathcal{F}_T) | \rho_t(X) \leq 0\}. \tag{2.2}$$

Definition 2.5. *A system $(\rho_t)_{0 \leq t \leq T}$ is called dynamic risk measure if ρ_t is a risk measure function defined as (2.1) for each $0 \leq t \leq T$.*

3 Minimal super-hedging prices

In the discrete-time model, let $(S_t)_{0 \leq t \leq T}$ be the discounted price process of asset where $S_t \in L^0(\mathbf{R}_+, \mathcal{F}_t)$. And $(\rho_t)_{0 \leq t \leq T}$ is dynamic risk measure defined in Definition 2.5. A contingent claim at time T is denoted by

a real-valued $\mathcal{F}_T$-measurable random variable h_T. The question is to find a self-financing strategy process $(\theta_t)_{0 \le t \le T}$ to super-replicate the contingent claim h_T. Here we use the concept of super-replication in the sense of acceptable set, that is the resulting risk is negative, instead of super-hedging almost surely as the most literatures did. In fact, super-replication almost surely usually can not be realized in a real market.

First let us start with the one step model, that is to super-replicate the contingent claim h_T at time $T - 1$. And the acceptable set $\mathcal{A}_{T-1}$ is assumed to be closed in this section. An notion of super-hedging with respect to the acceptable set is given as follows. In this paper, we just consider the contingent claims which can be super-hedged in the sense of the following definition.

Definition 3.1. *Contingent claim h_T is said to be super-hedged at time $T - 1$ if there exists some $P_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1})$ and strategy $\theta_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1})$ such that $P_{T-1} + \theta_{T-1}\Delta S_T - h_T \in \mathcal{A}_{T-1}$. And P_{T-1} are called the super-hedging prices of the contingent claim h_T at time $T - 1$.*

We show that h_T can be super-hedged if it satisfies the condition $h_T \le a_{T-1}S_T + b_{T-1}$ where $a_{T-1}, b_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1})$. In detail, take $\theta_{T-1} = a_{T-1}$ and $P_{T-1} = a_{T-1}S_{T-1} + b_{T-1}$, then $P_{T-1} + \theta_{T-1}\Delta S_T - h_T \ge 0$ such that $P_{T-1} + \theta_{T-1}\Delta S_T - h_T \in \mathcal{A}_{T-1}$ since $\mathcal{A}_t \cap L^0(\mathbf{R}, \mathcal{F}_t) = L^0(\mathbf{R}_+, \mathcal{F}_t)$.

The set $\mathcal{P}_{T-1}(h_T)$ consists of all super-hedging prices at time $T - 1$, that is

$$\mathcal{P}_{T-1}(h_T) := \{P_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1}) | \exists\, \theta_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1}) \text{ s.t.} P_{T-1} + \theta_{T-1}\Delta S_T - h_T \in \mathcal{A}_{T-1}\}.$$

Since we assume that the contingent claims of consideration can be super-hedged, that is to say, we may suppose that $\mathcal{P}_{T-1}(h_T) \neq \emptyset$. According to (2.2) and the cash invariance property of ρ_{T-1}, $P_{T-1} + \theta_{T-1}\Delta S_T - h_T \in \mathcal{A}_{T-1}$ if and only if $P_{T-1} \ge \theta_{T-1}S_{T-1} + \rho_{T-1}(\theta_{T-1}S_T - h_T)$. Then the set $\mathcal{P}_{T-1}(h_T)$ can be equivalently written as

$$\mathcal{P}_{T-1}(h_T) = \{\theta_{T-1}S_{T-1} + \rho_{T-1}(\theta_{T-1}S_T - h_T) : \theta_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1})\} + L^0(\mathbf{R}_+, \mathcal{F}_{T-1}).$$

Let

$$g(\omega, x) := xS_{T-1} + \rho_{T-1}(xS_T - h_T),$$

then the set of super-hedging prices can be expressed as

$$\mathcal{P}_{T-1}(h_T) = \{g(\theta_{T-1}) : \theta_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1})\} + L^0(\mathbf{R}_+, \mathcal{F}_{T-1}). \tag{3.3}$$

Actually, we may construct a jointly measurable version of the random function $g(\omega, x)$ such that $g(\theta_{T-1}) = \theta_{T-1}S_{T-1} + \rho_{T-1}(\theta_{T-1}S_T - h_T)$. And we can prove that $g(\omega, x)$ is convex and lower semi-continuous in x for almost all ω under the assumption that the acceptable set $\mathcal{A}_{T-1}$ is closed.

Lemma 3.2. *Let $\mathcal{G}_{T-1} = \{(X, Y) \in L^0(\mathbf{R}^2, \mathcal{F}_{T-1}) | Y \ge XS_{T-1} + \rho_{T-1}(XS_T - h_T)\}$. Then, $\mathcal{G}_{T-1}$ is a non-empty closed convex subset of $L^0(\mathbf{R}^2, \mathcal{F}_{T-1})$. Moreover, $\mathcal{G}_{T-1}$ is $\mathcal{F}_{T-1}$-decomposable such that $\mathcal{G}_{T-1} = L^0(G_{T-1}, \mathcal{F}_{T-1})$ for some non-empty $\mathcal{F}_{T-1}$-measurable random closed convex set G_{T-1}.*

Proof. Trivially $\mathcal{G}_{T-1}$ is closed and convex since $\mathcal{A}_{T-1}$ is supposed to be closed and a convex cone. And $\mathcal{G}_{T-1} \neq \emptyset$ since $\mathcal{P}_{T-1}(h_T) \neq \emptyset$ from the assumption. Moreover, $\mathcal{A}_{T-1}$ is $\mathcal{F}_{T-1}$-decomposable and so $\mathcal{G}_{T-1}$ is. Thus we can deduce that $\mathcal{G}_{T-1} = L^0(G_{T-1}, \mathcal{F}_{T-1})$ for some $\mathcal{F}_{T-1}$-measurable random closed set G_{T-1}, see [7, Proposition 5.4.3]. As $\mathcal{G}_{T-1}$ is not empty, we deduce that $G_{T-1} \neq \emptyset$ a.s. Moreover, there exists a Castaing representation of G_{T-1} such that $G_{T-1}(\omega) = \text{cl}\,\{Z^n(\omega) : n \ge 1\}$ for every $\omega \in \Omega$, where $(Z^n)_{n \ge 1}$ is a countable family of $\mathcal{G}_{T-1}$, see [21, Proposition 2.7]. Then, by a contradiction argument and using a measurable selection argument, we may show that G_{T-1} is convex as $\mathcal{G}_{T-1}$. □

