Open Mathematics

Research Article

Shu-Yu Cui* and Gui-Xian Tian*

Some improved bounds on two energy-like invariants of some derived graphs

https://doi.org/10.1515/math-2019-0069
Received August 16, 2018; accepted June 10, 2019

Abstract: Given a simple graph G, its Laplacian-energy-like invariant $LEL(G)$ and incidence energy $IE(G)$ are the sum of square root of its all Laplacian eigenvalues and signless Laplacian eigenvalues, respectively. This paper obtains some improved bounds on LEL and IE of the $\mathcal{R}$-graph and $\mathcal{Q}$-graph for a regular graph. Theoretical analysis indicates that these results improve some known results. In addition, some new lower bounds on LEL and IE of the line graph of a semiregular graph are also given.

Keywords: Spectrum; incidence energy; $\mathcal{R}$-graph; $\mathcal{Q}$-graph; Laplacian-energy-like invariant

MSC: 05C05; 05C12; 05C50

Introduction

We only consider finite simple graphs in this paper. Given a graph $G = (V, E)$ with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set E, then $d_i = d_G(v_i)$ denotes the degree of v_i. If $d_i = r$ for any $i = 1, 2, \ldots, n$, then G is called *r-regular*. If G is a bipartite graph and $V = V_1 \cup V_2$ is its bipartition, then G is said to (r_1, r_2)-*semiregular* whenever each vertex in V_1 has degree r_1 and each vertex in V_2 has degree r_2. The *adjacency matrix* of G, denoted by $A(G)$, is a square matrix whose (i, j)-entry is one if v_i and v_j are adjacent in G and zero otherwise. Let $D(G)$ be the degree diagonal matrix of G with diagonal entries $d_1, d_2, \ldots, d_n$. Then $L(G) = D(G) - A(G)$ is called *Laplacian matrix* of G and $Q(G) = D(G) + A(G)$ is called its *signless Laplacian matrix*.

Let F be an $n \times n$ matrix associated to G. Then its characteristic polynomial $\psi(F; x) = \det(xI_n - F)$ is called the *F-polynomial* of G, where I_n is the identity matrix of order n. The zeros of $\psi(F; x)$ is said to the *F-eigenvalues* of G. The set of all F-eigenvalues is called the *F-spectrum* of G. Specifically, if F is one of the Laplacian matrix $L(G)$ and signless Laplacian matrix $Q(G)$ of G, then the corresponding spectrum are called respective L-spectrum and Q-spectrum. Throughout we denote the respective L-spectrum and Q-spectrum by $Sp(L(G)) = \{\mu_1, \mu_2, \ldots, \mu_n\}$ and $Sp(Q(G)) = \{q_1, q_2, \ldots, q_n\}$, where $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$ and $q_1 \geq q_2 \geq \cdots \geq q_n \geq 0$ are the eigenvalues of $L(G)$ and $Q(G)$, respectively. For more details about L-spectrum and Q-spectrum of G, readers may refer to [1-6].

For the L-spectrum of G, Liu and Liu [7] put forward the concept of the Laplacian-energy-like invariant, that is,

$$LEL(G) = \sum_{i=1}^{n-1} \sqrt{\mu_i}. \tag{1}$$

***Corresponding Author: Shu-Yu Cui:** Xingzhi College, Zhejiang Normal University, Jinhua, Zhejiang, 321004, China, E-mail: cuishuyu@zjnu.cn
***Corresponding Author: Gui-Xian Tian:** Department of Mathematics, Zhejiang Normal University, Jinhua, Zhejiang, 321004, China, E-mail: gxtian@zjnu.cn or guixiantian@163.com

The motivation of this concept derived from the Laplacian energy [8], along with graph energy [9]. Recently, Stevanović et al. [10] pointed out that LEL has become a newly molecular descriptor. For more details about the mathematical properties of LEL, readers may refer to [11-17] and the references therein.

In 2007, Nikiforov also extended the definition of graph energy to any matrix M [18]. The energy of M is defined to the sum of all singular values of M, denoted by $E(M)$. Motivated further by above concepts $E(M)$ and LEL, Jooyandeh et al. [19] gave the definition of incidence energy $IE(G) = E(B(G))$ of a graph G, where $B(G)$ is the incidence matrix of G. It is easy to see that

$$IE(G) = E(B(G)) = \sum_{i=1}^{n} \sqrt{q_i}. \tag{2}$$

For more details about IE, see [19-22] and the references therein.

Regular graphs and semi-regular graphs are two important graph classes in graph theory and combinatorics, which play an important role in the study of spectral theory of graphs. In recent years, LEL and IE of some operations on regular graphs and semi-regular graphs have attracted people's attention. For example, some sharp bounds about LEL are obtained by Wang and Luo [15] for the line graph, subdivision graph and total graph of regular graphs. Pirzada et al. [23] also presented some new bounds about LEL for the line graph of semiregular graphs, the para-line graph, $\mathcal{R}$-graph, $\mathcal{Q}$-graph of regular graphs. In addition, Gutman et al. [21] presented some sharp bounds for IE of the line graph and iterated line graph of regular graphs. Wang et al. [24] gave some new bounds for IE of the subdivision graph and total graph of regular graphs. Wang and Yang [25] also presented some upper bounds on IE for the line graph of semiregular graphs, the para-line graph of a regular graph. Recently, Chen et al. [26] obtained some new bounds for LEL and IE of the line graph, subdivision graph and total graph of regular graphs. They pointed out that these results improved some known bounds in [15, 21].

Motivated by above researches, this paper gives some new bounds for LEL of the $\mathcal{R}$-graph, $\mathcal{Q}$-graph of regular graphs. Theoretical analysis indicates that these results improve some known results obtained by Pirzada et al. in [23]. We also obtain some new bounds for IE of the $\mathcal{R}$-graph, $\mathcal{Q}$-graph of regular graphs. These results are a useful supplement for the existing results on some bounds of LEL and IE of related graph operations of regular graphs in [26]. In addition, some new lower bounds are also presented on LEL and IE for the line graph of semiregular graphs.

2. Preliminaries

Some definitions of line graphs, $\mathcal{R}$-graph and $\mathcal{Q}$-graph are recalled in this section and some lemmas are listed, which shall be used in the following sections.

