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Lyapunov stable homoclinic classes for smooth vector fields

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Abstract: In this paper, we show that for generic C^1, if a flow X^t has the shadowing property on a bi-Lyapunov stable homoclinic class, then it does not contain any singularity and it is hyperbolic.

Keywords: homoclinic class; Lyapunov stable; shadowing; generic; hyperbolic

MSC: 37C50; 37C10; 37C20; 37C29; 37D05

1 Introduction

Let M be a compact smooth Riemannian manifold, and let Diff(M) be the space of diffeomorphisms of M endowed with the C^1 topology. Hyperbolicity and stability have been important topics in differentiable dynamical systems since they were introduced by Smale [1]. For instance, a diffeomorphism $f : M \to M$ is structurally stable if and only if it satisfies Axiom A and the strong transversality condition. A diffeomorphism $f : M \to M$ satisfies Axiom A if the nonwandering set $\Omega(f)$ is $\overline{P(f)}$ and is hyperbolic, where $P(f)$ is the set of all periodic points of f. A set of diffeomorphisms is generic (or residual) if it contains a countable intersection of dense open sets of Diff(M). Abraham and Smale [2] showed that the set of diffeomorphisms $f : M \to M$ satisfying Axiom A and the no-cycle condition is not dense in the space of Diff(M).

If a diffeomorphism $f : M \to M$ satisfies Axiom A, then from the work of Smale [1], the nonwandering set $\Omega(f) = \bigcup_{i=1}^{n} \Lambda_i$, where each Λ_i is a basic set. If a basic set contains a hyperbolic periodic point, then it is a homoclinic class. In general, a homoclinic class is not hyperbolic even in a generic sense. For a C^1 generic diffeomorphism $f : M \to M$, several extra conditions are imposed to obtain hyperbolicity of the homoclinic classes.

Let us give a short review of related results. Ahn *et al.* [3] proved that for generic C^1, if a diffeomorphism f has the shadowing property on a locally maximal homoclinic class, then it is hyperbolic. Lee [4] proved that for generic C^1, if a diffeomorphism f has the limit shadowing property on a locally maximal homoclinic class, then it is hyperbolic. Note that local maximality is quite a restrictive condition. Arbieto *et al.* [5] proved that for generic C^1, if a bi-Lyapunov stable homoclinic class is homogeneous and has the shadowing property, then it is hyperbolic. See [3, 4, 6–15] for related results.

We want to extend some of the above results for flows, that is, for a C^1 generic vector field $X \in \mathfrak{X}(M)$, a condition under which we can obtain hyperbolicity of homoclinic classes. Unfortunately, we cannot use the same arguments as in the diffeomorphism case.

We say that a diffeomorphism f satisfies the *star condition* if there is a C^1 neighborhood $\mathcal{U}(f) \subset$ Diff(M) such that for any $g \in \mathcal{U}(f)$, every periodic point of g is hyperbolic. Aoki [16] and Hayashi [17] showed that if a diffeomorphism f satisfies the star condition, then it is Axiom A and the no-cycle condition, that is, Ω stable.

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We say that a flow X^t satisfies the *star condition* if there is a C^1 neighborhood $\mathcal{U}(X) \subset \mathfrak{X}(M)$ such that for any $Y \in \mathcal{U}(X)$, every critical point of Y is hyperbolic. From the results of Guchenheimer [18], the Lorenz attractor satisfies the star condition, but it is not Ω-stable because the attractor contains a hyperbolic singular point. However, if a flow does not contain singularities and satisfies the star condition, then it is Ω stable (see [19]).