Proposition 3.3. *There exists a $\mathcal{F}_{T-1} \times \mathcal{B}(\mathbf{R})$-measurable function g_{T-1} such that $G_{T-1} = \{(x, y) : y \ge g(\omega, x)\}$ and $Y \ge XS_{T-1} + \rho_{T-1}(XS_T - h_T)$ if and only if $Y \ge g_{T-1}(X)$ where $X, Y \in L^0(\mathbf{R}, \mathcal{F}_{T-1})$. Moreover, $x \mapsto g(\omega, x)$ is a.s. convex and lower semi-continuous.*

Proof. Define the following random function

$$g(\omega, x) := \inf\{\alpha \in \mathbf{R} : (x, \alpha) \in G_{T-1}(\omega)\}. \tag{3.4}$$

We first show that g is $\mathcal{F}_{T-1} \times \mathcal{B}(\mathbf{R})$-measurable. To see it, since the x-sections of G_{T-1} are upper sets, we get that $g(\omega, x) := \inf\{\alpha \in \mathbf{Q} : (x, \alpha) \in G_{T-1}(\omega)\}$ where $\mathbf{Q}$ is the set of all rational numbers of $\mathbf{R}$. Let us define the $\mathcal{F}_{T-1} \times \mathcal{B}(\mathbf{R})$-measurable function $I(\omega, x) = 1$ if $(\omega, x) \in G_{T-1}$ and $I(\omega, x) = +\infty$ if $(\omega, x) \notin G_{T-1}$. Then, define, for each $\alpha \in \mathbf{Q}$, $\theta^\alpha(\omega, x) = \alpha I(\omega, x)$ with the convention $\mathbf{R} \times (+\infty) = +\infty$. As θ^α is $\mathcal{F}_{T-1} \times \mathcal{B}(\mathbf{R})$-measurable, we deduce that $g(\omega, x) = \inf_{\alpha \in \mathbf{Q}} \theta^\alpha(\omega, x)$ is also $\mathcal{F}_{T-1} \times \mathcal{B}(\mathbf{R})$-measurable.

Since G_{T-1} is closed, it is clear that $(x, g(\omega, x)) \in G_{T-1}(\omega)$ a.s. when $g(\omega, x) < \infty$ and, moreover, $g(\omega, x) > -\infty$ by Proposition 2.4. Therefore, $G_{T-1}(\omega)$ is the epigraph of the random function $x \mapsto g(\omega, x)$. As $Y \geq XS_{T-1} + \rho_{T-1}(XS_T - h_T)$ if and only if $(X, Y) \in \mathcal{G}_{T-1}$, or equivalently $(X, Y) \in G_{T-1}$ a.s., we deduce that it is equivalent to $Y \geq g(X)$.

Moreover, as G_{T-1} is convex, we deduce that $x \mapsto g(\omega, x)$ is a.s. convex. Let us show that $x \mapsto g(\omega, x)$ is a.s. lower-semi continuous. Consider a sequence $x^n \in \mathbf{R}$ which converges to $x_0 \in \mathbf{R}$. Let us denote $\beta^n := g(x^n)$. We have $(x^n, \beta^n) \in G_{T-1}$ from the above discussion. In the case where $\inf_n \beta^n = -\infty$, $g(\omega, x) - 1 > \beta_n$ for n large enough (up to a subsequence) hence $(x^n, g(\omega, x) - 1) \in G_{T-1}(\omega)$ since the x^n-sections of G_{T-1} are upper sets. As $n \to \infty$, we deduce that $(x, g(\omega, x) - 1) \in G_{T-1}(\omega)$. This contradicts the definition of g. Moreover, the inequality $g(x) \leq \liminf_n \beta^n$ is trivial when the right hand side is $+\infty$. Otherwise, $\beta^\infty := \liminf_n \beta^n < \infty$ and $(x_0, \beta^\infty) \in G_{T-1}$ as G_{T-1} is closed. It follows by definition of g that $g(x_0) \leq \liminf_n g(x^n)$, i.e. g is lower-semi continuous. $\qquad\square$

Corollary 3.4. *We have $g(X) = XS_{T-1} + \rho_{T-1}(XS_T - h_T)$ a.s. whatever $X \in L^0(\mathbf{R}, \mathcal{F}_{T-1})$.*

Proof. Consider a measurable selection $(x_{T-1}, y_{T-1}) \in \mathcal{G}_{T-1} \neq \emptyset$. We have $y_{T-1} \geq g(x_{T-1})$ by definition hence $g(x_{T-1}) < \infty$ a.s. Let us define $X_{T-1} = x_{T-1} 1_{\{g(X)=\infty\}} + X 1_{\{g(X)<\infty\}}$. Since we have

$$g(X_{T-1}) = g(x_{T-1}) 1_{\{g(X)=\infty\}} + g(X) 1_{\{g(X)<\infty\}},$$

is a.s. finite, $(X_{T-1}, g(X_{T-1})) \in G_{T-1}$ a.s. We deduce that

$$g(X_{T-1}) \geq X_{T-1} S_{T-1} + \rho_{T-1}(X_{T-1} S_T - h_T)$$

as $\mathcal{G}_{T-1} = L^0(G_{T-1}, \mathcal{F}_{T-1})$. Therefore, $g(X) \geq XS_{T-1} + \rho_{T-1}(XS_T - h_T)$ on the set $\{g(X) < \infty\}$. Moreover, the inequality trivially holds when $g(X) = +\infty$. Similarly, let us define

$$Y_{T-1} = \left(XS_{T-1} + \rho_{T-1}(XS_T - h_T)\right) 1_{\{XS_{T-1}+\rho_{T-1}(XS_T-h_T)<\infty\}}$$
$$+ y_{T-1} 1_{\{XS_{T-1}+\rho_{T-1}(XS_T-h_T)=+\infty\}}.$$