Recall that the line graph $\mathcal{L}(G)$ [2] of G is the graph whose vertex set is the edge set of G, and two vertices in $\mathcal{L}(G)$ are adjacent if and only if the corresponding edges in G have exactly a common vertex. Given an (r_1, r_2)-semiregular graph G of order n with m edges, then the L-spectrum [23] and Q-spectrum [25] of $\mathcal{L}(G)$ are, respectively,

$$Sp(L(\mathcal{L}(G))) = \left\{(r_1 + r_2)^{(m-n)}, r_1 + r_2 - \mu_1, \dots, r_1 + r_2 - \mu_n\right\} \tag{3}$$

and

$$Sp(Q(\mathcal{L}(G))) = \left\{(r_1 + r_2 - 4)^{(m-n)}, r_1 + r_2 - 4 + q_1, \dots, r_1 + r_2 - 4 + q_n\right\}, \tag{4}$$

where $a^{(b)}$ indicates that a is repeated b times, $\{\mu_1, \mu_2, \dots, \mu_n\}$ and $\{q_1, q_2, \dots, q_n\}$ are the L-spectrum and Q-spectrum of G, respectively.

The $\mathcal{R}$-graph [2] of G, denoted by $\mathcal{R}(G)$, is the graph derived from G by adding a vertex w_i corresponding to every edge $e_i = uv$ of G and by connecting every vertex w_i to the end vertices u and v of e_i. If G is an r-regular graph of order n with m edges, then L-polynomial [27, 28] and Q-polynomial [29] of $\mathcal{R}(G)$ are, respectively,

$$\psi(L(\mathcal{R}(G)), x) = x(x-2)^{m-n}(x-r-2)\prod_{i=1}^{n-1}[(x-2)(x-r-\mu_i) - 2r + \mu_i] \tag{5}$$

and

$$\psi(Q(\mathcal{R}(G)), x) = (x-2)^{m-n}(x^2-(2+3r)x+4r)\prod_{i=2}^{n}[(x-2)(x-r-q_i)-q_i]. \tag{6}$$

From (5) and (6), we obtain the following lemma easily.

Lemma 2.1 [27, 29] *If G is an r-regular graph of order n with m edges, then*
(i) If the L-spectrum of G is $Sp(L(G)) = \{\mu_1 \dots, \mu_n\}$, then the L-spectrum of $\mathcal{R}(G)$

$$Sp(L(\mathcal{R}(G))) = \left\{2^{(m-n)}, \frac{(r+2+\mu_i)\pm\sqrt{(r+2+\mu_i)^2-12\mu_i}}{2} \quad (i=1,2,\dots,n)\right\}.$$

(ii) If the Q-spectrum of G is $Sp(Q(G)) = \{q_1, \dots, q_n\}$, then the Q-spectrum of $\mathcal{R}(G)$

$$Sp(Q(\mathcal{R}(G))) = \left\{2^{(m-n)}, \frac{(r+2+q_i)\pm\sqrt{(r+2+q_i)^2-4(2r+q_i)}}{2} \quad (i=1,2,\dots,n)\right\}.$$

The $\mathcal{Q}$-graph [2] of G, denoted by $\mathcal{Q}(G)$, is the graph derived from G by plugging a vertex w_i to every edge $e_i = uv$ of G and by adding a new edge between two new vertices whenever these new vertices lie on adjacent edges of G. If G is an r-regular graph of order n with m edges, then L-polynomial [27, 28] and Q-polynomial [29] of $\mathcal{Q}(G)$ are, respectively,

$$\psi(L(\mathcal{Q}(G)), x) = x(x-2r-2)^{m-n}(x-r-2)\prod_{i=1}^{n-1}[(x-r)(x-2-\mu_i)-2r+\mu_i] \tag{7}$$

and

$$\psi(Q(\mathcal{Q}(G)), x) = (x-2r+2)^{m-n}[(x-r)(x-4r+2)-2r]\prod_{i=2}^{n}[(x-r)(x-2r+2-q_i)-q_i]. \tag{8}$$

Similarly, from (7) and (8), one obtains the following lemma easily.

Lemma 2.2 [27, 29] *If G is an r-regular graph of order n with m edges, then*
(i) If the L-spectrum of G is $Sp(L(G)) = \{\mu_1, \dots, \mu_n\}$, then the L-spectrum of $\mathcal{Q}(G)$

$$Sp(L(\mathcal{Q}(G))) = \left\{(2r+2)^{(m-n)}, \frac{(r+2+\mu_i)\pm\sqrt{(r+2+\mu_i)^2-4\mu_i(r+1)}}{2} \quad (i=1,2,\dots,n)\right\}.$$

(ii) If the Q-spectrum of G is $Sp(Q(G)) = \{q_1, \dots, q_n\}$, then the Q-spectrum of $\mathcal{Q}(G)$

$$Sp(Q(\mathcal{Q}(G))) = \left\{(2r-2)^{(m-n)}, \frac{(3r-2+q_i)\pm\sqrt{(3r-2+q_i)^2-4r(2r-2+q_i)+4q_i}}{2}\right\},$$

where $i = 1, \dots, n$.

Lemma 2.3 [26] *If G is an r-regular graph of order n, then*

$$\frac{nr}{\sqrt{r+1}} \le LEL(G) \le \sqrt{r+1} + \sqrt{(n-2)(nr-r-1)},$$

where both equalities hold if and only if G is the complete graph K_n.

Lemma 2.4 [30] *If G is any graph of order n, with at least one edge, then $\mu_1 = \mu_2 = \cdots = \mu_{n-1}$ if and only if G is the complete graph K_n.*

The following lemma for Q-spectrum is analogous to above Lemma 2.4 for L-spectrum. By Theorem 3.6 in [1], one obtains the following lemma easily.

Lemma 2.5 *If G is a graph of order n, with at least one edge, then $q_2 = q_3 = \cdots = q_n$ if and only if G is* *the complete graph K_n.*

The following lemma comes from [31], which is called the Ozeki's inequality.

Lemma 2.6 [31] *Let $\xi = (a_1, \ldots, a_n)$ and $\eta = (b_1, \ldots, b_n)$ be two positive n-tuples with $0 < p \le a_i \le P$ and $0 < q \le b_i \le Q$, where $i = 1, \ldots, n$. Then*

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \frac{1}{4} n^2 (PQ - pq)^2. \tag{9}$$

It is a remarkable fact that a refinement of Ozeki's inequality was obtained by Izumino et al.[32] below.