2 Basic notions and main theorem

Let M be a compact n (≥ 3)-dimensional smooth Riemannian manifold, and let d be the distance on M induced from a Riemannian metric $\|\cdot\|$ on the tangent bundle TM, and denote by $\mathfrak{X}(M)$ the set of C^1 vector fields on M endowed with the C^1 topology. Then, every $X \in \mathfrak{X}(M)$ generates a C^1 flow $X^t : M \times \mathbb{R} \to M$; that is, a C^1 map such that $X^t : M \to M$ is a diffeomorphism satisfying $X^0(x) = x$, and $X^{t+s}(x) = X^t(X^s(x))$ for all $s, t \in \mathbb{R}$ and $x \in M$. The flow of X will be denoted by $X^t,\ t \in \mathbb{R}$. For $X \in \mathfrak{X}(M)$, a point $x \in M$ is *singular* of X if $X(x) = 0$. Denote by $Sing(X)$ the set of all singular points of X. A point $x \in M$ is *regular* if $x \in M \backslash Sing(X)$. Denote by $R(M)$ the set of all regular points of X. A point $p \in M$ is *periodic* if there is $\pi(p) > 0$ such that $X^{\pi(p)}(p) = p$, where $\pi(p)$ is the prime period of p. Denote by $Per(X)$ the set of all closed orbits of X. Let $Crit(X) = Sing(X) \cup Per(X)$. For any $\delta > 0$, a sequence $\{(x_i, t_i) : x_i \in M, t_i \geq 1,\ \text{and}\ i \in \mathbb{Z}\}$ is a δ-pseudo-orbit of X if $d(X^{t_i}(x_i), x_{i+1}) < \delta$ for any $i \in \mathbb{Z}$.

An increasing homeomorphism $h : \mathbb{R} \to \mathbb{R}$ with $h(0) = 0$ is called a *reparametrization* of $\mathbb{R}$. Denote by $\mathrm{Rep}(\mathbb{R})$ the set of reparametrizations of $\mathbb{R}$. Fix $\epsilon > 0$ and define $\mathrm{Rep}(\epsilon)$ as follows:

$$\mathrm{Rep}(\epsilon) = \Big\{h \in \mathrm{Rep} : \Big|\frac{h(t)}{t} - 1\Big| < \epsilon\Big\}.$$

For a closed X^t-invariant set $\Lambda \subset M$, we say that X has the *shadowing property* on Λ if for any $\epsilon > 0$, there is $\delta > 0$ satisfying the following property: given any δ-pseudo-orbit $\xi = \{(x_i, t_i) : t_i \geq 1,\ i \in \mathbb{Z}\}$ with $x_i \in \Lambda$, there is a point $y \in M$ and an increasing homeomorphism $h \in \mathrm{Rep}(\epsilon)$ such that $d(X^{h(t)}(y), X^{t-s_i}(x_i)) < \epsilon$ for any $s_i < t < s_{i+1}$, where s_i is defined as

$$s_i = \begin{cases} t_0 + t_1 + \cdots + t_{i-1}, & \text{if } i > 0 \\ 0, & \text{if } i = 0 \\ -t_{-1} - t_{-2} - \cdots - t_i, & \text{if } i < 0. \end{cases}$$

The point $y \in M$ is said to be a *shadowing point* of ξ.

Let X^t be the flow of $X \in \mathfrak{X}(M)$, and let Λ be a X^t-invariant compact set. The set Λ is called *hyperbolic* for X^t if there are constants $C > 0,\ \lambda > 0$ and a splitting $T_xM = E^s_x \oplus \langle X(x)\rangle \oplus E^u_x$ such that the tangent flow $DX^t : TM \to TM$ leaves the continuous splitting invariant and

$$\|DX^t|_{E^s_x}\| \leq Ce^{-\lambda t} \ \text{ and } \ \|DX^{-t}|_{E^u_x}\| \leq Ce^{-\lambda t}$$

for $t > 0$ and $x \in \Lambda$. We say that $X \in \mathfrak{X}(M)$ is *Anosov* if M is hyperbolic for X.

Let γ be a hyperbolic closed orbit of a vector field $X \in \mathfrak{X}(M)$, and we define the *stable* and *unstable manifolds* of γ by

$$W^s(\gamma) = \{y \in M : \omega(y) = \gamma\}$$

and

$$W^u(\gamma) = \{y \in M : \alpha(y) = \gamma\}.$$

Let $X \in \mathfrak{X}(M)$, and let γ be a hyperbolic closed orbit of X^t. A point $x \in W^s(\gamma) \pitchfork W^u(\gamma)$ is called a *transversal homoclinic point* of X^t associated to γ. The closure of the transversal homoclinic points of X^t associated to γ is called the *homoclinic class* of X^t associated to γ, and it is denoted by

$$H_X(\gamma) = \overline{W^s(\gamma) \pitchfork W^u(\gamma)}.$$

It is clear that $H_X(\gamma)$ is a compact, transitive, and X^t-invariant set.