We have $(X_{T-1}, Y_{T-1}) \in \mathcal{G}_{T-1}$ a.s. hence, by definition of g, $g(X_{T-1}) \leq Y_{T-1}$. Then, $g(X) \leq XS_{T-1} + \rho_{T-1}(XS_T - h_T)$ on $\{XS_{T-1} + \rho_{T-1}(XS_T - h_T) < \infty\}$. The inequality being trivial on the complementary set, we finally conclude that the equality holds a.s. $\qquad\square$

The minimal super-hedging price is given in the sense of (conditional) essential infimum. A generalized concept and existence of conditional essential supremum (resp. conditional essential infimum) of a family of vector-valued random variables with respect to a random partial order are discussed in [22, 23]. Here we use the classical case with a natural partial order for a family of real-valued random variables (see Appendix A).

Definition 3.5. *The minimal super-hedging price of the contingent claim h_T at time $T-1$ is defined as*

$$P^\star_{T-1} := \operatorname*{ess\,inf}_{\theta_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1})} \mathcal{P}_{T-1}(h_T). \tag{3.5}$$

Omit $L^0(\mathbf{R}_+, \mathcal{F}_{T-1})$ and denote $\mathcal{P}'_{T-1}(h_T) := \{g(\theta_{T-1}) : \theta_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1})\}$, then

$$P^\star_{T-1} = \operatorname*{ess\,inf}_{\theta_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1})} \mathcal{P}_{T-1}(h_T) = \operatorname*{ess\,inf}_{\theta_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1})} \mathcal{P}'_{T-1}(h_T).$$

Lemma 3.6. *The set $\mathcal{P}'_{T-1}(h_T)$ is directed downward.*

Proof. For any $\theta_{T-1}^1, \theta_{T-1}^2 \in L^0(\mathbf{R}, \mathcal{F}_{T-1})$, define

$$\theta_{T-1} := \theta_{T-1}^1 \mathbf{1}_{\{g(\theta_{T-1}^1) \leq g(\theta_{T-1}^2)\}} + \theta_{T-1}^2 \mathbf{1}_{\{g(\theta_{T-1}^1) > g(\theta_{T-1}^2)\}} \in L^0(\mathbf{R}, \mathcal{F}_{T-1}).$$

Due to the convexity of g, it holds

$$g(\theta_{T-1}) \leq g(\theta_{T-1}^1) \mathbf{1}_{\{g(\theta_{T-1}^1) \leq g(\theta_{T-1}^2)\}} + g(\theta_{T-1}^2) \mathbf{1}_{\{g(\theta_{T-1}^1) > g(\theta_{T-1}^2)\}}$$
$$= g(\theta_{T-1}^1) \wedge g(\theta_{T-1}^2).$$

That implies that there exists $\theta_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1})$ such that $g(\theta_{T-1}) \in \mathcal{P}'_{T-1}(h_T)$ where $g(\theta_{T-1})$ is the lower bound of any pair $g(\theta_{T-1}^1)$ and $g(\theta_{T-1}^2)$ from the set $\mathcal{P}'_{T-1}(h_T)$. □

Theorem 3.7.

$$P_{T-1}^\star = \operatorname*{ess\,inf}_{\theta_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1})} g(\theta_{T-1}) = \lim_n \downarrow g(\theta_{T-1}^n) \tag{3.6}$$

for some sequence $\theta_{T-1}^n \in L^0(\mathbf{R}, \mathcal{F}_{T-1})$. Moreover, it holds

$$\operatorname*{ess\,inf}_{\theta_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1})} g(\theta_{T-1}) = \inf_{x \in \mathbf{R}} g(x). \tag{3.7}$$

Proof. The first equality (3.6) is a direct consequence of Lemma 3.6. In order to obtain (3.7), we first prove that $\inf_{x \in \mathbf{R}} g(x)$ is $\mathcal{F}_{T-1}$-measurable. Define

$$\mathrm{Dom}\, g(\omega) := \{x \in \mathbf{R} : g(\omega, x) < \infty\}$$
$$= \{x \in \mathbf{R} : \rho_{T-1}(xS_T - h_T) < \infty\}.$$

Observe that $\mathrm{Dom}\, g$ is an upper set, i.e. an interval. Since $\mathcal{P}_{T-1}(h_T) \neq \emptyset$, there exists a strategy $a_{T-1} \in \mathrm{Dom}\, g$ hence $\mathrm{Dom}\, g$ contains the interval $[a_{T-1}, \infty)$. Thus we can say that $\mathrm{Dom}\, g_{T-1}$ admits a non empty interior on which g_{T-1} is convex hence continuous. It follows that

$$\inf_{x \in \mathbf{R}} g(x) = \inf_{x \in \mathrm{Dom}\, g} g(x) = \inf_{x \in \mathrm{int}\, \mathrm{Dom}\, g} g(x) = \inf_{x \in \mathbf{Q} \cap \mathrm{int}\, \mathrm{Dom}\, g} g(x).$$

We deduce that $\inf_{x \in \mathbf{R}} g(x) \geq \inf_{x \in \mathbf{Q}} g(x)$ so that the equality holds and finally $\inf_{x \in \mathbf{R}} g(x)$ is $\mathcal{F}_{T-1}$-measurable. As $g(\theta_{T-1}) \geq \inf_{x \in \mathbf{R}} g(x)$ for any $\theta_{T-1} \in \mathbf{R}$, then $\operatorname*{ess\,inf}_{\theta_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1})} g(\theta_{T-1}) \geq \inf_{x \in \mathbf{R}} g(x)$ from the measurability of $\inf_{x \in \mathbf{R}} g(x)$. For the reverse, take $x_n \in \mathbf{R}$, of course $x_n \in L^0(\mathbf{R}, \mathcal{F}_{T-1})$ (basically x_n is a constant), then $g(x_n) \geq \operatorname*{ess\,inf}_{\theta_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1})} g(\theta_{T-1})$ such that $\inf_{x \in \mathbf{R}} g(x) = \inf_{n \in \mathbb{N}} g(x_n) \geq \operatorname*{ess\,inf}_{\theta_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1})} g(\theta_{T-1})$. Finally, the equality (3.7) holds. □

Actually, it is not very clear how to solve the optimization problem with the essential infimum. Now it has been transferred into a classical one just with infimum according to Theorem 3.7 so that we can know how to deal with it. Before characterizing the optimal solutions and studying the existence of optimal strategies, we first recall the concept of immediate profit (IP) as introduced in [18] and give a weak version of fundamental theorem of asset pricing to build a basic principle for the hedging and pricing.

