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Extinction of a two species competitive stage-structured system with the effect of toxic substance and harvesting

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Abstract: The extinction property of a two species competitive stage-structured phytoplankton system with harvesting is studied in this paper. Several sets of sufficient conditions which ensure that one of the components will be driven to extinction are established. Our results supplement and complement the results of Li and Chen [Extinction in periodic competitive stage-structured Lotka-Volterra model with the effects of toxic substances, J. Comput. Appl. Math., 2009, 231(1), 143-153] and Liu, Chen, Luo et al. [Extinction and permanence in nonautonomous competitive system with stage structure, J. Math. Anal. Appl., 2002, 274(2), 667-684].

Keywords: extinction, competition, stage-structured, toxicology, harvesting

MSC: 34-XX

1 Introduction

Throughout this paper, for a given function $g(t)$, we let g^L and g^M denote $\inf_{-\infty<t<\infty} g(t)$ and $\sup_{-\infty<t<\infty} g(t)$, respectively.

During the last two decades, ecosystem with stage structure become one of the most important research area, and some substantive progress has been made on this direction, see [1–11] and the references cited therein. For example, Chen et al. [2] showed that stage structure plays important role on the persistent property of the cooperative system. For the system without stage structure, the system always admits a unique positive equilibrium, which means the stable coexistence of the two species. However, if the stage structure is enough large, despite the cooperation between the two species, the species may still be driven to extinction. Xiao et al. [3] investigated the Hopf bifurcation and stability property of a Beddington-DeAngelis predator-prey model with stage structure for predator and time delay incorporating prey refuge. Among those works, many scholars ([1], [6–11]) done works on the stage structured competitive system. Also, competitive system with the effect of toxic substances is another important research area, many excellent results have been obtained, see [12–37] and the references cited therein. Li et al. [13] studied the stability property of a

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competitive system with the effect of toxic substances, they showed that the toxic substance have no influence to the stability property of the system, though it has influence on the position of the equilibrium. Their result is then generalized by Chen et al. [23] to the infinite delay case. Some scholars [17, 24, 29, 35] argued that it is better to describe the relationship between the competitive species by using the nonlinear function, and they obtained some interesting results, such as the extinction of the species, the existence, uniqueness and global stability of the periodic solution, etc.

Based on the traditional two species Lotka-Volterra competitive system, Liu et al. [6] first time proposed the following two-species competitive model with stage structure

$$\begin{aligned} \dot{x}_1(t) &= b_1 e^{-d_1\tau_1} x_1(t-\tau_1) - a_{11}x_1^2 - a_{12}x_1x_2, \\ \dot{y}_1(t) &= b_1x_1 - d_1y_1 - b_1e^{-d_1\tau_1}x_1(t-\tau_1), \\ \dot{x}_2(t) &= b_2 e^{-d_2\tau_2} x_2(t-\tau_2) - a_{22}x_2^2 - a_{21}x_1x_2, \\ \dot{y}_2(t) &= b_2x_2 - d_2y_2 - b_2e^{-d_2\tau_2}x_2(t-\tau_2). \end{aligned} \tag{1.1}$$

As as pointed out by Liu et al. [6], to study the dynamic behaviors of the system (1.1), it is enough to study the asymptotic behavior of the following subsystem of system (1.1)

$$\begin{aligned} \dot{x}_1(t) &= b_1 e^{-d_1\tau_1} x_1(t-\tau_1) - a_{11}x_1^2 - a_{12}x_1x_2, \\ \dot{x}_2(t) &= b_2 e^{-d_2\tau_2} x_2(t-\tau_2) - a_{22}x_2^2 - a_{21}x_1x_2. \end{aligned} \tag{1.2}$$

System (1.2) admits three non-negative equilibria.

$$E_0(0,0),\ E_1\Big(\frac{b_1e^{-d_1\tau_1}}{a_{11}}, 0\Big),\ E_2\Big(0, \frac{b_2e^{-d_2\tau_2}}{a_{22}}\Big). \tag{1.3}$$

Concerned with the stability property of E_1 and E_2, the authors obtained the following results.

Theorem A. *E_1 is globally asymptotically stable provided*

$$\frac{b_1e^{-d_1\tau_1}}{b_2e^{-d_2\tau_2}} > \frac{a_{11}}{a_{21}} \text{ and } \frac{b_1e^{-d_1\tau_1}}{b_2e^{-d_2\tau_2}} > \frac{a_{12}}{a_{22}}. \tag{1.4}$$

Theorem B. *E_2 is globally asymptotically stable provided*

$$\frac{b_1e^{-d_1\tau_1}}{b_2e^{-d_2\tau_2}} < \frac{a_{11}}{a_{21}} \text{ and } \frac{b_1e^{-d_1\tau_1}}{b_2e^{-d_2\tau_2}} < \frac{a_{12}}{a_{22}}. \tag{1.5}$$

Liu et al. [7] proposed the following n-species nonautonomous stage-structured competitive system,

$$\begin{aligned} \dot{x}_i(t) &= b_i(t-\tau_i)e^{\int_{t-\tau_i}^t d_i(s)ds} x_i(t-\tau_i) - x_i(t)\sum_{j=1}^n x_j(t), \\ \dot{y}_i(t) &= b_i(t)x_i(t) - d_i(t)y_i(t) - b_i(t-\tau_i)e^{\int_{t-\tau_i}^t d_i(s)ds} x_i(t-\tau_i), \end{aligned} \tag{1.6}$$

where $i = 1, 2, \dots, n$, τ_i are nonnegative constants. $b_i(t), a_{ij}(t), d_i(t) (i, j = 1, 2, \dots, n)$ are all nonnegative continuous and ω-periodic functions. $b_i(t), a_{ii}(t), d_i(t) > 0$ for all $t \in [0, \omega]$. Set

$$B_i(t) = b_i(t-\tau_i)e^{\int_{t-\tau_i}^t d_i(s)ds}.$$

Then, Liu et al. [7] obtained the following results.

Theorem C. *For system (1.6) in the case $n = 2$, assume*

$$B_1^L > B_2^M \frac{a_{12}^M}{a_{22}^L} \text{ and } B_1^L > B_2^M \frac{a_{11}^M}{a_{21}^L}. \tag{1.7}$$

Then $\lim\limits_{t\to+\infty} x_2(t) = \lim\limits_{t\to+\infty} y_2(t) = 0$.