For two hyperbolic closed orbits γ_1 and γ_2 of X^t, we say that γ_1 and γ_2 are *homoclinic related*, denoted by $\gamma_1 \sim \gamma_2$, if $W^s(\gamma_1) \pitchfork W^u(\gamma_2) \neq \emptyset$ and $W^u(\gamma_1) \pitchfork W^s(\gamma_2) \neq \emptyset$. It is clear that if $\gamma_1 \sim \gamma$, then $\mathrm{index}(\gamma_1) = \mathrm{index}(\gamma)$, where $\mathrm{index}(\gamma) = \dim W^s(\gamma)$. Note that if γ is a hyperbolic closed orbit of X^t, then there exist a C^1 neighborhood $\mathcal{U}(X)$ of X and a neighborhood U of γ such that for any $Y \in \mathcal{U}(X)$, there exists a unique hyperbolic closed orbit γ_Y that equals $\bigcap_{t\in\mathbb{R}} Y^t(U)$. The hyperbolic closed orbit γ_Y is called the *continuation* of γ with respect to Y, and $\mathrm{index}(\gamma) = \mathrm{index}(\gamma_Y)$.

A closed invariant set Λ is *Lyapunov stable* if for any neighborhood U of Λ, there is a neighborhood V of Λ such that $X^t(V) \subset U$ for all $t > 0$. We say that Λ is *bi-Lyapunov stable* if it is Lyapunov stable for X and for $-X$.

We say that a subset $\mathcal{G} \subset \mathfrak{X}(M)$ is *residual* if $\mathcal{G}$ contains the intersection of a countable family of open and dense subsets of $\mathfrak{X}(M)$. In this case $\mathcal{G}$ is dense in $\mathfrak{X}(M)$. A property "P" is said to be C^1*-generic* if "P" holds for all vector fields that belong to some residual subset of $\mathfrak{X}(M)$. We write *for C^1 generic $X \in \mathfrak{X}(M)$* in the sense that there is a residual set $\mathcal{G} \subset \mathfrak{X}(M)$ for any $X \in \mathcal{G}$. In this paper, we prove the following theorem, which is an extension of a result of Arbieto *et al.* [5] for flows.

Theorem. *For C^1 generic $X \in \mathfrak{X}(M)$, if a flow X^t has the shadowing property on a bi-Lyapunov stable homoclinic class $H_X(\gamma)$, then $H_X(\gamma) \cap Sing(X) = \emptyset$ and $H_X(\gamma)$ is hyperbolic.*

3 Proof of the Theorem

Let M be as previously, and let $X \in \mathfrak{X}(M)$. We define the strong stable and unstable manifolds of a hyperbolic periodic point p respectively as follows:

$$W^{ss}(p) = \{y \in M : d(X^t(y), X^t(p)) \to 0 \text{ as } t \to \infty\}$$

and

$$W^s(Orb(p))) = \bigcup_{t\in\mathbb{R}} W^{ss}(X^t(p)),$$

where $Orb(p)$ is the orbit of p. If $\epsilon > 0$, the local strong stable manifold is defined as

$$W^{ss}_{\epsilon(p)}(p) = \{y \in M : d(X^t(y), X^t(p)) < \epsilon, \text{ if } t \geq 0\}.$$

By the stable manifold theorem, there is an $\epsilon = \epsilon(p) > 0$ such that

$$W^{ss}(p) = \bigcup_{t\geq 0} X^{-t}(W^{ss}_{\epsilon(p)}(X^t(p))).$$

We can define the unstable manifolds similarly. If σ is a hyperbolic singularity of X, then there exists an $\epsilon = \epsilon(\sigma) > 0$ such that

$$W^s_\epsilon(\sigma) = \{x \in M : d(X^t(x), \sigma) < \epsilon \text{ as } t \geq 0\}$$

and

$$W^s(\sigma) = \bigcup_{t\geq 0} X^t(W^s_\epsilon(\sigma)).$$

Analogous definitions hold for unstable manifolds.