4 Weak fundamental theorem of asset pricing

Let us extend the acceptable set $\mathcal{A}_t$ to $\mathcal{A}_{t,t+s} \subseteq L^0(\mathbf{R}, \mathcal{F}_{t+s})$ by the same axiomatic conditions in Definition 2.1. In what follows, all acceptable sets are supposed to be closed. The risk measure ρ_t is defined on $L^0(\mathbf{R}, \mathcal{F}_{t+s})$ for some $s \geq 0$ instead of $L^0(\mathbf{R}, \mathcal{F}_T)$, the risk measure function is

$$\rho_t(X) = \operatorname{ess\,inf}\{Y \in L^0(\mathbf{R}, \mathcal{F}_t) | X + Y \in \mathcal{A}_{t,t+s}\}$$

and the corresponding acceptable set is

$$\mathcal{A}_{t,t+s} = \{X \in L^0(\mathbf{R}, \mathcal{F}_{t+s}) | \rho_t(X) \le 0\}.$$

First we consider the general one-step model from t to $t + 1$, super-hedging the contingent claim h_{t+1} at time t means that there exists some $P_t \in L^0(\mathbf{R}, \mathcal{F}_t)$ and strategy $\theta_t \in L^0(\mathbf{R}, \mathcal{F}_t)$ such that $P_t + \theta_t \Delta S_{t+1} - h_{t+1}$ is acceptable with respect to the acceptable set $\mathcal{A}_{t,t+1}$. Similarly we can express the set of all super-hedging prices as

$$\mathcal{P}_t(h_{t+1}) = \{\theta_t S_t + \rho_t(\theta_t S_{t+1} - h_{t+1}) : \theta_t \in L^0(\mathbf{R}, \mathcal{F}_t)\} + L^0(\mathbf{R}_+, \mathcal{F}_t).$$

The minimal super-hedging price at time t for this one-step model is

$$P_t^\star := \operatorname*{ess\,inf}_{\theta_t \in L^0(\mathbf{R}, \mathcal{F}_t)} \mathcal{P}_t(h_{t+1}). \tag{4.8}$$

For the contingent claim h_T we define recursively

$$P_T^\star = h_T \text{ and } P_t^\star := \operatorname*{ess\,inf}_{\theta_t \in L^0(\mathbf{R}, \mathcal{F}_t)} \mathcal{P}_t(P_{t+1}^\star)$$

where $P_{t+1}^\star$ can be regarded as the contingent claim h_{t+1}.

Let us recall the concept of immediate profit as introduced in [18], which means that it is possible to super-replicate contingent claim zero with a negative price.

Definition 4.1. *Absence of Immediate Profit (AIP) holds if*

$$\mathcal{P}_t(0) \cap L^0(\mathbf{R}_-, \mathcal{F}_t) = \{0\} \tag{4.9}$$

for any $0 \le t \le T$.

It is obvious that (AIP) property automatically holds at time T since $\mathcal{P}_T(0) = L^0(\mathbf{R}_+, \mathcal{F}_T)$. Next we characterize (AIP) for general model with $t \le T - 1$.

Theorem 4.2. *(Weak Fundamental theorem of asset pricing) (AIP) property holds if and only if*

$$-\rho_t(S_{t+1}) \le S_t \le \rho_t(-S_{t+1}) \tag{4.10}$$

for all $0 \le t \le T - 1$.

Proof. For the backward recursion starting from $P_T^\star = h_T = 0$, the set of super-hedging prices for contingent claim zero at time $T - 1$ is

$$\mathcal{P}_{T-1}(0) = \{\theta_{T-1} S_{T-1} + \rho_{T-1}(\theta_{T-1} S_T) : \theta_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1})\} + L^0(\mathbf{R}_+, \mathcal{F}_{T-1})$$

and the minimal super-hedging price is $P_{T-1}^\star = \operatorname*{ess\,inf}_{\theta_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1})} \mathcal{P}_{T-1}(0)$. From Theorem 3.7 we know

$$P_{T-1}^\star = \operatorname*{ess\,inf}_{\theta_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1})} g(\theta_{T-1}) = \inf_{x \in \mathbf{R}} g(x)$$

where $g(x) = xS_{T-1} + \rho_{T-1}(xS_T)$ for the case $h_T = 0$. Now it is easy to see that

$$g(x) = x[S_{T-1} + \rho_{T-1}(S_T)]1_{x \ge 0} + x[S_{T-1} - \rho_{T-1}(-S_T)]1_{x<0}.$$

Denote $\Lambda_{T-1} := \{-\rho_{T-1}(S_T) \le S_{T-1} \le \rho_{T-1}(-S_T)\}$, then we can deduce that

$$P_{T-1}^\star = (0)1_{\Lambda_{T-1}} + (-\infty)1_{\Omega \setminus \Lambda_{T-1}}.$$

Now (AIP) at time $T - 1$ implies that the set $\Omega \setminus \Lambda_{T-1}$ is empty, that is

$$-\rho_{T-1}(S_T) \le S_{T-1} \le \rho_{T-1}(-S_T) \tag{4.11}$$

holds almost surely. By repeating the procedure for time $T - 2, T - 3, \ldots$ we can get the conclusion. □