Theorem D. *For system (1.6) in the case $n = 2$, assume*

$$B_2^L > B_1^M \frac{a_{22}^M}{a_{12}^L} \text{ and } B_2^L > B_1^M \frac{a_{21}^M}{a_{11}^L}. \tag{1.8}$$

Then $\lim_{t\to+\infty} x_1(t) = \lim_{t\to+\infty} y_1(t) = 0.$

One could easily see that Theorem C and D generalize Theorem A and B to the non-autonomous case.

Based on the works of [5] and [6], Li and Chen [10] proposed the following two species periodic competitive stage-structured system with the effects of toxic substances:

$$\begin{aligned}
\dot{x}_1(t) &= b_1(t-\tau_1)e^{-\int_{t-\tau_1}^{t} r_1(s)ds}x_1(t-\tau_1)\\
&\quad -a_{11}(t)x_1^2(t) - a_{12}(t)x_1(t)x_2(t) - d_1(t)x_1^2(t)x_2(t),\\
\dot{y}_1(t) &= b_1(t)x_1(t) - r_1(t)y_1(t) - b_1(t-\tau_1)e^{-\int_{t-\tau_1}^{t} r_1(s)ds}x_1(t-\tau_1),\\
\dot{x}_2(t) &= b_2(t-\tau_2)e^{-\int_{t-\tau_2}^{t} r_2(s)ds}x_2(t-\tau_2) - a_{22}(t)x_2^2(t)\\
&\quad -a_{21}(t)x_1(t)x_2(t) - d_2(t)x_1(t)x_2^2(t),\\
\dot{y}_2(t) &= b_2(t)x_2(t) - r_2(t)y_2(t) - b_2(t-\tau_2)e^{-\int_{t-\tau_2}^{t} r_2(s)ds}x_2(t-\tau_2),
\end{aligned} \tag{1.9}$$

where $x_i(t)$ and $y_i(t)(i = 1, 2)$ represent the density of mature and immature species at time $t > 0$, respectively; $b_i(t)$, $a_{ij}(t)$, $r_i(t)$, $d_i(t)(i, j = 1, 2)$ are all nonnegative continuous and ω-periodic functions. Li and Chen [10] obtained the following result.

Theorem E. *If the coefficients of system (1.9) satisfy*

$$\frac{b_1^L e^{-r_1^M\tau_1}}{b_2^M e^{-r_2^L\tau_2}} > \frac{a_{12}^M}{a_{22}^L}, \quad \frac{b_1^L e^{-r_1^M\tau_1}}{b_2^M e^{-r_2^L\tau_2}} \geq \frac{a_{11}^M}{a_{21}^L}, \quad \frac{b_1^L e^{-r_1^M\tau_1}}{b_2^M e^{-r_2^L\tau_2}} \geq \frac{d_1^M}{d_2^L}. \tag{1.10}$$

Then second species will be driven to extinction while the first one is global attractive to a positive periodic solution of a stage-structured single species system.

Comparing Theorem A, C and E, one could see that the first two inequalities of Theorem E is the same as that of the Theorem C. Noting that the authors of [10] is to investigated the dynamic behaviors of a stage-structured system with toxic substance, hence, one could see that the idea behind that of Theorem E is to assume that the second species in the system without toxic substance is driven to extinction, and to find out the suitable restrictions on the coefficients of toxic substances term, to ensure the second species still be driven to extinction.

Now, one of the interesting issue proposed: What would happen if the first two inequalities in Theorem E hold, while the third inequality does not holds?

To bring some hints on this issue, let's consider the following example.

Example 1.1. Consider the following equations

$$\begin{aligned}
\dot{x}_1(t) &= 3e^{-0.2}x_1(t-0.2) - (1.5+0.5\cos(t))x_1^2(t)\\
&\quad -(2+\sin(t))x_1(t)x_2(t) - 0.2x_1^2(t)x_2(t),\\
\dot{y}_1(t) &= 3x_1(t) - y_1(t) - 3e^{-0.2}x_1(t-0.2),\\
\dot{x}_2(t) &= 2e^{-0.2}x_2(t-0.2) - (3.5+0.5\cos(t))x_2^2(t)\\
&\quad -2x_1(t)x_2(t) - 0.1x_1(t)(x_2(t))^2,\\
\dot{y}_2(t) &= 2x_2(t) - y_2(t) - 2e^{-0.2}x_2(t-0.2),
\end{aligned} \tag{1.11}$$

where $\tau_1 = 0.2, \tau_2 = 0.2, b_1(t) = 4, r_1(t) = 1, a_{11}(t) = 1.5 + 0.5\cos(t), a_{12}(t) = 2 + \sin(t), d_1(t) = 0.2, d_2(t) = 0.1, b_2(t) = 2, r_2(t) = 1, a_{21}(t) = 2, a_{22}(t) = 3.5 + 0.5\cos(t)$.

One could easily see that

$$\begin{aligned}
\frac{b_1^L e^{-r_1^M \tau_1}}{b_2^M e^{-r_2^L \tau_2}} &= \frac{3e^{-0.2}}{2e^{-0.2}} = \frac{3}{2} > \frac{3}{3} = 1 = \frac{a_{12}^M}{a_{22}^L},\\
\frac{b_1^L e^{-r_1^M \tau_1}}{b_2^M e^{-r_2^L \tau_2}} &= \frac{3}{2} > \frac{2}{2} = 1 = \frac{a_{11}^M}{a_{21}^L},\\
\frac{b_1^L e^{-r_1^M \tau_1}}{b_2^M e^{-r_2^L \tau_2}} &= \frac{3}{2} < \frac{0.2}{0.1} = 2 = \frac{d_1^M}{d_2^L}.
\end{aligned} \tag{1.12}$$

Inequalities (1.12) show that the coefficients of the system (1.11) satisfies the first two inequalities in (1.10), while the third inequality no longer holds. Numeric simulation (Fig. 1) shows that in this case, species 2 will be driven to extinction while species 1 is globally attractive.

Above example enlighten us to revisit the dynamic behaviors of the system (1.9), and to find out some new sufficient conditions which ensure the extinction of some of the species in system (1.9).

On the other hand, based on the traditional two species competitive system with toxic substance, Kar and Chaudhuri [36] proposed the following non-selective harvesting system

$$\begin{aligned}
\frac{dx}{dt} &= r_1 x(1 - \tfrac{x}{k_1}) - \alpha_1 xy - \gamma_1 x^2 y - q_1 E x,\\
\frac{dy}{dt} &= r_2 y(1 - \tfrac{y}{k_2}) - \alpha_2 xy - \gamma_2 x y^2 - q_2 E y,
\end{aligned} \tag{1.13}$$

where q_1, q_2 are the catchability coefficients of the two species. The authors gave a thoroughly investigation of the dynamical behaviour about system.