3.1 Transversal intersection and the absence of singularities

The following lemma states that there are transversal intersections between invariant manifolds of hyperbolic closed orbits and singularities.

Lemma 3.1. *Let γ be a hyperbolic closed orbit of X. If a flow X^t has the shadowing property on $H_X(\gamma)$, then for every hyperbolic $\sigma \in H_X(\gamma) \cap Crit(X)$, we have*

$$W^s(\gamma) \cap W^u(\sigma) \neq \emptyset \quad \text{and} \quad W^u(\gamma) \cap W^s(\sigma) \neq \emptyset.$$

Proof. First, we assume that $\eta \in H_X(\gamma) \cap Per(X)$. Let $p \in \gamma$ and $q \in \eta$. Take $\epsilon = \min\{\epsilon(p), \epsilon(q)\}$ and let $0 < \delta \leq \epsilon$ be given by the shadowing property according to ϵ. Since $H_X(\gamma)$ is transitive, there is $x \in H_X(\gamma)$ such that $\omega(x) = H_X(\gamma)$. Then, there are $t_1 > 0$ and $t_2 > 0$ such that $X^{t_1}(x) \in B_\delta(p)$ and $X^{t_2}(x) \in B_\delta(q)$. Assume that $t_2 = t_1 + k$ for some $k > 0$. Then, the sequence

$$\{p, X^{t_1}(x), X^{t_1+1}(x), \ldots, X^{t_1+k-1}(x), q\} \subset H_X(\gamma)$$

is a finite δ- pseudo-orbit of X. We construct a δ-pseudo-orbit $\{(x_i, t_i) : t_i \geq 1, i \in \mathbb{Z}\} \subset H_X(\gamma)$ as follows:
(i) $X^{-i}(p) = x_{-i}$ for $i \geq 0$;
(ii) $X^{t_1+i}(x) = x_i$ for $i = 1, \ldots, k-1$; and
(iii) $X^i(q) = x_{k+i}$ for all $i \geq 0$.
Since X^t has the shadowing property on $H_X(\gamma)$, there is $y \in M$ and an increasing homeomorphism $h : \mathbb{R} \to \mathbb{R}$ with $h(0) = 0$ such that

$$d(X^{h(t)}(y), X^{t-s_i}(x_i)) < \epsilon \text{ and } d(X^{h(t)}(y), X^{t-s_{-i}}(x_{-i}) < \epsilon,$$

where $s_i < t < s_{i+1}$ and $s_{-i} < t < s_{-i+1}$ for all $t \in \mathbb{R}$ and $i \in \mathbb{Z}$. Then $y \in W^u_\epsilon(p)$ and there is $\tau > 0$ such that $X^\tau(y) \subset W^s_\epsilon(q)$. Thus, we have

$$Orb(y) \cap W^u(\gamma) \cap W^s(\eta) \neq \emptyset.$$

The other case is similar.

Now, we assume that $\sigma \in H_X(\gamma) \cap Sing(X)$. Let $p \in \gamma$. Take $\epsilon = \min\{\epsilon(p), \epsilon(\sigma)\}$ and let $0 < \delta \leq \epsilon$ be given by the shadowing property according to ϵ. Since $H_X(\gamma)$ is transitive, there is $x \in H_X(\gamma)$ such that $\omega(x) = H_X(\gamma)$. Then, there are $t_1 > 0$ and $t_2 > 0$ such that $X^{t_1}(x) \in B_\delta(\sigma)$ and $X^{t_2}(x) \in B_\delta(p)$. Assume that $t_2 = t_1 + k$ for some $k > 0$. We can thus construct a δ-pseudo-orbit $\{(x_i, t_i) : t_i \geq 1, i \in \mathbb{Z}\} \subset H_X(\gamma)$ as follows:
(i) $\sigma = x_{-i}$, $t_{-i} = 1$ for $i \geq 0$;
(ii) $X^{t_1+i}(x) = x_i$ for $i = 1, \ldots, k-1$; and
(iii) $X^i(p) = x_{k+i}$ for all $i \geq 0$.
Since X^t has the shadowing property on $H_X(\gamma)$, as in the proof of previous arguments, we have $W^u(\sigma) \cap W^s(\gamma) \neq \emptyset$. The other case is similar. □