Lemma 2.7 [32] *Let $\xi = (a_1, \ldots, a_n)$ and $\eta = (b_1, \ldots, b_n)$ be two n-tuples with $0 \le p \le a_i \le P$, $0 \le q \le b_i \le Q$ and $PQ \ne 0$, where $i = 1, \ldots, n$. Take $\alpha = p/P$ and $\beta = q/Q$. If $(1+\alpha)(1+\beta) \ge 2$, then (9) still holds.*

Remark that if G is 1-regular, then G is isomorphic to $\frac{n}{2}K_2$. For avoiding the triviality, we always suppose that $r \ge 2$ for an r-regular graph. For an (r_1, r_2)-semiregular graph G, G is isomorphic to $\frac{n}{3}P_3$ whenever $r_1 + r_2 = 3$. Next we also suppose that $r_1 + r_2 \ge 4$ for an (r_1, r_2)-semiregular graph throughout this paper. In addition, it is well known [2, 3] that the largest Laplacian eigenvalue $\mu_1 \le 2r$ and largest signless Laplacian eigenvalue $q_1 = 2r$ for an r-regular graph. From Lemma 3.3 in [23], we also see that $\mu_1 = r_1 + r_2$ for an (r_1, r_2)-semiregular graph.

3. The Laplacian-energy-like invariant

In this section, we shall give some improved bounds for LEL of $\mathcal{R}$-graph and $\mathcal{Q}$-graph of regular graphs, as well as for the line graph of semiregular graphs. Now we first consider LEL of $\mathcal{R}$-graph of regular graphs.

Theorem 3.1 *If G is an r-regular graph of order n with m edges, then*
(i)

$$LEL(\mathcal{R}(G)) \le \tfrac{n(r-2)}{2}\sqrt{2} + \sqrt{r+2} + (n-1)\sqrt{r+2+\tfrac{nr}{n-1}+\tfrac{2\sqrt{3}}{n-1}LEL(G)}, \tag{10}$$

where the equality holds in (10) if and only if G is the complete graph K_n.
(ii)

$$LEL(\mathcal{R}(G)) \ge \tfrac{n(r-2)}{2}\sqrt{2} + \sqrt{r+2} + (n-1)\sqrt{\tfrac{3}{4}(r+2)+\tfrac{nr}{n-1}+\tfrac{2\sqrt{3}}{n-1}LEL(G)}. \tag{11}$$

Proof. Suppose that $Sp(L(G)) = \{\mu_1, \mu_2, \ldots, \mu_n\}$ is the L-spectrum of G. Then from (1) and the (i) in Lemma 2.1, one gets

$$\begin{aligned} LEL(\mathcal{R}(G)) &= \sum_{i=1}^{n-1}\left(\sqrt{\tfrac{(r+2+\mu_i)+\sqrt{(r+2+\mu_i)^2-12\mu_i}}{2}}\right) + \sum_{i=1}^{n-1}\left(\sqrt{\tfrac{(r+2+\mu_i)-\sqrt{(r+2+\mu_i)^2-12\mu_i}}{2}}\right) + (m-n)\sqrt{2} + \sqrt{r+2} \\ &= \sum_{i=1}^{n-1}\sqrt{\left(\sqrt{\tfrac{(r+2+\mu_i)+\sqrt{(r+2+\mu_i)^2-12\mu_i}}{2}} + \sqrt{\tfrac{(r+2+\mu_i)-\sqrt{(r+2+\mu_i)^2-12\mu_i}}{2}}\right)^2} + (m-n)\sqrt{2} + \sqrt{r+2} \\ &= \sum_{i=1}^{n-1}\sqrt{r+2+\mu_i+2\sqrt{3\mu_i}} + (m-n)\sqrt{2} + \sqrt{r+2}. \end{aligned} \tag{12}$$

Notice that $\sum_{i=1}^{n-1}\mu_i = 2m = nr$. Applying the Cauchy-Schwarz inequality, one obtains

$$\begin{aligned} LEL(\mathcal{R}(G)) &\le \sqrt{(n-1)\sum_{i=1}^{n-1}(r+2+\mu_i+2\sqrt{3\mu_i})} + (m-n)\sqrt{2} + \sqrt{r+2} \\ &= (n-1)\sqrt{r+2+\tfrac{nr}{n-1}+\tfrac{2\sqrt{3}}{n-1}LEL(G)} + \tfrac{n(r-2)}{2}\sqrt{2} + \sqrt{r+2}, \end{aligned}$$

where above equality holds if and only if $\mu_1 = \mu_2 = \cdots = \mu_{n-1}$. It follows from Lemma 2.4 that G is the complete graph K_n. Hence the proof of the (i) is completed.

Now we prove the (ii). Assume that $a_i = \sqrt{r+2+\mu_i+2\sqrt{3\mu_i}}$ and $b_i = 1$, $i = 1,\dots,n-1$. Take $P = \sqrt{3r+2+2\sqrt{6r}}$, $p = \sqrt{r+2}$ and $Q = q = 1$. Since $0 \le \mu_i \le 2r$, then $0 < p \le a_i \le P$, $0 < q \le b_i \le Q$ and

$$(PQ-pq)^2 = (\sqrt{3r+2+2\sqrt{6r}} - \sqrt{r+2})^2 = (\sqrt{3r}+\sqrt{2}-\sqrt{r+2})^2 \le r+2.$$

By Lemma 2.6, we obtain

$$\begin{aligned}\sum_{i=1}^{n-1}\sqrt{r+2+\mu_i+2\sqrt{3\mu_i}} &\ge \sqrt{(n-1)\sum_{i=1}^{n-1}(r+2+\mu_i+2\sqrt{3\mu_i}) - \tfrac{1}{4}(n-1)^2(PQ-pq)^2}\\ &\ge (n-1)\sqrt{\tfrac{3}{4}(r+2)+\tfrac{nr}{n-1}+\tfrac{2\sqrt{3}}{n-1}LEL(G)}.\end{aligned}$$

From (12), one obtains the required result (ii). □

Corollary 3.2 *If G is an r-regular graph of order n with m edges, then*
(i)

$$LEL(\mathcal{R}(G)) \le \tfrac{n(r-2)}{2}\sqrt{2}+\sqrt{r+2}+(n-1)\sqrt{r+2+\tfrac{nr}{n-1}+\tfrac{2\sqrt{3}(\sqrt{r+1}+\sqrt{(n-2)(nr-r-1)})}{n-1}},$$

where above equality holds if and only if G is the complete graph K_n.
(ii)

$$LEL(\mathcal{R}(G)) > \tfrac{n(r-2)}{2}\sqrt{2}+\sqrt{r+2}+(n-1)\sqrt{\tfrac{3}{4}(r+2)+\tfrac{nr}{n-1}+\tfrac{2\sqrt{3}}{n-1}\tfrac{nr}{\sqrt{r+1}}}.$$