Example 4.3. *For the classical one-step super-hedging problem, i.e., a contingent claim h_T can be super-replicated at time $T - 1$ means that there exist some $P_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1})$ and strategy $\theta_{T-1} \in L^0(\mathbf{R}, \mathcal{F}_{T-1})$ such that $P_{T-1} + \theta_{T-1}\Delta S_T - h_T \geq 0$ almost surely. In this case the acceptable set $\mathcal{A}_{T-1}$ is as follows:*

$$\mathcal{A}_{T-1} = \{X \in L_T^0 | X \geq 0\}$$
$$= \{X \in L_T^0 | \operatorname{ess\,inf}_{\mathcal{F}_{T-1}} X \geq 0\}$$
$$= \{X \in L_T^0 | - \operatorname{ess\,inf}_{\mathcal{F}_{T-1}} X \leq 0\}.$$

This also implies that $\rho_{T-1}(X) = - \operatorname{ess\,inf}_{\mathcal{F}_{T-1}} X$. Then from Theorem 4.2 AIP property can be expressed as the same equivalent condition:

$$\operatorname{ess\,inf}_{\mathcal{F}_{T-1}} S_T \leq S_{T-1} \leq \operatorname{ess\,sup}_{\mathcal{F}_{T-1}} S_T. \tag{4.12}$$

Indeed, $S_{T-1} \geq -\rho_{T-1}(S_T) = \operatorname{ess\,inf}_{\mathcal{F}_{T-1}} S_T$ and $S_{T-1} \leq \rho_{T-1}(-S_T) = - \operatorname{ess\,inf}_{\mathcal{F}_{T-1}}(-S_T) = \operatorname{ess\,sup}_{\mathcal{F}_{T-1}} S_T$. Thus the second equivalent condition of (AIP) in [18, Theorem 3.4] is one of the special cases in our paper when taking the worst-case risk measure $\rho_{T-1}(X) = - \operatorname{ess\,inf}_{\mathcal{F}_{T-1}} X$.

Remark 4.4. *The condition (4.11) implies (4.12) trivially. Actually, the risk at time $T-1$ of position $X \in L^0(\mathbf{R}, \mathcal{F}_T)$ given by $\rho_{T-1}(X) = - \operatorname{ess\,inf}_{\mathcal{F}_{T-1}} X$ is the worst-case (maximum) one. Indeed, from (2.1), we can easily see that*

$$\rho_{T-1}(X) \leq \operatorname{ess\,sup}_{\mathcal{F}_{T-1}}(-X) = - \operatorname{ess\,inf}_{\mathcal{F}_{T-1}} X$$

since $X + \operatorname{ess\,sup}_{\mathcal{F}_{T-1}}(-X) \in \mathcal{A}_{T-1}$ and $\operatorname{ess\,sup}_{\mathcal{F}_{T-1}}(-X)$ is $\mathcal{F}_{T-1}$-measurable. By considering $-X$ it holds that

$$\rho_{T-1}(-X) \leq \operatorname{ess\,sup}_{\mathcal{F}_{T-1}}(X)$$

such that we can get by taking $X = S_T$ that

$$\operatorname{ess\,inf}_{\mathcal{F}_{T-1}} S_T \leq -\rho_{T-1}(S_T) \leq S_{T-1} \leq \rho_{T-1}(-S_T) \leq \operatorname{ess\,sup}_{\mathcal{F}_{T-1}} S_T.$$

5 Price representation

In this section, the study is restricted to bounded non-negative contingent claims. The main purpose is to give the specific expression of minimal super-hedging prices in the sense of risk management.

Notice that the risk measure ρ_t is based on the space L^0 and its dual representation is not used in the previous content. Next we give a new risk measure defined on the space L^∞ under which the minimal super-hedging price of a bounded contingent claim is just the risk of its opposite payoff.

Let us recall the general axiomatic definition of conditional coherent risk measure $\rho_t : L^\infty(\mathbf{R}, \mathcal{F}_T) \to L^0(\mathbf{R}, \mathcal{F}_t)$ (see Definition 1,2 and 3 in [24]):

Definition 5.1. *([24]) A map $\rho_t : L^\infty(\mathbf{R}, \mathcal{F}_T) \to L^0(\mathbf{R}, \mathcal{F}_t)$ is said to be a conditional coherent risk measure if it satisfies the following properties:*
-Normalization: $\rho_t(0) = 0$;
-Conditional translation invariance: for all $X \in L^\infty(\mathbf{R}, \mathcal{F}_T)$ and $m_t \in L^\infty(\mathbf{R}, \mathcal{F}_t)$,

$$\rho_t(X + m_t) = \rho_t(X) - m_t;$$

-Monotonicity: for all $X, X \in L^\infty(\mathbf{R}, \mathcal{F}_T)$, $X \leq X'$ means $\rho_t(X) \geq \rho_t(X')$;
-Subadditivity: for all $X, X' \in L^\infty(\mathbf{R}, \mathcal{F}_T)$, $\rho_t(X + X') \leq \rho_t(X) + \rho_t(X')$;
-Conditional positive homogeneity: for all $k \in L^\infty(\mathbf{R}_+, \mathcal{F}_t)$, $\rho_t(kX) = k\rho_t(X)$.

Let us define recursively $(\tilde{\rho}_t)_{0 \leq t \leq T}$ for some bounded position $Y \in L^\infty(\mathbf{R}, \mathcal{F}_T)$ based on the given dynamic risk measure $(\rho_t)_{0 \leq t \leq T}$ as

$$\tilde{\rho}_T(Y) = -Y \text{ and } \tilde{\rho}_t(Y) = \inf_{x \in \mathbf{R}} \rho_t(x\Delta S_{t+1} - \tilde{\rho}_{t+1}(Y)).$$

Actually, it can be proved that $\tilde{\rho}_t$ are conditional coherent risk measures defined in Definition 5.1 for all $0 \le t \le T$ and $(\tilde{\rho}_t)_t$ is time-consistent, that is for all $X, Y \in L^\infty(\mathbf{R}, \mathcal{F}_T)$ and $0 \le t \le T$, $\tilde{\rho}_{t+1}(X) = \tilde{\rho}_{t+1}(Y)$ implies $\tilde{\rho}_t(X) = \tilde{\rho}_t(Y)$ (see Section 5 in [24]). Then the pricing problem is naturally equivalent to measure the risk of contingent claim under the conditional coherent risk measure $\tilde{\rho}_t$, that is

$$P_t^\star = \tilde{\rho}_t(-h_T) \tag{5.13}$$

which is the time-consistent price process.