We say that X is *Kupka–Smale* if every $\sigma \in Crit(X)$ is hyperbolic, and their invariant manifolds intersect transversally. Denote by $\mathcal{KS}$ the set of all Kupka–Smale vector fields. It is known that $\mathcal{KS} \subset \mathfrak{X}(M)$ is a residual subset (see [20]).

Lemma 3.2. *There is a residual set $\mathcal{G}_1 \subset \mathfrak{X}(M)$ such that for any $X \in \mathcal{G}_1$, if a flow X^t has the shadowing property on $H_X(\gamma)$, then for all $\eta \in H_X(\gamma) \cap Crit(X)$, we have*

$$W^s(\gamma) \pitchfork W^u(\eta) \neq \emptyset \quad \text{and} \quad W^u(\gamma) \pitchfork W^s(\eta) \neq \emptyset.$$

Proof. Let $X \in \mathcal{G}_1 = \mathcal{KS}$ and let $\eta \in H_X(\gamma) \cap Crit(X)$. Since a flow X^t has the shadowing property on $H_X(\gamma)$, by Lemma 3.1, $W^s(\gamma) \cap W^u(\eta) \neq \emptyset$ and $W^u(\gamma) \cap W^s(\eta) \neq \emptyset$. Since $X \in \mathcal{KS}$, $W^s(\gamma) \pitchfork W^u(\eta) \neq \emptyset$ and $W^u(\gamma) \pitchfork W^s(\eta) \neq \emptyset$. □

Proposition 3.3. *For any $X \in \mathcal{G}_1$, if a flow X^t has the shadowing property on $H_X(\gamma)$, then we have*

$$H_X(\gamma) \cap Sing(X) = \emptyset.$$

Proof. Let $X \in \mathcal{KS}$ and let γ be a hyperbolic periodic orbit of X in $H_X(\gamma)$ with index j. Suppose that X has a hyperbolic singularity $\sigma \in H_X(\gamma)$ with index i. If $j < i$, then $\dim W^u(\sigma) + \dim W^s(\gamma) \leq \dim M$. Since X is a Kupka–Smale vector field, we have $\dim W^u(\sigma) + \dim W^s(\gamma) = \dim M$. By assumption, we can take $x \in W^u(\sigma) \cap W^s(\gamma)$.

Then $Orb(x) \in W^u(\sigma) \cap W^s(\gamma)$ and we can split

$$T_x(W^u(\sigma)) = T_x(Orb(x)) \oplus E^1 \quad \text{and} \quad T_x(W^s(\gamma)) = Tx(Orb(x)) \oplus E^2.$$

Thus, we know that

$$\dim(T_x(W^u(\sigma)) + T_x(W^s(\gamma))) < \dim W^u(\sigma) + \dim W^s(\gamma) = \dim M.$$

This is a contradiction, because X is a Kupka–Smale vector field. If $j \geq i$, then

$$\dim W^s(\sigma) + \dim W^u(\gamma) \leq \dim M.$$

By the previous arguments, we have a contradiction. Thus, $H_X(\gamma) \cap Sing(X) = \emptyset$. □