Proof. Theorem 3.1 (i) and Lemma 2.3 together imply (i) in the corollary. Again, from Lemma 2.3 and (11), one gets

$$LEL(\mathcal{R}(G)) \ge \tfrac{n(r-2)}{2}\sqrt{2}+\sqrt{r+2}+(n-1)\sqrt{\tfrac{3}{4}(r+2)+\tfrac{nr}{n-1}+\tfrac{2\sqrt{3}}{n-1}\tfrac{nr}{\sqrt{r+1}}}. \quad (13)$$

Suppose that the equality in (13) holds. From Lemma 2.3, we have G is the complete graph K_n. But for the complete graph K_n, the inequality (10) implies that the equality is false in (11). This completes the proof. □

Remark 1 Given an r-regular graph G of order n, Pirzada et al. [23] proved that

$$\tfrac{n(r-2)}{2}\sqrt{2}+n\sqrt{r+2} < LEL(\mathcal{R}(G)) \le \tfrac{n(r-2)}{2}\sqrt{2}+\sqrt{r+2}+(n-1)(\sqrt{3r}+\sqrt{2}), \quad (14)$$

where the equality on the right of (14) holds if and only if G is the complete graph K_2. Notice that these bounds in Corollary 3.2 improve those in (14). In fact, by direct computation, we have

$$\begin{aligned}\sqrt{r+2+\tfrac{nr}{n-1}+\tfrac{2\sqrt{3}(\sqrt{r+1}+\sqrt{(n-2)(nr-r-1)})}{n-1}} &\le \sqrt{r+2+\tfrac{nr}{n-1}+\tfrac{2\sqrt{3}\sqrt{(n-1)nr}}{n-1}}\\ &= \sqrt{r+2+\tfrac{nr}{n-1}+2\sqrt{3}\sqrt{\tfrac{nr}{n-1}}}\\ &\le \sqrt{r+2+2r+2\sqrt{3}\sqrt{2r}}\\ &= \sqrt{3r}+\sqrt{2},\end{aligned}$$

which implies that the upper bound in Corollary 3.2 is an improvement on that in (14). For the lower bound, it is easy to see that

$$\sqrt{\frac{3}{4}(r+2)+\frac{nr}{n-1}+\frac{2\sqrt{3}}{n-1}\frac{nr}{\sqrt{r+1}}} > \sqrt{\frac{3}{4}(r+2)+r} > \sqrt{r+2}.$$

Hence the lower bound in Corollary 3.2 is also an improvement on that in (14).

Next we consider the Laplacian-energy-like invariant of $\mathcal{Q}$-graph of a regular graph.

Theorem 3.3 *If G is an r-regular graph of order n with m edges, then*
(i)

$$LEL(\mathcal{Q}(G)) \le \tfrac{n(r-2)}{2}\sqrt{2r+2} + \sqrt{r+2} + (n-1)\sqrt{r+2+\tfrac{nr}{n-1}+\tfrac{2\sqrt{r+1}}{n-1}LEL(G)}, \tag{15}$$

where the equality holds in (15) if and only if G is the complete graph K_n.
(ii)

$$LEL(\mathcal{Q}(G)) > \tfrac{n(r-2)}{2}\sqrt{2r+2} + \sqrt{r+2} + (n-1)\sqrt{r+2+\tfrac{nr}{n-1}+\tfrac{2\sqrt{r+1}}{n-1}LEL(G)-\tfrac{3}{4}r}. \tag{16}$$

Proof. Assume that $Sp(L(G)) = \{\mu_1, \ldots, \mu_n\}$ is L-spectrum of G. By the (i) in Lemma 2.2 and (1), one has

$$\begin{aligned} LEL(\mathcal{Q}(G)) &= \sum_{i=1}^{n-1}\left(\sqrt{\tfrac{(r+2+\mu_i)+\sqrt{(r+2+\mu_i)^2-4(r+1)\mu_i}}{2}}\right) + \sum_{i=1}^{n-1}\left(\sqrt{\tfrac{(r+2+\mu_i)-\sqrt{(r+2+\mu_i)^2-4(r+1)\mu_i}}{2}}\right) + (m-n)\sqrt{2r+2} + \sqrt{r+2} \\ &= \sum_{i=1}^{n-1}\sqrt{\left(\sqrt{\tfrac{(r+2+\mu_i)+\sqrt{(r+2+\mu_i)^2-4(r+1)\mu_i}}{2}} + \sqrt{\tfrac{(r+2+\mu_i)-\sqrt{(r+2+\mu_i)^2-4(r+1)\mu_i}}{2}}\right)^2} + (m-n)\sqrt{2r+2} + \sqrt{r+2} \\ &= \sum_{i=1}^{n-1}\sqrt{r+2+\mu_i+2\sqrt{(r+1)\mu_i}} + (m-n)\sqrt{2r+2} + \sqrt{r+2}. \end{aligned} \tag{17}$$

Notice that $\sum_{i=1}^{n-1}\mu_i = 2m = nr$. Applying the Cauchy-Schwarz inequality, one obtains

$$\begin{aligned} LEL(\mathcal{Q}(G)) &\le \sqrt{(n-1)\sum_{i=1}^{n-1}(r+2+\mu_i+2\sqrt{(r+1)\mu_i})} + (m-n)\sqrt{2r+2} + \sqrt{r+2} \\ &= (n-1)\sqrt{r+2+\tfrac{nr}{n-1}+\tfrac{2\sqrt{r+1}}{n-1}LEL(G)} + \tfrac{n(r-2)}{2}\sqrt{2r+2} + \sqrt{r+2} \end{aligned}$$

with the equality holding if and only if $\mu_1 = \mu_2 = \cdots = \mu_{n-1}$. By Lemma 2.4, G is the complete graph K_n. The proof of the (i) is completed.

Next we prove the (ii). Assume that $a_i = \sqrt{r+2+\mu_i+2\sqrt{(r+1)\mu_i}}$ and $b_i = 1$, where $i = 1, \ldots, n-1$. Take $P = \sqrt{3r+2+2\sqrt{2r(r+1)}}$, $p = \sqrt{r+2}$ and $Q = q = 1$. Since $0 \le \mu_i \le 2r$ and $P = \sqrt{3r+2+2\sqrt{2r(r+1)}} \le \sqrt{7r+2}$. Then $0 < p \le a_i \le P$, $0 < q \le b_i \le Q$ and

$$(PQ-pq)^2 = (\sqrt{3r+2+2\sqrt{2r(r+1)}} - \sqrt{r+2})^2 \le (\sqrt{7r+2} - \sqrt{r+2})^2 < 3r.$$

From (17) and Lemma 2.6, one has

$$\begin{aligned} \sum_{i=1}^{n-1}\sqrt{r+2+\mu_i+2\sqrt{(r+1)\mu_i}} &\ge \sqrt{(n-1)\sum_{i=1}^{n-1}(r+2+\mu_i+2\sqrt{(r+1)\mu_i}) - \tfrac{1}{4}(n-1)^2(PQ-pq)^2} \\ &> (n-1)\sqrt{r+2+\tfrac{nr}{n-1}+\tfrac{2\sqrt{r+1}}{n-1}LEL(G)-\tfrac{3}{4}r}. \end{aligned}$$

From (17), one obtains the required result (ii). □

By Theorem 3.3, we obtain Corollary 3.4 immediately.