Lemma 5.2. *Assume the condition (AIP) holds, then $\tilde{\rho}_t$ are conditional coherent risk measures for all $0 \le t \le T$ on L^∞. Moreover, $(\tilde{\rho}_t)_t$ is time-consistent whenever the underlying dynamic risk measure $(\rho_t)_t$ is or not.*

Proof. Indeed, $\tilde{\rho}_T(\cdot)$ trivially satisfies the conditions in the Definition 5.1 such that $\tilde{\rho}_T(\cdot)$ is a conditional coherent risk measure. And all the other properties except normalization for $\tilde{\rho}_t$ with $0 \le t \le T - 1$ are easy to be inherited from ρ_t by the induction. Here we just need to prove the normalization. Assume $\tilde{\rho}_{t+1}(0) = 0$, then

$$\begin{aligned}
\tilde{\rho}_t(0) &= \inf_{x \in \mathbf{R}} \rho_t(x \Delta S_{t+1}) \\
&= \inf_{x \in \mathbf{R}} [x \rho_t(\Delta S_{t+1}) 1_{x \ge 0} - x \rho_t(-\Delta S_{t+1}) 1_{x < 0}] \\
&= 0
\end{aligned}$$

as (AIP) implies that $\rho_t(\Delta S_{t+1})$ and $\rho_t(-\Delta S_{t+1})$ are both non-negative. The time-consistency can be easily deduced from the definition of $(\tilde{\rho}_t)_t$. $\qquad\square$

Next we can give the expression of $P_t^\star$ in the sense of robust representation for conditional coherent risk measure $\tilde{\rho}_t$. First let us give the following sets of probability measures for all $0 \le t \le T$ as:

$$\mathcal{Q}_t := \{Q \text{ is a probability measure}|Q \ll P \text{ and } Q = P|_{\mathcal{F}_t}\}. \tag{5.14}$$

Theorem 5.3. *Assume (AIP) property holds, then the minimal super-hedging price can be represented as*

$$P_t^\star = \text{ess sup}_{Q \in \mathcal{Q}_t^\star} \mathbb{E}_Q(h_T|\mathcal{F}_t)$$

where

$$\mathcal{Q}_t^\star := \{Q \in \mathcal{Q}_t|\mathbb{E}_Q(Y|\mathcal{F}_t) \ge -\tilde{\rho}_t(Y), \forall\, Y \in L^\infty(\mathbf{R}, \mathcal{F}_T)\}. \tag{5.15}$$

Proof. From Lemma 5.2 $\tilde{\rho}_t$ is a conditional coherent risk measure. And the lower semi-continuity of $\tilde{\rho}_t$ is inherited from the underlying risk measure ρ_t. Thus the following robust representation (see [24]) can be obtained

$$\tilde{\rho}_t(Y) = \text{ess sup}_{Q \in \mathcal{Q}_t^\star} \{-\mathbb{E}_Q(Y|\mathcal{F}_t)\}$$

where $\mathcal{Q}_t$ and $\mathcal{Q}_t^\star$ are defined as (5.14) and (5.15). Then let $Y = -h_T$ it is easy to conclude from (5.13). $\qquad\square$

Appendix

A. Conditional essential supremum/infimum

Given a measurable probability space $(\Omega, \mathcal{F}, P)$ and $\mathcal{H}$ is a sub-σ-algebra of $\mathcal{F}$. Recall the concept of generalized conditional essential supremum (see [22], Definition 3.1) in $L^0(\mathbf{R}^d)$ as well as the existence and uniqueness for the case where $d = 1$ (see [22], Lemma 3.9). A similar result holds for the conditional essential infimum.

Lemma 3.9([22]) Let $\Gamma \ne \emptyset$ be a subset of $L^0(\mathbf{R} \cup \{+\infty\}, \mathcal{F})$. Then there exist a unique $\mathcal{H}$-measurable random variable $\hat{\gamma} \in L^0(\mathbf{R} \cup \{+\infty\}, \mathcal{H})$ denoted as ess sup$_\mathcal{H} \Gamma$ such that the following conditions hold:
(i) $\hat{\gamma} \ge \gamma$ a.s. for any $\gamma \in \Gamma$;
(ii) if $\tilde{\gamma} \in L^0(\mathbf{R} \cup \{+\infty\}, \mathcal{H})$ satisfies $\tilde{\gamma} \ge \gamma$ a.s. for any $\gamma \in \Gamma$, then $\tilde{\gamma} \ge \hat{\gamma}$ a.s.

B. Measurable subsequences

First, let us recall the existence of convergent subsequences of the random sequence from $L^0(\mathbf{R}^d)$, see [7, Lemma 2.1.2]. The technical constructions of these convergent subsequences can be found in the proof of this lemma.

Lemma 2.1.2([7]) Let $\eta^n \in L^0(\mathbf{R}^d)$ be such that $\underline{\eta} := \liminf |\eta^n| < \infty$. Then there are $\tilde{\eta}^k \in L^0(\mathbf{R}^d)$ such that for all ω the sequence of $\tilde{\eta}^k(\omega)$ is a convergent subsequence of the sequence of $\eta^n(\omega)$.

It is worth noting that the subsequence $\tilde{\eta}^k$ is random due to the fact that

$$\tilde{\eta}^k(\omega) = \eta^{n_k(\omega)}(\omega) = \sum_{p \geq k}^{\infty} \eta^p(\omega) 1_{n_k=p}.$$

The more detailed results about the random convergent subsequence can be found in [25, Section 6.3]. Let $(\mathcal{K}, d)$ be a compact metric space and $\mathbb{N}$ be the set of all natural numbers.