3.2 Chain recurrent class and homoclinic class

For any $x, y \in M$, we say that $x \rightsquigarrow y$ if for any $\delta > 0$, there is a finite δ-pseudo-orbit $\{(x_i, t_i) : 0 \leq i < n\}$ with $n > 1$ such that $x_0 = x$ and $d(X^{t_{n-1}}(x_{n-1}), y) < \delta$ and a δ-pseudo-orbit $\{(z_i, s_i) : 0 \leq i < m\}$ with $m > 1$ such that $z_0 = y$ and $d(X^{s_{m-1}}(z_{m-1}), x) < \delta$. It is easy to see that $\leftrightsquigarrow$ gives an equivalent relation on the chain recurrent set $\mathcal{CR}(X)$. We denoted the equivalence class as

$$C_X(\gamma) = \{x \in M : x \rightsquigarrow \gamma \quad \text{and} \quad \gamma \rightsquigarrow x\}$$

and called the *chain recurrence class* associated to γ. It is known that $H_X(\gamma) \subset C_X(\gamma)$, but the converse is not true in general. We now summarize some results about homoclinic classes and chain recurrence classes.

Lemma 3.4. *There is a residual set $\mathcal{G}_2 \subset \mathfrak{X}(M)$ such that every $X \in \mathcal{G}_2$ satisfies:*
(a) the chain recurrence class $C_X(\gamma) = H_X(\gamma)$ (see [21]);
(b) if a closed orbit $\eta \in H_X(\gamma)$, then $H_X(\gamma) = H_X(\eta)$ (see [22]);
(c) $H_X(\gamma) = \overline{W^s(\gamma)} \cap \overline{W^u(\gamma)}$ (see [22]);
(d) $\overline{W^s(\gamma)}$ is Lyapunov stable for $-X$ and $\overline{W^u(\gamma)}$ is Lyapunov stable for X (see [22]);
(e) if $H_X(\gamma)$ is Lyapunov stable for X, then there is a C^1 neighborhood $\mathcal{U}(X)$ of X such that for every $Y \in \mathcal{U}(X)$, $H_Y(\gamma_Y)$ is Lyapunov stable (see [23]);
(f) there exist a C^1 neighborhood $\mathcal{U}(X)$ of X and an interval of natural numbers $[\alpha, \beta]$ such that for every $Y \in \mathcal{U}(X)$, $H_Y(\gamma_Y)$ has closed orbits of every index in $[\alpha, \beta]$; moreover, every closed orbit in $H_Y(\gamma_Y)$ has its index in that interval (see [24]).

Let $X \in \mathfrak{X}(M)$ have no singularities and let $N \subset TM$ be the sub-bundle such that the fiber N_x at $x \in M$ is the orthogonal linear subspace of $\langle X(x)\rangle$ in T_xM, that is, $N_x = \langle X(x)\rangle^{\perp}$. Here $\langle X(x)\rangle$ is the linear subspace spanned by $X(x)$ for $x \in M$. Let $\pi : TN \to N$ be the projection along X, and let

$$P_t^X(v) = \pi(D_xX^t(v)),$$

for $v \in N_x$ and $x \in M$. Let Λ be a closed X^t-invariant regular set. We say that Λ is *hyperbolic* if the bundle N_Λ has a P_t^X-invariant splitting $\Delta^s \oplus \Delta^u$ and there exists an $l > 0$ such that

$$\left\| P_l^X|_{\Delta_x^s} \right\| \leq \frac{1}{2} \quad \text{and} \quad \left\| P_{-l}^X|_{\Delta^u_{X^l(x)}} \right\| \leq \frac{1}{2},$$

for all $x \in \Lambda$. Then, Doering [25] proved the following result, which is a method of proof for hyperbolicity.

Proposition 3.5. *Let $\Lambda \subset M$ be a compact invariant set of X^t. Then, Λ is a hyperbolic set of X^t if and only if the linear Poincaré flow restriction on Λ has a hyperbolic splitting $N_\Lambda = \Delta^s \oplus \Delta^u$, where $N = \bigcup_{x \in M_X} N_x$.*