Corollary 3.4 *If G is an r-regular graph of order n with m edges, then*
(i)

$$LEL(\mathcal{Q}(G)) \le (n-1)\sqrt{r+2+\tfrac{nr}{n-1}+\tfrac{2\sqrt{r+1}(\sqrt{r+1}+\sqrt{(n-2)(nr-r-1)})}{n-1}} + \tfrac{n(r-2)}{2}\sqrt{2r+2} + \sqrt{r+2},$$

where the equality holds if and only if G is the complete graph K_n.
(ii)

$$LEL(\mathcal{Q}(G)) > \tfrac{n(r-2)}{2}\sqrt{2r+2} + \sqrt{r+2} + (n-1)\sqrt{(\tfrac{3n}{n-1}+\tfrac{1}{4})r+2}.$$

Remark 2 Given an r-regular graph G, Pirzada et al. [23] proved that

$$\tfrac{n(r-2)}{2}\sqrt{2r+2} + n\sqrt{r+2} < LEL(\mathcal{Q}(G)) \le (n-1)\sqrt{r} + \sqrt{r+2} + \tfrac{(nr-2)\sqrt{2r+2}}{2}, \tag{18}$$

where the equality on the right of (18) holds if and only if G is the complete graph K_2. Since $(\frac{3n}{n-1}+\frac{1}{4})r > r$, then the lower bound in Corollary 3.4 is an improvement on that in (18). For the upper bound, one has

$$\begin{aligned}\sqrt{r+2+\tfrac{nr}{n-1}+\tfrac{2\sqrt{r+1}(\sqrt{r+1}+\sqrt{(n-2)(nr-r-1)})}{n-1}} &\le \sqrt{r+2+\tfrac{nr}{n-1}+\tfrac{2\sqrt{r+1}\sqrt{(n-1)nr}}{n-1}}\\ &= \sqrt{r+2+\tfrac{nr}{n-1}+2\sqrt{r+1}\sqrt{\tfrac{nr}{n-1}}}\\ &< \sqrt{r+2+2r+2\sqrt{r+1}\sqrt{2r}}\\ &= \sqrt{2r+2}+\sqrt{r}.\end{aligned}$$

Hence, the upper bound in Corollary 3.4 is also an improvement on that in (18).

We finally consider the LEL of line graph of an (r_1, r_2)-semiregular graph. Pirzada et al. [23] presented the following an upper bound on LEL of line graph $\mathcal{L}(G)$ for an (r_1, r_2)-semiregular graph G, that is,

$$LEL(\mathcal{L}(G)) \le (\frac{nr_1r_2}{r_1+r_2}-n+1)\sqrt{r_1+r_2}+(n-2)\sqrt{\frac{n-1}{n-2}(r_1+r_2)-\frac{2nr_1r_1}{(n-2)(r_1+r_2)}}.$$

Next we shall give a lower bound on LEL of its line graph $\mathcal{L}(G)$.

Theorem 3.5 *If G is an (r_1, r_2)-semiregular graph of order n with m edges, then*

$$LEL(\mathcal{L}(G)) \ge (\frac{nr_1r_2}{r_1+r_2}-n+1)\sqrt{r_1+r_2}+(n-2)\sqrt{\frac{3n-2}{4n-8}(r_1+r_2)-\frac{2nr_1r_2}{(n-2)(r_1+r_2)}}.$$

Proof. Suppose that $Sp(L(G)) = \{\mu_1, \ldots, \mu_n\}$ is the L-spectrum of G. Since $\mu_1 = r_1 + r_2$ and $\mu_n = 0$, then from (1) and (3), one gets

$$LEL(\mathcal{L}(G)) = (m-n+1)\sqrt{r_1+r_2}+\sum_{i=2}^{n-1}\sqrt{r_1+r_2-\mu_i}.$$

Now, assume that $a_i = \sqrt{r_1+r_2-\mu_i}$ and $b_i = 1$, $i = 2, \ldots, n-1$. Take $P = \sqrt{r_1+r_2}$, $p = 0$ and $Q = q = 1$. Obviously, $0 \le p \le a_i \le P$, $0 \le q \le b_i \le Q$, $PQ \ne 0$ and $(1 + p/P)(1 + q/Q) \ge 2$. From Lemma 2.7, we have

$$\begin{aligned}LEL(\mathcal{L}(G)) &\ge (m-n+1)\sqrt{r_1+r_2}+\sqrt{(n-2)\sum_{i=2}^{n-1}(r_1+r_2-\mu_i)-\tfrac{1}{4}(n-2)^2(r_1+r_2)}\\ &= (m-n+1)\sqrt{r_1+r_2}+(n-2)\sqrt{r_1+r_2-\tfrac{2m-(r_1+r_2)}{n-2}-\tfrac{1}{4}(r_1+r_2)}\\ &= (m-n+1)\sqrt{r_1+r_2}+(n-2)\sqrt{\tfrac{3n-2}{4n-8}(r_1+r_2)-\tfrac{2m}{n-2}},\end{aligned}$$

which yields the required result as $m = nr_1r_2/(r_1 + r_2)$. □

4. The incidence energy

In this section, we shall give some new bounds for IE of $\mathcal{R}$-graph and $\mathcal{Q}$-graph of regular graphs, as well as for the line graph of semiregular graphs. Now we first consider IE of $\mathcal{R}$-graph of regular graphs.