Definition 6.3.1([25]) An $\mathbb{N}$-valued, $\mathcal{F}$-measurable function is called a random time. A strictly increasing sequence $(\tau_k)_{k=1}^{\infty}$ of random times is called a measurably parameterised subsequence or simply a measurable subsequence.

Lemma 6.3.2([25]) Let $(f_n)_{n=1}^{\infty}$ be a sequence of $\mathcal{F}$-measurable function $f_n : \Omega \to \mathcal{K}$. Let $\tau : \Omega \to \{1, 2, 3, \cdots\}$ be an $\mathcal{F}$-measurable random time, then $g(\omega) = f_{\tau(\omega)}(\omega)$ is $\mathcal{F}$-measurable.

Proposition 6.3.3([25]) For a sequence $(f_n)_{n=1}^{\infty} \in L^0(\Omega, \mathcal{F}, P; \mathcal{K})$ we may find a measurably parameterised subsequence $(\tau_k)_{k=1}^{\infty}$ such that $(f_{\tau_k})_{k=1}^{\infty}$ converges for all $\omega \in \Omega$.

Proposition 6.3.4([25]) Under the assumptions of Proposition 6.3.3 we have in addition:
(i) Let $x_0 \in \mathcal{K}$ and define

$$B = \{\omega \in \Omega : x_0 \text{ is an accumulation point of } (f_n(\omega))_{n=1}^{\infty}\}.$$

Then the sequence $(\tau_k)_{k=1}^{\infty}$ in Proposition 6.3.3 may be chosen such that

$$\lim_k f_{\tau_k(\omega)}(\omega) = x_0, \text{ for each } \omega \in B.$$

(ii) Let $f_0 \in L^0(\Omega, \mathcal{F}, P; \mathcal{K})$ and define

$$C = \{\omega \in \Omega : f_0 \text{ is not the limit of } (f_n(\omega))_{n=1}^{\infty}\},$$

where the above means that either the limit does not exist or, if it exists, it is different from $f_0(\omega)$. Then the sequence $(\tau_k)_{k=1}^{\infty}$ in Proposition 6.3.3 may be chosen such that

$$\lim_k f_{\tau_k(\omega)}(\omega) \neq f_0(\omega), \text{ for each } \omega \in C.$$

C. Proof of Proposition 2.4

The first five conventional properties are directly deduced from the definition of ρ_t in (2.1). Let us prove the "Moreover" part under the assumption that the acceptable set $\mathcal{A}_t$ is closed.

First, we can prove that $\rho_t(X) > -\infty$ a.s for all $X \in L^0(\mathbf{R}, \mathcal{F}_T)$. Indeed, by Lemma 2.3, for any $X \in L^0(\mathbf{R}, \mathcal{F}_T)$ there exists a sequence $Y_n \in \mathcal{A}_t^X$, i.e. $Y_n \in L^0(\mathbf{R}, \mathcal{F}_t)$ satisfying $X + Y_n \in \mathcal{A}_t$ such that $\rho_t(X) = \lim_n \downarrow Y_n$. Suppose that $P\{\rho_t(X) = -\infty\} > 0$. Denote the $\mathcal{F}_t$-measurable set $\Lambda_t := \{\omega \in \Omega : \rho_t(X) = -\infty\} = \{\omega \in \Omega : \lim_n \downarrow Y_n = -\infty\}$. Let us consider it by the following steps:

Step 1: By taking $\mathcal{K} = \mathbf{R} \cup \{-\infty\}$ and $x_0 = -\infty$ in [25, Proposition 6.3.4 (i)], there is a $\mathcal{F}_t$-measurably parameterised subsequence $(\tau_k)_{k=1}^{\infty}$ such that the subsequence $(L_k)_{k=1}^{\infty} := (Y_{\tau_k})_{k=1}^{\infty}$ diverges to $-\infty$ on the set Λ_t of positive probability. Since $(X(\omega) + L_k(\omega)) 1_{\Lambda_t} = (X(\omega) + Y_{\tau_k(\omega)}(\omega)) 1_{\Lambda_t} = (X(\omega) + \sum_{p \geq k}^{\infty} Y_p(\omega) 1_{\tau_k=p}) 1_{\Lambda_t} =$

$\sum_{p\geq k}^{\infty}(X(\omega)+Y_p(\omega))1_{\tau_k=p}1_{\Lambda_t}$ and τ_k is $\mathcal{F}_t$-measurable, then we can deduce from the additivity and the positive homogeneity of $\mathcal{A}_t$ that $X+L_k\in\mathcal{A}_t$ on the set Λ_t.

Step 2: By the normalization procedure $\bar{X}_k:=\frac{X}{|\bar{L}_k|}$ and $\bar{L}_k:=\frac{L_k}{|\bar{L}_k|}$, we get that $\bar{X}_k+\bar{L}_k\in\mathcal{A}_t$ on the set Λ_t. Applying [25, Proposition 6.3.3] to the sequence $(\bar{L}_k)_{k=1}^{\infty}$, there is a $\mathcal{F}_t$-measurably parameterised subsequence $(\sigma_i)_{i=1}^{\infty}$ such that the subsequence $(\bar{L}_{\sigma_i})_{i=1}^{\infty}$ converges to some $\bar{L}$. As $|\bar{L}_k|=1$ for any $k\geq 1$, we can see that $|\bar{L}|=1$. Actually, $\bar{L}=-1$ a.s. as $\bar{L}_{\sigma_i}<0$ for large enough i.

Step 3: Next we can say that -1 is also the limit of the sequence $(\bar{L}_k)_{k=1}^{\infty}$ a.s. Otherwise, the set $C:=\{\omega\in\Omega: -1$ is not the limit of $(\bar{L}_k(\omega))_{k=1}^{\infty}\}$ has positive probability. By taking $f_0=-1$ in [25, Proposition 6.3.4 (ii)], then $\mathcal{F}_t$-measurably parameterised subsequence $(\sigma_i)_{i=1}^{\infty}$ may be chosen such that $\lim_i\bar{L}_{\sigma_i(\omega)}(\omega)\neq -1$ for each $\omega\in C$. This is contradicted with the above statement $\bar{L}=-1$. Thus, we can deduce that $\lim_k\bar{L}_k=-1$.