Recently, Gupta et al. [37] made the following assumption: the two species are being harvested by different agencies, both the species are harvested with harvesting efforts E_1 and E_2, respectively. This leads to the following modeling

$$\begin{aligned}
\frac{dx}{dt} &= r_1 x(1 - \tfrac{x}{k_1}) - \alpha_1 xy - \gamma_1 x^2 y - q_1 E_1 x,\\
\frac{dy}{dt} &= r_2 y(1 - \tfrac{y}{k_2}) - \alpha_2 xy - \gamma_2 x y^2 - q_2 E_2 y,
\end{aligned} \tag{1.14}$$

The authors showed that the system (1.14) may exists two saddle-node bifurcations for different bifurcation parameters.

Now stimulated by the works of [36, 37], it is natural to incorporating the harvesting efforts to system (1.9), here, without loss of generality, we may assume that we only harvest the mature species, and this leads to the following system:

$$\begin{aligned}
\dot{x}_1(t) &= b_1(t-\tau_1)e^{-\int_{t-\tau_1}^t r_1(s)ds} x_1(t-\tau_1)\\
&\quad - a_{11}(t)x_1^2(t) - a_{12}(t)x_1(t)x_2(t) - d_1(t)x_1^2(t)x_2(t) - q_1(t)E_1(t)x_1(t),\\
\dot{y}_1(t) &= b_1(t)x_1(t) - r_1(t)y_1(t) - b_1(t-\tau_1)e^{-\int_{t-\tau_1}^t r_1(s)ds} x_1(t-\tau_1),\\
\dot{x}_2(t) &= b_2(t-\tau_2)e^{-\int_{t-\tau_2}^t r_2(s)ds} x_2(t-\tau_2) - a_{22}(t)x_2^2(t)\\
&\quad - a_{21}(t)x_1(t)x_2(t) - d_2(t)x_1(t)x_2^2(t) - q_2(t)E_2(t)x_2(t),\\
\dot{y}_2(t) &= b_2(t)x_2(t) - r_2(t)y_2(t) - b_2(t-\tau_2)e^{-\int_{t-\tau_2}^t r_2(s)ds} x_2(t-\tau_2),
\end{aligned} \tag{1.15}$$

where $x_i(t)$ and $y_i(t) (i = 1, 2)$ represent the density of mature and immature species at time $t > 0$, respectively; $b_i(t), a_{ij}(t), r_i(t), d_i(t), q_i(t), E_i(t) (i, j = 1, 2)$ are all nonnegative continuous and ω-periodic functions.

Already, there are many scholars investigated the extinction property of the competitive system with toxic substance, see [12, 13, 17–24], however, all of those works did not consider the influence of harvesting.

The aim of this paper is, by further developing the analysis technique of Li and Chen [10], Chen et al. [35] and Montes De Oca and Vivas [32], to investigate the extinction property of the system (1.15).

The initial conditions for system (1.15) take the form

$$x_i(\theta) = \phi_i(\theta) > 0, \quad y_i(\theta) = \psi_i(\theta) > 0, -\tau \le \theta \le 0, \quad i = 1, 2, \tag{1.16}$$

where $\tau = \max\{\tau_1, \tau_2\}$. For the continuity of the solutions of system (1.15), in this paper, we always assume

$$y_i(0) = \psi_i(0) = \int_{-\tau_i}^{0} b_i(s)\phi_i(s)e^{-\int_s^0 r_i(u)du} ds, \quad i = 1, 2. \tag{1.17}$$

The organization of this paper is as follows. In Section 2, we introduce some useful lemmas. In Section 3, we study the extinction property of system (1.15). In Section 4, several numeric examples are carried out to illustrate the feasibility of the main results. We end this paper by a briefly discussion.

2 Preliminaries

Now let us state several lemmas which will be useful in the proof of our main results.

Lemma 2.1. *Solutions of system (1.15) with initial conditions (1.16) and (1.17) are positive for all* $t > 0$.

Proof. The proof of Lemma 2.1 is similar to that of Lemma 3.1 [5], and we omit the detail proof here.

Lemma 2.2. [7] *Consider the following equations:*

$$\begin{aligned} x'(t) &= bx(t-\delta) - a_1 x(t) - a_2 x^2(t), \\ x(t) &= \phi(t) > 0, \quad -\delta \le t \le 0, \end{aligned}$$

and assume that $b, a_2 > 0$, $a_1 \ge 0$ *and* $\delta \ge 0$ *are constants, then:*

(i) If $b \ge a_1$, *then* $\lim_{t\to+\infty} x(t) = \dfrac{b-a_1}{a_2}$;

(ii) If $b \le a_1$, *then* $\lim_{t\to+\infty} x(t) = 0$.

Lemma 2.3. *Let* $(x_1(t), y_1(t), x_2(t), y_2(t))^T$ *be any solution of system (1.15) with initial conditions (1.16) and (1.17). Then for* $i = 1, 2$

$$\limsup_{t\to+\infty} x_i(t) \le M_i, \quad \limsup_{t\to+\infty} y_i(t) \le N_i, \tag{2.1}$$

where

$$M_i = \frac{b_i^M e^{-r_i^L \tau_i} - q_i^L E_i^L}{a_{ii}^L}, \quad N_i = \frac{b_i^M M_i}{r_i^L}\left(1 - e^{-r_i^M \tau_i}\right). \tag{2.2}$$

Proof. It follows from the first or third equation of system (1.15) that

$$\begin{aligned} \dot{x}_i(t) &\le b_i(t-\tau_1)e^{-\int_{t-\tau_i}^{t} r_i(s)ds} x_i(t-\tau_i) - a_{ii}(t)x_i^2(t) - q_i(t)E_i(t)x_i(t) \\ &\le b_i^M e^{-r_i^L \tau_i} x_i(t-\tau_i) - a_{ii}^L x_i^2(t) - q_i^L E_i^L x_i(t). \end{aligned}$$

Consider the following equation

$$\dot{u}_i(t) = b_i^M e^{-r_i^L \tau_i} u_i(t-\tau_i) - a_{ii}^L u_i^2(t) - q_i^L E_i^L u_i(t)$$

with $u_i(t) = x_i(t) (-\tau \le t \le 0)$, $i = 1, 2$. By Lemma 2.2, $\lim_{t\to+\infty} u_i(t) = \dfrac{b_i^M e^{-r_i^L \tau_i} - q_i^L E_i^L}{a_{ii}^L}$, and so,

$$\limsup_{t\to+\infty} x_i(t) \le \frac{b_i^M e^{-r_i^L \tau_i} - q_i^L E_i^L}{a_{ii}^L} \overset{\text{def}}{=} M_i, \quad i = 1, 2. \tag{2.3}$$

The rest of the proof is similar to that of the proof of Lemma 2.3 in [10], and we omit the detail here.