Theorem 4.1 *If G is an r-regular graph of order n with m edges, then*
(i)

$$IE(\mathcal{R}(G)) \le \tfrac{n(r-2)}{2}\sqrt{2}+\sqrt{3r+2+4\sqrt{r}}+(n-1)\sqrt{\tfrac{2n-3}{n-1}r+2\sqrt{\tfrac{3n-4}{n-1}r+2}}, \quad (19)$$

where the equality holds if and only if G is the complete graph K_n.
(ii)

$$IE(\mathcal{R}(G)) > \tfrac{n(r-2)}{2}\sqrt{2}+\sqrt{3r+2+4\sqrt{r}}+(n-1)\sqrt{(\tfrac{2n-3}{n-1}-\tfrac{2-\sqrt{3}}{2})r+2\sqrt{(\tfrac{3n-4}{n-1}-\tfrac{3-2\sqrt{2}}{2})r+2}}. \quad (20)$$

Proof. Assume that $Sp(Q(G)) = \{q_1, \dots, q_n\}$ is the Q-spectrum of G. Notice that $q_1 = 2r$ as G is r-regular. Then from (2) and the (ii) in Lemma 2.1, we obtain, by a simple calculation,

$$IE(\mathcal{R}(G)) = \sum_{i=2}^{n} \sqrt{r + q_i + 2 + 2\sqrt{2r + q_i}} + (m - n)\sqrt{2} + \sqrt{3r + 2 + 4\sqrt{r}}. \tag{21}$$

Clearly, $\sum_{i=2}^{n} q_i = 2m - 2r = (n-2)r$. Applying the Cauchy-Schwarz inequality, one obtains

$$\begin{aligned}\sum_{i=2}^{n} \sqrt{r + 2 + q_i + 2\sqrt{2r + q_i}} &\le \sqrt{(n-1)\sum_{i=2}^{n} (r + 2 + q_i + 2\sqrt{2r + q_i})} \\ &= (n-1)\sqrt{\tfrac{2n-3}{n-1} r + 2 + \tfrac{2}{n-1}\sum_{i=2}^{n} \sqrt{2r + q_i}} \\ &\le (n-1)\sqrt{\tfrac{2n-3}{n-1} r + 2 + \tfrac{2}{n-1}\sqrt{(n-1)\sum_{i=2}^{n} (2r + q_i)}} \\ &= (n-1)\sqrt{\tfrac{2n-3}{n-1} r + 2 + 2\sqrt{\tfrac{3n-4}{n-1} r}}.\end{aligned}$$

From (21), we obtain the desired upper bound (19). Moreover, above equality occurs if and only if $q_1 = 2r$ and $q_2 = q_3 = \cdots = q_n$. Thus by Lemma 2.5, G is the complete graph K_n. The proof of the (i) is completed.

Next we prove the (ii). Assume that $a_i = \sqrt{r + 2 + q_i + 2\sqrt{2r + q_i}}$ and $b_i = 1$, $i = 2, \dots, n$. Take $P = \sqrt{3r + 2 + 4\sqrt{r}}$, $p = \sqrt{r + 2 + 2\sqrt{2r}}$ and $q = Q = 1$. Since $0 \le q_i \le 2r$, then $0 < p \le a_i \le P$, $0 < q \le b_i \le Q$. By a simple computation, one has

$$(PQ - pq)^2 = \left(\sqrt{3r + 2 + 4\sqrt{r}} - \sqrt{r + 2 + 2\sqrt{2r}}\right)^2 < (4 - 2\sqrt{3})r.$$

Then by Lemma 2.6, one has

$$\begin{aligned}\sum_{i=2}^{n} \sqrt{r + 2 + q_i + 2\sqrt{2r + q_i}} &\ge \sqrt{(n-1)\sum_{i=2}^{n} (r + 2 + q_i + 2\sqrt{2r + q_i}) - \tfrac{1}{4}(n-1)^2(PQ - pq)^2} \\ &> (n-1)\sqrt{\tfrac{(2n-3)r}{n-1} + 2 + \tfrac{2}{n-1}\sum_{i=2}^{n} \sqrt{2r + q_i} - \tfrac{1}{4}(4 - 2\sqrt{3})r}.\end{aligned}$$

Similarly, assume that $a_i = \sqrt{2r + q_i}$ and $b_i = 1$, $i = 2, \dots, n$. Take $P = 2\sqrt{r}$, $p = \sqrt{2r}$ and $Q = q = 1$. Since $0 \le q_i \le 2r$, then $0 < p \le a_i \le P$, $0 < q \le b_i \le Q$. Again by Lemma 2.6, one has

$$\begin{aligned}\sum_{i=2}^{n} \sqrt{2r + q_i} &\ge \sqrt{(n-1)\sum_{i=2}^{n} (2r + q_i) - \tfrac{1}{4}(n-1)^2(2\sqrt{r} - \sqrt{2r})^2} \\ &= (n-1)\sqrt{(\tfrac{3n-4}{n-1} - \tfrac{3-2\sqrt{2}}{2})r}.\end{aligned}$$

Hence,

$$\sum_{i=2}^{n} \sqrt{r + 2 + q_i + 2\sqrt{2r + q_i}} > (n-1)\sqrt{(\tfrac{2n-3}{n-1} - \tfrac{2-\sqrt{3}}{2})r + 2\sqrt{(\tfrac{3n-4}{n-1} - \tfrac{3-2\sqrt{2}}{2})r} + 2},$$

which, along with (21), implies the required result (ii). □

Next we shall consider IE of $\mathcal{Q}$-graph for regular graphs.

Theorem 4.2 *If G is an r-regular graph of order n with m edges, then*
(i)

$$IE(\mathcal{Q}(G)) \le \tfrac{n(r-2)}{2}\sqrt{2r + 2} + \sqrt{5r - 2 + 4\sqrt{r(r-1)}} + (n-1)\sqrt{\tfrac{4n-5}{n-1} r + 2\sqrt{\tfrac{3n-4}{n-1} r(r-1)} - 2}, \tag{22}$$

where the equality holds in (22) if and only if G is the complete graph K_n.
(ii)

$$IE(\mathcal{Q}(G)) > \frac{n(r-2)}{2}\sqrt{2r+2} + \sqrt{5r-2+4\sqrt{r(r-1)}} + (n-1)\sqrt{\left(\frac{4n-5}{n-1}-\frac{1}{4}\right)r + 2\sqrt{\left(\frac{3n-4}{n-1}-\frac{3-2\sqrt{2}}{2}\right)r(r-1)} - 2}. \tag{23}$$

Proof. Assume that $Sp(Q(G)) = \{q_1, \dots, q_n\}$ is the Q-spectrum of G. Notice that $q_1 = 2r$ as G is r-regular. Then from (2) and the (ii) in Lemma 2.2, it is easy to see that, by a simple calculation,

$$IE(Q(G)) = \sum_{i=2}^{n}\sqrt{3r+q_i-2+2\sqrt{r(2r+q_i-2)-q_i}} + (m-n)\sqrt{2r-2} + \sqrt{5r-2+4\sqrt{r(r-1)}}. \tag{24}$$