Step 4: On the other hand, $\bar{X}_k=\frac{X}{|\bar{L}_k|}$ trivially converges to zero as L_k diverges to $-\infty$. Finally, we deduce that $\lim_k(\bar{X}_k+\bar{L}_k)=-1\in\mathcal{A}_t$ on the set Λ_t if $\mathcal{A}_t$ is closed. This is contradicted with the third condition: $\mathcal{A}_t\cap L^0(\mathbf{R},\mathcal{F}_t)=L^0(\mathbf{R}_+,\mathcal{F}_t)$ in the Definition 2.1. Thus, the assumption $\rho_t(X)=-\infty$ with a positive probability is impossible, that is $\rho_t(X)>-\infty$ with probability one.

Since we assume that $\rho_t(X)<+\infty$ for any $X\in L^0(\mathbf{R},\mathcal{F}_T)$, then it holds $\rho_t(X)\in L^0(\mathbf{R},\mathcal{F}_t)$. By Lemma 2.3, we know that $\rho_t(X)=\lim_n\downarrow Y_n$ a.s. where $Y_n\in L^0(\mathbf{R},\mathcal{F}_t)$ satisfying $X+Y_n\in\mathcal{A}_t$. As the set $\mathcal{A}_t$ is closed, $\mathcal{F}_t$-decomposable and contains 0, we deduce that $X+\rho_t(X)\in\mathcal{A}_t$ for any $X\in L^0(\mathbf{R},\mathcal{F}_T)$. Now let us prove the lower semi-continuity of ρ_t. Consider a sequence $X_n\in L^0(\mathbf{R},\mathcal{F}_T)$ which converges to X_0. Denote $\alpha_n:=\rho_t(X_n)$, then $X_n+\alpha_n\in\mathcal{A}_t$. Our goal is to prove the inequality $\rho_t(X_0)\leq\liminf\alpha_n$ a.s. Let us divide it into the following three cases:

a) As for the case where $\liminf\alpha_n=+\infty$, the inequality $\rho_t(X_0)\leq\liminf\alpha_n$ holds trivially. Thus we may assume w.l.o.g. that $\liminf\alpha_n<+\infty$.

b) Let us consider the case where $\liminf\alpha_n=-\infty$. Suppose that the $\mathcal{F}_t$-measurable set $\Gamma_t:=\{\omega\in\Omega: \liminf\alpha_n=-\infty\}$ has a positive probability. Obviously, $-\infty$ is an accumulation point of $(\alpha_n)_{n=1}^{\infty}$ on the set Γ_t. For convenience, denote $\alpha_\infty:=\liminf\alpha_n$. Again, [25, Proposition 6.3.4 (i)] implies that there is a $\mathcal{F}_t$-measurably parameterised subsequence $(\mu_k)_{k=1}^{\infty}$ such that the subsequence $(\beta_k)_{k=1}^{\infty}:=(\alpha_{\mu_k})_{k=1}^{\infty}$ diverges to $-\infty$ on the set Γ_t of positive probability. Let $(Z_k)_{k=1}^{\infty}:=(X_{\mu_k})_{k=1}^{\infty}$ be the corresponding subsequence of the sequence X_n. Then we can see that $Z_k+\beta_k\in\mathcal{A}_t$ on the set Γ_t as $(Z_k(\omega)+\beta_k(\omega))1_{\Gamma_t}=(X_{\mu_k(\omega)}(\omega)+\alpha_{\mu_k(\omega)}(\omega))1_{\Gamma_t}=\sum_{p\geq k}^{\infty}(X_p(\omega)+\alpha_p(\omega))1_{\mu_k=p}1_{\Gamma_t}$. Then, using the normalization procedure $\tilde{Z}_k:=\frac{Z_k}{|\beta_k|}$ and $\tilde{\beta}_k:=\frac{\beta_k}{|\beta_k|}$, we get that $\tilde{Z}_k+\tilde{\beta}_k\in\mathcal{A}_t$ on the set Γ_t. By passing once again to a measurably parameterised subsequence, we may assume that $\tilde{\beta}_k$ converges to -1 according to the similar statements in the above Step 2 and Step 3. Note that $Z_k=X_{\mu_k}$ converges to X_0 and β_k diverges to $-\infty$ such that $\tilde{Z}_k$ converges to zero, we finally get that $\lim_k(\tilde{Z}_k+\tilde{\beta}_k)=-1\in\mathcal{A}_t$ on the set Γ_t if the set $\mathcal{A}_t$ is closed. This contradicts with the third condition in the Definition 2.1. Thus, $\alpha_\infty=\liminf\alpha_n>-\infty$ with probability one.

c) Combining the cases a) and b), we can assume w.l.o.g. that $\alpha_\infty\in L^0(\mathbf{R},\mathcal{F}_t)$ and $X_0+\alpha_\infty\in\mathcal{A}_t$. It follows that $\rho_t(X_0)\leq\alpha_\infty=\liminf\rho_t(X_n)$ a.s.

At last, if the set $\mathcal{A}_t$ is closed, the acceptable set $\mathcal{A}_t$ can be represented as $\mathcal{A}_t=\{X\in L^0(\mathbf{R},\mathcal{F}_T)|\rho_t(X)\leq 0\}$. Indeed, it is clear that $\rho_t(X)\leq 0$ for all $X\in\mathcal{A}_t$. Reciprocally, if $\rho_t(X)\leq 0$, we get that $X=-\rho_t(X)+a_t$ where $a_t\in\mathcal{A}_t$. Finally we can deduce that $X\in\mathcal{A}_t$ since $0\leq-\rho_t(X)\in\mathcal{A}_t$ and $\mathcal{A}_t+\mathcal{A}_t\subseteq\mathcal{A}_t$. $\square$

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