Remark 2.1. If $b_i^M e^{-r_i^L\tau_i} \le q_i^L E_i^L$, $i = 1, 2$, then $\lim_{t\to+\infty} x_i(t) = 0$, and consequently, $\lim_{t\to+\infty} y_i(t) = 0$, $i = 1, 2$. That is, overfishing will lead to the extinction of both species.

Lemma 2.4. [32] *(Fluctuation lemma) Let $x(t)$ be a bounded differentiable function on (α, ∞), then there exist sequences $\tau_n \to \infty$, $\sigma_n \to \infty$ such that*

$$(a)\ \dot{x}(\tau_n) \to 0 \ and\ x(\tau_n) \to \limsup_{t\to\infty} x(t) = \overline{x} \ as\ n \to \infty,$$

$$(b)\ \dot{x}(\sigma_n) \to 0 \ and\ x(\sigma_n) \to \liminf_{t\to\infty} x(t) = \underline{x} \ as\ n \to \infty.$$

3 Main results

As indicated by the Remark 2.1, overfishing will leads to the extinction of both species, hence, from now on, we make the following assumption:

$$b_i^L e^{-r_i^M\tau_i} > q_i^M E_i^M, \quad i = 1, 2. \tag{3.1}$$

Before stating the main results of this section, we introduce a set of conditions

$$\frac{b_1^L e^{-r_1^M\tau_1} - q_1^M E_1^M}{b_2^M e^{-r_2^L\tau_2} - q_2^L E_2^L} > \max\left\{\frac{a_{12}^M}{a_{22}^L}, \frac{a_{11}^M}{a_{21}^L}, \frac{d_1^M}{d_2^L}\right\} \tag{3.2}$$

$$\frac{b_1^L e^{-r_1^M\tau_1} - q_1^M E_1^M}{b_2^M e^{-r_2^L\tau_2} - q_2^L E_2^L} > \max\left\{\frac{a_{12}^M}{a_{22}^L}, \frac{a_{11}^M + d_1^M M_2}{a_{21}^L}\right\} \tag{3.3}$$

$$\frac{b_1^L e^{-r_1^M\tau_1} - q_1^M E_1^M}{b_2^M e^{-r_2^L\tau_2} - q_2^L E_2^L} > \max\left\{\frac{a_{12}^M + d_1^M M_1}{a_{22}^L}, \frac{a_{11}^M}{a_{21}^L}\right\} \tag{3.4}$$

$$\frac{b_2^L e^{-r_2^M\tau_2} - q_2^M E_2^M}{b_1^M e^{-r_1^L\tau_1} - q_1^L E_1^L} > \max\left\{\frac{a_{21}^M}{a_{11}^L}, \frac{a_{22}^M}{a_{12}^L}, \frac{d_2^M}{d_1^L}\right\} \tag{3.5}$$

$$\frac{b_2^L e^{-r_2^M\tau_2} - q_2^M E_2^M}{b_1^M e^{-r_1^L\tau_1} - q_1^L E_1^L} > \max\left\{\frac{a_{21}^M}{a_{11}^L}, \frac{a_{22}^M + d_2^M M_1}{a_{12}^L}\right\} \tag{3.6}$$

$$\frac{b_2^L e^{-r_2^M\tau_2} - q_2^M E_2^M}{b_1^M e^{-r_1^L\tau_1} - q_1^L E_1^L} > \max\left\{\frac{a_{21}^M + d_2^M M_2}{a_{11}^L}, \frac{a_{22}^M}{a_{12}^L}\right\} \tag{3.7}$$

where M_i, $i = 1, 2$ are defined by (2.2).

Before we begin to prove the main results, we need several Lemmas again.

Lemma 3.1. *Let $(x_1(t), y_1(t), x_2(t), y_2(t))^T$ be any solution of system (1.15) with initial conditions (1.16) and (1.17). Assume that (3.2) or (3.3) or (3.4) holds, then there exists a $\alpha > 0$ such that $x_1(t) \ge \alpha$ for all $t \ge 0$.*

Proof. We first show that the conclusion of Lemma 3.1 holds under the assumption (3.2). It follows from Lemma 2.3 that $\limsup_{t\to+\infty} x_2(t) \le \frac{b_2^M e^{-r_2^L\tau_1} - q_2^L E_2^L}{a_{22}^L}$. Given $\varepsilon = \frac{1}{2}\left(\frac{b_1^L e^{-r_1^M\tau_1} - q_1^M E_1^M}{a_{12}^M} - \frac{b_2^M e^{-r_2^L\tau_2} - q_2^M E_2^M}{a_{22}^L}\right)$, there exists a $T \ge 0$ such that for all $t \ge T$

$$x_2(t) \le \frac{b_2^M e^{-r_2^L\tau_1} - q_2^L E_2^L}{a_{22}^L} + \varepsilon = \frac{1}{2}\left(\frac{b_1^L e^{-r_1^M\tau_1} - q_1^M E_1^M}{a_{12}^M} + \frac{b_2^M e^{-r_2^L\tau_2} - q_2^M E_2^M}{a_{22}^L}\right).$$

So, for all $t \geq T$, from the first equation of system (1.15), it follows that

$$\begin{aligned}
\dot{x_1}(t) &\geq b_1^L e^{-r_1^M\tau_1} x_1(t-\tau_1) - a_{11}^M x_1^2(t) - a_{12}^M x_1(t)x_2(t) \\
&\quad -d_1^M x_1^2(t)x_2(t) - q_1^M E_1^M x_1(t) \\
&\geq b_1^L e^{-r_1^M\tau_1} x_1(t-\tau_1) - \left(a_{11}^M + \frac{d_1^M}{2}\left(\frac{b_1^L e^{-r_1^M\tau_1} - q_1^M E_1^M}{a_{12}^M} + \frac{b_2^M e^{-r_2^L\tau_2} - q_2^M E_2^M}{a_{22}^L}\right)\right)x_1^2(t) \\
&\quad -\left(\frac{1}{2}\left(b_1^L e^{-r_1^M\tau_1} - q_1^M E_1^M + a_{12}^M \frac{b_2^M e^{-r_2^L\tau_2} - q_2^M E_2^M}{a_{22}^L}\right) + q_1^M E_1^M\right)x_1(t) \\
&\stackrel{\text{def}}{=} A x_1(t-\tau_1) - B x_1(t) - C x_1^2(t).
\end{aligned}$$