Clearly, $\sum_{i=2}^{n} q_i = 2m - 2r = (n-2)r$. Applying the Cauchy-Schwarz inequality, one obtains

$$\begin{aligned}
\sum_{i=2}^{n}\sqrt{3r+q_i-2+2\sqrt{r(2r-2+q_i)-q_i}} &\le \sqrt{(n-1)\sum_{i=2}^{n}\left(3r+q_i-2+2\sqrt{r(2r-2+q_i)-q_i}\right)}\\
&= (n-1)\sqrt{\frac{4n-5}{n-1}r - 2 + \frac{2}{n-1}\sum_{i=2}^{n}\sqrt{r(2r-2+q_i)-q_i}}\\
&\le (n-1)\sqrt{\frac{4n-5}{n-1}r - 2 + \frac{2}{n-1}\sqrt{(n-1)\sum_{i=2}^{n}[r(2r-2+q_i)-q_i]}}\\
&= (n-1)\sqrt{\frac{4n-5}{n-1}r - 2 + 2\sqrt{\frac{3n-4}{n-1}r(r-1)}},
\end{aligned}$$

which, along with (24), implies the desired upper bound. Moreover, above equality occurs if and only if $q_1 = 2r$ and $q_2 = q_3 = \cdots = q_n$. Thus from Lemma 2.5, G is the complete graph K_n. The proof of the (i) is completed.

Now we prove the (ii). Assume that $a_i = \sqrt{3r-2+q_i+2\sqrt{r(2r-2+q_i)-q_i}}$ and $b_i = 1$, where $i = 2, \dots, n$. Take $P = \sqrt{5r-2+4\sqrt{r(r-1)}}$, $p = \sqrt{3r-2+2\sqrt{2r(r-1)}}$ and $Q = q = 1$. Since $0 \le q_i \le 2r$, then $0 < p \le a_i \le P$, $0 < q \le b_i \le Q$. By a simple computation, one has

$$(PQ-pq)^2 = \left(\sqrt{5r-2+4\sqrt{r(r+1)}} - \sqrt{3r-2+2\sqrt{2r(r-1)}}\right)^2 < r.$$

Then by Lemma 2.6, one has

$$\begin{aligned}
\sum_{i=2}^{n}\sqrt{3r-2+q_i+2\sqrt{r(2r-2+q_i)-q_i}} &\ge \sqrt{(n-1)\sum_{i=2}^{n}\left(3r-2+q_i+2\sqrt{r(2r-2+q_i)-q_i}\right) - \frac{1}{4}(n-1)^2(PQ-pq)^2}\\
&> \sqrt{(n-1)\sum_{i=2}^{n}\left(3r-2+q_i+2\sqrt{r(2r-2+q_i)-q_i}\right) - \frac{1}{4}(n-1)^2 r}\\
&= (n-1)\sqrt{\left(\frac{4n-5}{n-1}-\frac{1}{4}\right)r - 2 + \frac{2}{n-1}\sum_{i=2}^{n}\sqrt{r(2r-2+q_i)-q_i}}.
\end{aligned}$$

Similarly, suppose that $a_i = \sqrt{r(2r-2+q_i)-q_i}$ and $b_i = 1$, $i = 2, 3, \dots, n$. Take $P = 2\sqrt{r(r-1)}$, $p = \sqrt{2r(r-1)}$ and $Q = q = 1$. Since $0 \le q_i \le 2r$, then $0 < p \le a_i \le P$, $0 < q \le b_i \le Q$. Again, from Lemma 2.6, one obtains

$$\begin{aligned}
\sum_{i=2}^{n}\sqrt{r(2r-2+q_i)-q_i} &\ge \sqrt{(n-1)\sum_{i=2}^{n}[r(2r-2+q_i)-q_i] - \frac{1}{4}(n-1)^2(PQ-pq)^2}\\
&= (n-1)\sqrt{2r(r-1) + \frac{n-2}{n-1}r(r-1) - \frac{1}{4}(2-\sqrt{2})^2 r(r-1)}\\
&= (n-1)\sqrt{\left(\frac{3n-4}{n-1}-\frac{3-2\sqrt{2}}{2}\right)r(r-1)}.
\end{aligned}$$

Hence,

$$\sum_{i=2}^{n}\sqrt{3r-2+q_i+2\sqrt{r(2r-2+q_i)-q_i}} > (n-1)\sqrt{\left(\frac{4n-5}{n-1}-\frac{1}{4}\right)r + 2\sqrt{\left(\frac{3n-4}{n-1}-\frac{3-2\sqrt{2}}{2}\right)r(r-1)} - 2}.$$

It follows from (24) that the (ii) holds. □

We finally consider the incidence energy of line graph for an (r_1, r_2)-semiregular graph. In [25], an upper bound on IE of line graph $\mathcal{L}(G)$ for an (r_1, r_2)-semiregular graph G was obtained as follows:

$$IE(\mathcal{L}(G)) \leq (\tfrac{nr_1r_2}{r_1+r_2} - n + 1)\sqrt{r_1 + r_2 - 4} + \sqrt{2(r_1 + r_2) - 4} + (n-2)\sqrt{\tfrac{n-3}{n-2}(r_1 + r_2) + \tfrac{2nr_1r_2}{(n-2)(r_1+r_2)} - 4}.$$

Below one gives a lower bound for IE of its line graph $\mathcal{L}(G)$.

Theorem 4.3 *If G is an (r_1, r_2)-semiregular graph with n vertices and m edges, then*

$$IE(\mathcal{L}(G)) \geq (\tfrac{nr_1r_2}{r_1+r_2} - n + 1)\sqrt{r_1 + r_2 - 4} + \sqrt{2(r_1 + r_2) - 4} + (n-2)\sqrt{\tfrac{3n-10}{4n-8}(r_1 + r_2) + \tfrac{2nr_1r_2}{(n-2)(r_1+r_2)} - 4}.$$

Proof. Suppose that $Sp(Q(G)) = \{q_1, \ldots, q_n\}$ is the Q-spectrum of G. Notice that $q_1 = r_1 + r_2$ and $q_n = 0$ as G is bipartite. Then from (2) and (4), one gets

$$IE(L(G)) = (m - n + 1)\sqrt{r_1 + r_2 - 4} + \sqrt{2(r_1 + r_2) - 4} + \sum_{i=2}^{n-1} \sqrt{r_1 + r_2 - 4 + q_i}. \tag{25}$$

Now, assume that $a_i = \sqrt{r_1 + r_2 - 4 + q_i}$ and $b_i = 1$, $i = 2, \ldots, n-1$. Take $P = \sqrt{2(r_1 + r_2) - 4}$, $p = \sqrt{r_1 + r_2 - 4}$ and $Q = q = 1$. Obviously, $0 \leq p \leq a_i \leq P$, $0 \leq q \leq b_i \leq Q$, $PQ \neq 0$ and $(1 + p/P)(1 + q/Q) \geq 2$. By a simple computation, one has

$$(PQ - pq)^2 = (\sqrt{2(r_1 + r_2) - 4} - \sqrt{r_1 + r_2 - 4})^2 \leq r_1 + r_2.$$