3.3 Weak hyperbolic periodic points

An exponential map $\exp_p : T_pM(1) \to M$ is well defined for all $p \in M$, where $T_pM(\delta)$ denotes the ball $\{v \in T_pM : \|v\| \le \delta\}$. For every regular point $x \in R(X)$, let $N_x = \langle X(x)\rangle^{\perp} \subset T_xM$, and $N_x(\delta)$ be the δ-ball in N_x. Let $\mathcal{N}_{x,r} = \exp_x(N_x(r))$. Given any point $x \in R(M)$ and $t \in \mathbb{R}$, there are $r > 0$ and a C^1 map $\tau : \mathcal{N}_{x,t} \to \mathbb{R}$ with $\tau(x) = t$ such that $X^{\tau(y)}(y) \in \mathcal{N}_{X^{\tau}(x),1}$ for any $y \in \mathcal{N}_{x,r}$. We define the *Poincaré map* as

$$f_{x,t} : \mathcal{N}_{x,r} \to \mathcal{N}_{X^t(x),1}$$
$$y \mapsto f_{x,t}(y) := X^{\tau(y)}(y).$$

Let $X \in \mathfrak{X}(M)$, and suppose $p \in \gamma \in Per(X)(X^{\pi(p)}(p) = p$, where $\pi(p) > 0$ is the prime period. If $f : \mathcal{N}_{p,r_0} \to \mathcal{N}_p$ is the Poincaré map ($r_0 > 0$), then $f(p) = p$. Note that γ is hyperbolic if and only if p is a hyperbolic fixed point of f.

The following lemma states that by perturbation of vector fields, we can gain some control on eigenvalues of the Poincaré map.

Lemma 3.6. *Let $p \in \eta \in H_X(\gamma) \cap Per(X)$ and let $f : \mathcal{N}_{p,r} \to \mathcal{N}_p (r > 0)$ be the Poincaré map of X^t. For any $\delta > 0$, if the eigenvalue λ of D_pf is $1 < \lambda < 1 + \delta$, then there is g that is C^1 close to f such that D_pg has an eigenvalue μ with $\mu \le 1 - \delta$, where g is the Poincaré map associated to Y.*

Proof. Let Δ_p be the eigenspace corresponding to λ with index$(p) = i$, and let $N_p = \Delta_p \oplus \Delta_p^{\perp}$. For the splitting, we have

$$Df(p) = \begin{pmatrix} Df(p)|_{\Delta_p} & A^1(f) \\ O & A^2(f) \end{pmatrix}.$$

Applying Gourmelon's result [26] (see also [5, Theorem 2.5]), we define the map $T : [0, 1] \to \Gamma_i$ as follows

$$T(t) = \begin{pmatrix} (1-t)Df(p)|_{\Delta_p} + t(\frac{1-\delta}{1+\delta})Df(p)|_{\Delta_p} & A^1(f) \\ O & A^2(f) \end{pmatrix},$$

for $t \in [0, 1]$. Then, we have

$$D_pf = T(0) = \begin{pmatrix} Df(p)|_{\Delta_p} & A^1(f) \\ O & A^2(f) \end{pmatrix}$$

and

$$D_pg = T(1) = \begin{pmatrix} (\frac{1-\delta}{1+\delta})Df(p)|_{\Delta_p} & A^1(f) \\ O & A^2(f) \end{pmatrix}.$$

Thus, one can see that

$$\left(\frac{1-\delta}{1+\delta}\right)Df(p)|_{\Delta_p} \le \left(\frac{1-\delta}{1+\delta}\right)(1+\delta) = (1-\delta).$$

The proof is complete. □

For any $\delta > 0$, we say that a point $p \in \gamma \in Per(X)$ is *δ-weak hyperbolic periodic* if there is an eigenvalue λ of D_pf such that $(1-\delta) < |\lambda| < (1+\delta)$, where $f : \mathcal{N}_{p,r} \to \mathcal{N}_p$ is the Poincaré map associated to X^t.

Let $X \in \mathcal{G}_2$ and let $H_X(\gamma)$ be bi-Lyapunov stable with index$(\gamma) = i(0 < i < \dim M - 1)$. Then, there is $\mathcal{U}(X)$ of X such that for any $Y \in \mathcal{U}(X)$, $H_Y(\gamma_Y)$ is bi-Lyapunov stable and every closed orbit in $H_Y(\gamma_Y)$ has the same index i. From this fact, we have the following result.