Let $u(t)$ be a solution of the following equation

$$\dot{u}(t) = Au(t-\tau_1) - Bu(t) - Cu^2(t),$$

with $u(T+\tau_1) = x_1(T+\tau_1)$. It follows from condition (3.1) that

$$A - B = \frac{1}{2}\left(b_1^L e^{-r_1^M\tau_1} - q_1^M E_1^M - a_{12}^M \frac{b_2^M e^{-r_2^L\tau_2} - q_2^M E_2^M}{a_{22}^L}\right) > 0.$$

From Lemma 2.2

$$\lim_{t\to+\infty} u(t) = \frac{A-B}{C} = \alpha_1 > 0.$$

Therefore, we obtain

$$\underline{x}_1 = \liminf_{t\to+\infty} x_1(t) \geq \alpha_1 > 0.$$

Given $\varepsilon = \frac{1}{2}\alpha_1$, there exists a $T_1 \geq T$ such that

$$x_1(t) \geq \underline{x}_1 - \frac{\alpha_1}{2} \geq \alpha_1 - \frac{1}{2}\alpha_1 = \frac{\alpha_1}{2}, \quad t \geq T_1.$$

Let $\alpha_2 = \min\{x_1(t) : 0 \leq t \leq T_1\} > 0$ and $\alpha = \min\{\frac{\alpha_1}{2}, \alpha_2\} > 0$. It follows that $x_1(t) \geq \alpha > 0$ for all $t \geq 0$.

Noting that above proof only use the fact

$$\frac{b_1^L e^{-r_1^M\tau_1} - q_1^M E_1^M}{a_{12}^M} > \frac{b_2^M e^{-r_2^L\tau_2} - q_2^M E_2^M}{a_{22}^L}.$$

Condition (3.3) and (3.4) all implies this inequality holds, hence, under the assumption of (3.2) or (3.3) or (3.4), the conclusion of Lemma 3.1 holds. This ends the proof of Lemma 3.1.

Our main results are the following Theorems.

Theorem 3.1. *Assume that (3.2) holds. Then*

$$\begin{aligned}
&m_1 \leq \liminf_{t\to+\infty} x_1(t) \leq \limsup_{t\to+\infty} x_1(t) \leq M_1. \\
&n_1 \leq \liminf_{t\to+\infty} y_1(t) \leq \limsup_{t\to+\infty} y_1(t) \leq N_1. \\
&\lim_{t\to+\infty} x_2(t) = 0, \quad \lim_{t\to+\infty} y_2(t) = 0,
\end{aligned}$$

where

$$m_1 = \frac{b_1^L e^{-r_1^M\tau_1} - q_1^M E_1^M}{a_{11}^M}, \quad n_1 = \frac{b_1^L m_1}{r_1^M}\left(1 - e^{-r_1^L\tau_1}\right).$$

Theorem 3.2. *Assume that (3.3) holds. Then the conclusions of Theorem 3.1 hold.*

Theorem 3.3. *Assume that (3.4) holds. Then the conclusions of Theorem 3.1 hold.*

Noting that system (1.9) is the special case of system (1.15), ($q_i(t) \equiv 0, E_i(t) \equiv 0, i = 1, 2$) Then as a direct corollary of Theorem 3.1, we have

Corollary 3.1. *Assume that in system (1.9)*

$$\frac{b_1^L e^{-r_1^M \tau_1}}{b_2^M e^{-r_2^L \tau_2}} > \max\left\{ \frac{a_{12}^M}{a_{22}^L}, \frac{a_{11}^M}{a_{21}^L}, \frac{d_1^M}{d_2^L} \right\}$$

hold. Then

$$m_1 \le \liminf_{t\to+\infty} x_1(t) \le \limsup_{t\to+\infty} x_1(t) \le M_1.$$
$$n_1 \le \liminf_{t\to+\infty} y_1(t) \le \limsup_{t\to+\infty} y_1(t) \le N_1.$$
$$\lim_{t\to+\infty} x_2(t) = 0, \quad \lim_{t\to+\infty} y_2(t) = 0,$$

where

$$m_1 = \frac{b_1^L e^{-r_1^M \tau_1}}{a_{11}^M}, \quad n_1 = \frac{b_1^L m_1}{r_1^M}\left(1 - e^{-r_1^L \tau_1}\right).$$

Remark 3.1. Corollary 3.1 is Theorem 3.1 of Li and Chen [10], hence we generalize the main result of [10].

As a direct corollary of Theorem 3.2 and 3.3, we have

Corollary 3.2.*Assume that in system (1.9)*

$$\frac{b_1^L e^{-r_1^M \tau_1}}{b_2^M e^{-r_2^L \tau_2}} > \max\left\{ \frac{a_{12}^M}{a_{22}^L}, \frac{a_{11}^M + d_1^M M_{11}}{a_{21}^L} \right\}$$

hold, where $M_{11} = \dfrac{b_1^M e^{-r_1^L \tau_1}}{a_{11}^L}$. *Then the conclusion of Corollary 3.1 holds.*

Corollary 3.3. *Assume that in system (1.9)*

$$\frac{b_1^L e^{-r_1^M \tau_1}}{b_2^M e^{-r_2^L \tau_2}} > \max\left\{ \frac{a_{12}^M + d_1^M M_{22}}{a_{22}^L}, \frac{a_{11}^M}{a_{21}^L} \right\} \tag{3.8}$$

hold, where $M_{22} = \dfrac{b_2^M e^{-r_2^L \tau_1}}{a_{22}^L}$. *Then the conclusion of Corollary 3.1 holds.*

Remark 3.2. As was showed in Example 1.1, though the conditions of Theorem 3.1 in [10] are not satisfied, the second species still be possible of driving to extinction. Corollary 3.2 and 3.3 are two set of new sufficient conditions which ensure the extinction of the second species, hence, Corollary 3.2 and 3.3 supplement and complement the main results of [10].