From Lemma 2.7, one has

$$\begin{aligned}\sum_{i=2}^{n-1} \sqrt{r_1 + r_2 - 4 + q_i} &\geq \sqrt{(n-2)\sum_{i=2}^{n-1}(r_1 + r_2 - 4 + q_i) - \tfrac{1}{4}(n-2)^2(r_1 + r_2)} \\ &= (n-2)\sqrt{r_1 + r_2 - 4 + \tfrac{2m-(r_1+r_2)}{n-2} - \tfrac{1}{4}(r_1 + r_2)} \\ &= (n-2)\sqrt{\tfrac{3n-10}{4n-8}(r_1 + r_2) + \tfrac{2m}{n-2} - 4},\end{aligned}$$

which, along with (25), implies the required result as $m = nr_1r_2/(r_1 + r_2)$. □

Acknowledgements We would like to thank the anonymous referees for careful reading of our manuscript and for invaluable comments. We are also very grateful to the managing editor for the editing and typesetting work thoughtfully for this paper. This work was supported by the National Natural Science Foundation of China (11801521).

References

[1] Cvetković D., New theorems for signless Laplacians eigenvalues, *Bull. Acad. Serbe Sci. Arts, Cl. Sci. Math. Natur., Sci. Math.*, 2008, 137(33), 131-146.
[2] Cvetković D., Doob M., Sachs H., *Spectra of Graphs: Theory and Application*, 1980, New York: Academic press.
[3] Cvetković D., Rowinson P., Simić H., *An introduction to the Theory of Graph Spectra*, 2010, Cambridge: Cambridge University Press.
[4] Cvetković D., Rowlinson P., Simić S.K., Signless Laplacians of finite graphs, *Linear Algebra Appl.*, 2007, 423, 155-171.
[5] Grone R., Merris R., Sunder V.S., The Laplacian spectrum of a graph, *SIAM J. Matrix Anal. Appl.*, 1990, 11, 218-238.
[6] Merris R., Laplacian matrices of graphs: a survey, *Linear Algebra Appl.*, 1994, 197-198, 143-176.
[7] Liu J., Liu B., A Laplacian-energy-like invariant of a graph, *MATCH Commun. Math. Comput. Chem.*, 2008, 59, 397-419.
[8] Gutman I., Zhou B., Laplacian energy of a graph, *Linear Algebra Appl.*, 2006, 414, 29-37.
[9] Li X., Shi Y., Gutman I., *Graph Energy*, 2012, New York: Springer.
[10] Stevanović D., Ilić A., Onisor C., Diudea M., LEL-a newly designed molecular descriptor, *Acta Chim. Slov.*, 2009, 56, 410-417.

[11] Das K.C., Gutman I., Çevik A.S., On the Laplacian-energy-like invariant, *Linear Algebra Appl.*, 2014, 442, 58-68.

[12] Gutman I., Zhou B., Furtula B., The Laplacian-energy-like invariant is an energy like invariant, *MATCH Commun. Math. Comput. Chem.*, 2010, 64, 85-96.

[13] Liu B., Huang Y., You Z., A survey on the Laplacian-energy-like invariant, *MATCH Commun. Math. Comput. Chem.*, 2011, 66, 713-730.

[14] Liu J.B., Pan X.F., Hu F.T., Hu F.F., Asymptotic Laplacian-energy-like invariant of lattices, *Appl. Math. Comput.*, 2015, 253, 205-214.

[15] Wang W., Luo Y., On Laplacian-energy-like invariant of a graph, *Linear Algebra Appl.*, 2012, 437, 713-721.

[16] Xu K., Das K.C., Extremal Laplacian-energy-like invariant of graphs with given matching number, *Electron J. Linear Algebra*, 2013, 26, 131-140.

[17] Zhu B.X., The Laplacian-energy like of graphs, *Appl. Math. Lett.*, 2011, 24, 1604-1607.

[18] Nikiforov V., The energy of graphs and matrices, *J. Math. Anal. Appl.*, 2007, 326, 1472-1475.

[19] Jooyandeh M.R., Kiani D., Mirzakhah M., Incidence energy of a graph, *MATCH Commun. Math. Comput. Chem.*, 2009, 62, 561-572.

[20] Gutman I., Kiani D., Mirzakhah M., On incidence energy of graphs, *MATCH Commun. Math. Comput. Chem.*, 2009, 62, 573-580.

[21] Gutman I., Kiani D., Mirzakhah M., Zhou B., On incidence energy of a graph, *Linear Algebra Appl.*, 2009, 431, 1223-1233.

[22] Zhou B., More upper bounds for the incidence energy, *MATCH Commun. Math. Comput. Chem.*, 2010, 64, 123-128.

[23] Pirzada S., Ganie H.A., Gutman I., On Laplacian-energy-like invariant and Kirchhoff index, *MATCH Commun. Math. Comput. Chem.*, 2015, 73, 41-59.

[24] Wang W., Luo Y., Gao X., On incidence energy of some graphs, *Ars Combin.*, 2014, 114, 427-436.

[25] Wang W., Yang D., Bounds for incidence energy of some graphs, *J. Appl. Math.*, 2013, article ID 757542.

[26] Chen X., Hou Y., Li J., On two energy-like invariants of line graphs and related graph operations, *J. Inequal. Appl.*, 2016, 51, 1-15.

[27] Deng A., Kelmans A., Meng J., Laplacian spectra of regular graph transformations, *Discrete Appl. Math.*, 2013, 161, 118-133.

[28] Wang W., Yang D., Luo Y., The Laplacian polynomial and Kirchhoff index of graphs derived from regular graphs, *Discrete Appl. Math.*, 2013, 161, 3063-3071.

[29] Li J., Zhou B., Signless Laplacian characteristic polynomials of regular graph transformations, 2013, *arXiv*: 1303.5527v1.

[30] Das K.C., A sharp upper bound for the number of spanning trees of a graph, *Graphs Comb.*, 2007, 23, 625-632.

[31] Ozeki N., On the estimation of the inequalities by the maximum, or minimum values, *J. College Arts Sci. Chiba Univ.*, 1968, 5, 199-203.

[32] Izumino S., Mori H., Seo Y., On Ozeki's inequality, *J. Inequal. Appl.*, 1998, 2, 235-253.