Lemma 3.7. *There is a residual set $\mathcal{G}_3 \subset \mathfrak{X}(M)$ such that for any $X \in \mathcal{G}_3$, if a homoclinic class $H_X(\gamma)$ is bi-Lyapunov stable, then $H_X(\gamma)$ does not contain a δ- weak hyperbolic periodic point.*

Proof. Let $X \in \mathcal{G}_3 = \mathcal{G}_1 \cap \mathcal{G}_2$ have the shadowing property on $H_X(\gamma)$. Since X has the shadowing property on $H_X(\gamma)$, by Lemmas 3.1 and 3.2, we have $\eta \sim \gamma$ for every $\eta \in H_X(\gamma) \cap Per(X)$. Suppose, by contradiction, that for any $\delta > 0$ there is $p \in \eta \in H_X(\gamma) \cap Per(X)$ such that p is a δ weak hyperbolic periodic point. Then, there is an eigenvalue λ of $D_p f$ such that

$$1 - \delta < |\lambda| < 1 + \delta,$$

where $f : \mathcal{N}_{p,r} \to \mathcal{N}_p$ is the Poincaré map corresponding to the flow X^t. Assume that $1 < \lambda < 1 + \delta$ (the other case is similar). Let $p \in \gamma$ and $q \in \eta \in H_X(\gamma) \cap Per(X)$. Take $x \in W^{ss}(p) \cap W^{uu}(q)$ and choose a neighborhood U of q such that:
(i) $U \cap \{\gamma\} = \emptyset$;
(ii) $U \cap Orb^+(x) = \emptyset$; and
(iii) $Orb^-(x) \subset U$.

Then, by [5, Theorem 2.5] and Lemma 3.6, there is g C^1 close to f such that:
(i) index(η_Y) > index(γ_Y);
(ii) it preserves the i strong stable manifold of $q_g \in \eta_Y$ outside U; and
(iii) $W^{uu}(p_g) \cap W^{ss}(q_g) \neq \emptyset$;
where γ_Y is the continuation of γ, η_Y is the continuation of η, Y^t is the flow corresponding to g, and $p_g \in \gamma_Y$. Using the λ-lemma, we have $q_g \in \overline{W^{uu}(p_g)}$. Since $H_Y(\gamma_Y)$ is Lyapunov stable for Y, we have $\overline{W^u(\gamma_Y)} \subset H_Y(\gamma_Y)$, and so $q_g \in \eta_Y \subset H_Y(\gamma_Y)$. This is a contradiction. Since $X \in \mathcal{G}_3$, if every $\eta \in H_X(\gamma) \cap Per(X)$ has index i, then by Lemma 3.4, every $\eta_Y \in H_Y(\gamma_Y) \cap Per(Y)$ has index i. □

By Proposition 3.3, $H_X(\gamma) \cap Sing(X) = \emptyset$. Then, we have the following lemma, which is a flow version of the result proved by Wang [27].

Lemma 3.8. *There is a residual set $\mathcal{G}_4 \subset \mathfrak{X}(M)$ such that for any $X \in \mathcal{G}_4$, if a homoclinic class $H_X(\gamma)$ is not Hyperbolic, then for any $\delta > 0$, there is a periodic point $q \in \eta \subset H_X(\gamma) \cap Per(X)$ such that $\eta \sim \gamma$ and q is a δ-weak hyperbolic periodic point.*

Proof of the Theorem. Let $X \in \mathcal{G}_3 \cap \mathcal{G}_4$ have the shadowing property on $H_X(\gamma)$. Suppose, by contradiction, that $H_X(\gamma)$ is not hyperbolic. Since X has the shadowing property on $H_X(\gamma)$, by Lemma 3.2 we have $\eta \sim \gamma$, for all $\eta \in H_X(\gamma) \cap Per(X)$. Then, by Lemma 3.8, for any $\delta > 0$ there is $q \in \eta \in H_X(\gamma) \cap Per(X)$ such that q is a weak hyperbolic periodic point. This is a contradiction by Lemma 3.7. □

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