Proof of Theorem 3.1. It follows from Lemma 2.1 and 2.3 that $x_i(t)$, $i = 1, 2$ are bounded and positive for all $t \ge 0$. Let $\underline{x}_1 = \liminf_{t\to+\infty} x_1(t)$ and $\overline{x}_2 = \limsup_{t\to+\infty} x_2(t)$. For above $\varepsilon_1 > 0$, it follows from Lemma 2.3 that

$$\underline{x}_1 < M_1 + \varepsilon_1, \quad \overline{x}_2 < M_2 + \varepsilon_1. \tag{3.9}$$

From Lemma 3.1 we know that $\underline{x}_1 \ge \alpha > 0$. Obviously, $\overline{x}_2 \ge 0$. To prove $\lim_{t\to+\infty} x_2(t) = 0$, it suffices to show that $\overline{x}_2 = 0$. In order to get a contradiction, we suppose that $\overline{x}_2 > 0$. According to the Fluctuation lemma (Lemma 2.4), there exist sequences $\gamma_n \to +\infty$, $\sigma_n \to +\infty$ such that $x_1'(\gamma_n) \to 0$, $x_2'(\sigma_n) \to 0$, $x_1(\gamma_n) \to \underline{x}_1$ and $x_2(\sigma_n) \to \overline{x}_2$ as $n \to +\infty$. It follows from the first equation of system (1.15) that

$$\begin{aligned}
\dot{x_1}(\gamma_n) &= b_1(\gamma_n - \tau_1)e^{-\int_{\gamma_n-\tau_1}^{\gamma_n} r_1(s)ds} x_1(\gamma_n - \tau_1) - a_{11}(\gamma_n)x_1^2(\gamma_n) \\
&\quad - a_{12}(\gamma_n)x_1(\gamma_n)x_2(\gamma_n) - d_1(\gamma_n)x_1^2(\gamma_n)x_2(\gamma_n) - q_1(\gamma_n)E_1(\gamma_n)x_1(\gamma_n) \\
&\ge b_1^L e^{-r_1^M \tau_1} \inf_{t \ge \gamma_n - \tau_1} x_1(t) - a_{11}^M x_1^2(\gamma_n) \\
&\quad - a_{12}^M x_1(\gamma_n) \sup_{t\ge\gamma_n} x_2(t) - d_1^M x_1^2(\gamma_n) \sup_{t\ge\gamma_n} x_2(t) - q_1^M E_1^M x_1(\gamma_n).
\end{aligned} \tag{3.10}$$

By taking the limit of the above inequality as $n \to +\infty$, we obtain the inequality

$$b_1^L e^{-r_1^M \tau_1} - q_1^M E_1^M \le a_{11}^M \underline{x}_1 + a_{12}^M \overline{x}_2 + d_1^M \underline{x}_1 \overline{x}_2. \tag{3.11}$$

From the third equation of system (1.15), by a similar argument as above, we obtain

$$b_2^M e^{-r_2^L \tau_2} - q_2^L E_2^L \ge a_{21}^L \underline{x}_1 + a_{22}^L \overline{x}_2 + d_2^L \underline{x}_1 \overline{x}_2. \tag{3.12}$$

(3.11) is equivalent to

$$1 \le \frac{a_{11}^M}{b_1^L e^{-r_1^M \tau_1} - q_1^M E_1^M} \underline{x}_1 + \frac{a_{12}^M}{b_1^L e^{-r_1^M \tau_1} - q_1^M E_1^M} \overline{x}_2 + \frac{d_1^M}{b_1^L e^{-r_1^M \tau_1} - q_1^M E_1^M} \underline{x}_1 \overline{x}_2. \tag{3.13}$$

(3.12) is equivalent to

$$1 \ge \frac{a_{21}^L}{b_2^M e^{-r_2^L \tau_2} - q_2^L E_2^L} \underline{x}_1 + \frac{a_{22}^L}{b_2^M e^{-r_2^L \tau_2} - q_2^L E_2^L} \overline{x}_2 + \frac{d_2^L}{b_2^M e^{-r_2^L \tau_2} - q_2^L E_2^L} \underline{x}_1 \overline{x}_2. \tag{3.14}$$

(3.13) together with (3.14) leads to

$$A_1 \underline{x}_1 + A_2 \overline{x}_2 + A_3 \underline{x}_1 \overline{x}_2 \ge 0, \tag{3.15}$$

where

$$A_1 = \frac{a_{11}^M}{b_1^L e^{-r_1^M \tau_1} - q_1^M E_1^M} - \frac{a_{21}^L}{b_2^M e^{-r_2^L \tau_2} - q_2^L E_2^L},$$

$$A_2 = \frac{a_{12}^M}{b_1^L e^{-r_1^M \tau_1} - q_1^M E_1^M} - \frac{a_{22}^L}{b_2^M e^{-r_2^L \tau_2} - q_2^L E_2^L},$$

$$A_3 = \frac{d_1^M}{b_1^L e^{-r_1^M \tau_1} - q_1^M E_1^M} - \frac{d_2^L}{b_2^M e^{-r_2^L \tau_2} - q_2^L E_2^L}.$$

It follows from (3.2) that $A_i < 0$, $i = 1, 2, 3$, this together with the fact $\underline{x}_1 > 0, \overline{x}_2 > 0$ leads to

$$A_1 \underline{x}_1 + A_2 \overline{x}_2 + A_3 \underline{x}_1 \overline{x}_2 < 0, \tag{3.16}$$

which is contradiction with (3.15). Then we obtain $\lim_{t \to +\infty} x_2(t) = 0$. Since

$$y_2(t) = \int_{t-\tau_2}^{t} b_2(s) x_2(s) e^{\int_t^s r_2(u)du} ds,$$

it immediately follows that

$$\lim_{t \to +\infty} y_2(t) = 0.$$

Above analysis shows that for $0 < \varepsilon < \dfrac{(b_1^L e^{-r_1^M \tau_1} - q_1^M E_1^M)}{a_{12}^M}$, there exists a $T_1 > 0$, such that for all $t \ge T_1$, $y_2(t) < \varepsilon$. Lemma 2.3 had showed that

$$\limsup_{t \to +\infty} x_1(t) \le M_1, \quad \limsup_{t \to +\infty} y_1(t) \le N_1.$$

To end the proof of Theorem 3.1, it's enough to show that

$$\liminf_{t \to +\infty} x_1(t) \ge m_1, \quad \liminf_{t \to +\infty} y_1(t) \ge n_1.$$

For $t \ge T_1 + \tau$, from the first equation of system (1.15), we have

$$\dot{x}_1(t) \ge b_1^L e^{-r_1^M \tau_1} x_1(t - \tau_1) - (a_{11}^M + d_1^M \varepsilon) x_1^2(t) - (a_{12}^M \varepsilon + q_1^M E_1^M) x_1(t). \tag{3.17}$$

Let $u(t)$ be the solution of the equation

$$\dot{u} = b_1^L e^{-r_1^M \tau_1} u_1(t-\tau_1) - (a_{11}^M + d_1^M \varepsilon) u_1^2(t) - (a_{12}^M \varepsilon + q_1^M E_1^M) u(t)$$

with $u(T_1 + \tau) = x_1(T_1 + \tau)$. It follows from Lemma 2.2 that

$$\lim_{t \to +\infty} u(t) = \frac{b_1^L e^{-r_1^M \tau_1} - (a_{12}^M \varepsilon + q_1^M E_1^M)}{a_{11}^M + d_1^M \varepsilon}.$$

Therefore, we have

$$\liminf_{t \to +\infty} x_1(t) \geq \frac{b_1^L e^{-r_1^M \tau_1} - (a_{12}^M \varepsilon + q_1^M E_1^M)}{a_{11}^M + d_1^M \varepsilon}.$$

Setting $\varepsilon \to 0$, it follows that

$$\liminf_{t \to +\infty} x_1(t) \geq \frac{b_1^L e^{-r_1^M \tau_1} - q_1^M E_1^M}{a_{11}^M} \overset{\text{def}}{=} m_1 .$$

Noting that

$$y_1(t) = \int_{t-\tau_2}^{t} b_1(s) x_1(s) e^{\int_t^s r_1(u) du} ds.$$

From this, one could easily obtain

$$\liminf_{t \to +\infty} y_1(t) \geq \frac{b_1^L m_1}{r_1^M} (1 - e^{-r_1^L \tau_1}).$$

The proof of Theorem 3.1 is completed.

Proof of Theorem 3.2. Let $(x_1(t), y_1(t), x_2(t), y_2(t))^T$ be any solution of system (1.15) with initial conditions (1.16) and (1.17). It follows from (3.3) that there exists a $\varepsilon_2 > 0$ enough small, such that

$$\frac{b_1^L e^{-r_1^M \tau_1} - q_1^M E_1^M}{b_2^M e^{-r_2^L \tau_2} - q_2^L E_2^L} > \max\left\{ \frac{a_{12}^M}{a_{22}^L}, \frac{a_{11}^M + d_1^M (M_2 + \varepsilon_2)}{a_{21}^L} \right\}. \tag{3.18}$$

Let $\underline{x}_1$ and $\overline{x}_2$ be defined as that of Lemma 3.3. For above $\varepsilon_2 > 0$, it follows from Lemma 2.3 that

$$\underline{x}_1 < M_1 + \varepsilon, \quad \overline{x}_2 < M_2 + \varepsilon_2. \tag{3.19}$$

From Lemma 3.1 we know that $\underline{x}_1 \geq \alpha > 0$. Obviously, $\overline{x}_2 \geq 0$. To prove $\lim_{t \to +\infty} x_2(t) = 0$, it suffices to show that $\overline{x}_2 = 0$. In order to get a contradiction, we suppose that $\overline{x}_2 > 0$. Already, by using the Fluctuation lemma, we had established the inequalities (3.11) and (3.12). Now, from (3.11) and (3.19), we have

$$b_1^L e^{-r_1^M \tau_1} - q_1^M E_1^M \leq \left(a_{11}^M + d_1^M (M_2 + \varepsilon)\right) \underline{x}_1 + a_{12}^M \overline{x}_2, \tag{3.20}$$

which is equivalent to

$$1 \leq \frac{a_{11}^M + d_1^M (M_2 + \varepsilon)}{b_1^L e^{-r_1^M \tau_1} - q_1^M E_1^M} \underline{x}_1 + \frac{a_{12}^M}{b_1^L e^{-r_1^M \tau_1} - q_1^M E_1^M} \overline{x}_2. \tag{3.21}$$

Also, it follows from (3.12) that

$$1 \geq \frac{a_{21}^L}{b_2^M e^{-r_2^L \tau_2} - q_2^L E_2^L} \underline{x}_1 + \frac{a_{22}^L}{b_2^M e^{-r_2^L \tau_2} - q_2^L E_2^L} \overline{x}_2. \tag{3.22}$$

(3.21) combine with (3.22) leads to

$$B_1 \underline{x}_1 + B_2 \overline{x}_2 \geq 0, \tag{3.23}$$

where

$$B_1 = \frac{a_{11}^M + d_1^M (M_2 + \varepsilon)}{b_1^L e^{-r_1^M \tau_1} - q_1^M E_1^M} - \frac{a_{21}^L}{b_2^M e^{-r_2^L \tau_2} - q_2^L E_2^L}.$$

$$B_2 = \frac{a_{12}^M}{b_1^L e^{-r_1^M \tau_1} - q_1^M E_1^M} - \frac{a_{22}^L}{b_2^M e^{-r_2^L \tau_2} - q_2^L E_2^L}.$$

Condition (3.18) implies that $B_i < 0, i = 1, 2$. This together with the fact $\underline{x}_1 > 0, \overline{x}_2 > 0$ leads to

$$B_1 \underline{x}_1 + B_2 \overline{x}_2 < 0, \tag{3.24}$$

which is contradiction with (3.23). Then we obtain $\lim_{t \to +\infty} x_2(t) = 0$. The rest of the proof is similar to that of the proof of Theorem 3.1, and we omit the detail here.

Proof of Theorem 3.3. Let $(x_1(t), y_1(t), x_2(t), y_2(t))^T$ be any solution of system (1.15) with initial conditions (1.16) and (1.17). It follows from (3.4) that there exists a $\varepsilon_3 > 0$ enough small, such that

$$\frac{b_1^L e^{-r_1^M \tau_1} - q_1^M E_1^M}{b_2^M e^{-r_2^L \tau_2} - q_2^L E_2^L} > \max\left\{ \frac{a_{12}^M + d_1^M (M_1 + \varepsilon_3)}{a_{22}^L}, \frac{a_{11}^M}{a_{21}^L} \right\}. \tag{3.25}$$

Let $\underline{x}_1$ and $\overline{x}_2$ be defined as that of Lemma 3.3. For above $\varepsilon_3 > 0$, it follows from Lemma 2.3 that

$$\underline{x}_1 < M_1 + \varepsilon_3, \quad \overline{x}_2 < M_2 + \varepsilon_3. \tag{3.26}$$

From Lemma 3.1 we know that $\underline{x}_1 \geq \alpha > 0$. Obviously, $\overline{x}_2 \geq 0$. To prove $\lim_{t \to +\infty} x_2(t) = 0$, it suffices to show that $\overline{x}_2 = 0$. In order to get a contradiction, we suppose that $\overline{x}_2 > 0$. Already, by using the Fluctuation lemma, we had established the inequalities (3.11) and (3.12). Now, from (3.11) and (3.26), we have

$$b_1^L e^{-r_1^M \tau_1} - q_1^M E_1^M \leq a_{11}^M \underline{x}_1 + \left(a_{12}^M + d_1^M (M_1 + \varepsilon_3)\right) \overline{x}_2, \tag{3.27}$$

which is equivalent to

$$1 \leq \frac{a_{11}^M}{b_1^L e^{-r_1^M \tau_1} - q_1^M E_1^M} \underline{x}_1 + \frac{a_{12}^M + d_1^M (M_1 + \varepsilon_3)}{b_1^L e^{-r_1^M \tau_1} - q_1^M E_1^M} \overline{x}_2. \tag{3.28}$$

Also, it follows from (3.12) that

$$1 \geq \frac{a_{21}^L}{b_2^M e^{-r_2^L \tau_2} - q_2^L E_2^L} \underline{x}_1 + \frac{a_{22}^L}{b_2^M e^{-r_2^L \tau_2} - q_2^L E_2^L} \overline{x}_2. \tag{3.29}$$

(3.28) combine with (3.29) leads to

$$C_1 \underline{x}_1 + C_2 \overline{x}_2 \geq 0, \tag{3.30}$$

where

$$C_1 = \frac{a_{11}^M}{b_1^L e^{-r_1^M \tau_1} - q_1^M E_1^M} - \frac{a_{21}^L}{b_2^M e^{-r_2^L \tau_2} - q_2^L E_2^L}.$$

$$C_2 = \frac{a_{12}^M + d_1^M (M_1 + \varepsilon_3)}{b_1^L e^{-r_1^M \tau_1} - q_1^M E_1^M} - \frac{a_{22}^L}{b_2^M e^{-r_2^L \tau_2} - q_2^L E_2^L}.$$

Condition (3.25) implies that $C_i < 0, i = 1, 2$. This together with the fact $\underline{x}_1 > 0, \overline{x}_2 > 0$ leads to

$$C_1 \underline{x}_1 + C_2 \overline{x}_2 < 0, \tag{3.31}$$

which is contradiction with (3.30). Then we obtain $\lim_{t \to +\infty} x_2(t) = 0$. The rest of the proof is similar to that of the proof of Theorem 3.1, and we omit the detail here.

Concerned with the extinction of the first species, we have the following result.

Theorem 3.4. *Assume that (3.5) or (3.6) or (3.7) hold. Then*

$$m_2 \leq \liminf_{t \to +\infty} x_2(t) \leq \limsup_{t \to +\infty} x_2(t) \leq M_2.$$

$$n_2 \leq \liminf_{t \to +\infty} y_2(t) \leq \limsup_{t \to +\infty} y_2(t) \leq N_2.$$

$$\lim_{t \to +\infty} x_1(t) = 0, \quad \lim_{t \to +\infty} y_1(t) = 0,$$

where

$$m_2 = \frac{b_2^L e^{-r_2^M \tau_2} - q_2^M E_2^M}{a_{22}^M}, \quad n_2 = \frac{b_2^L m_2}{r_2^M}\left(1 - e^{-r_2^L \tau_2}\right).$$

Since the proof of Theorem 3.4 is similar to that of Theorems 3.1-3.3, we omit the detail here.

As a direct corollary of Theorem 2.4, we have

Corollary 3.4. *Assume that in system (1.9), one of the following three inequalities holds.*

$$\frac{b_2^L e^{-r_2^M \tau_2}}{b_1^M e^{-r_1^L \tau_1}} > \max\left\{\frac{a_{21}^M}{a_{11}^L}, \frac{a_{22}^M}{a_{12}^L}, \frac{d_2^M}{d_1^L}\right\},$$

$$\frac{b_2^L e^{-r_2^M \tau_2}}{b_1^M e^{-r_1^L \tau_1}} > \max\left\{\frac{a_{21}^M}{a_{11}^L}, \frac{a_{22}^M + d_2^M M_{22}}{a_{12}^L}\right\},$$

$$\frac{b_1 2^L e^{-r_2^M \tau_2}}{b_1^M e^{-r_1^L \tau_1}} > \max\left\{\frac{a_{21}^M + d_2^M M_{11}}{a_{11}^L}, \frac{a_{22}^M}{a_{12}^L}\right\},$$

where $M_{ii} = \dfrac{b_i^M e^{-r_i^L \tau_i}}{a_{ii}^L}$, $i = 1, 2$. *Then*

$$m_2 \le \liminf_{t\to+\infty} x_2(t) \le \limsup_{t\to+\infty} x_2(t) \le M_2.$$
$$n_2 \le \liminf_{t\to+\infty} y_2(t) \le \limsup_{t\to+\infty} y_2(t) \le N_2.$$
$$\lim_{t\to+\infty} x_1(t) = 0, \quad \lim_{t\to+\infty} y_1(t) = 0,$$

where

$$m_2 = \frac{b_2^L e^{-r_2^M \tau_2}}{a_{22}^M}, \quad n_2 = \frac{b_2^L m_2}{r_2^M}\left(1 - e^{-r_2^L \tau_2}\right).$$

4 Examples

In this section we shall give two examples to illustrate the feasibility of main results in the previous section.

Example 4.1. Consider Example 1.1 in the introduction Section. Already, we had verified

$$\frac{b_1^L e^{-r_1^M \tau_1}}{b_2^M e^{-r_2^L \tau_2}} = \frac{3}{2} > 1 = \frac{a_{12}^M}{a_{22}^L}. \tag{4.1}$$

Noting that

$$M_{11} = \frac{b_1^M e^{-r_1^L \tau_1}}{a_{11}^L} = \frac{3e^{-0.2}}{1}.$$

Thus,

$$\frac{a_{11}^M + d_1^M M_1}{a_{21}^L} = \frac{2 + 0.2 \times 3e^{-0.2}}{2} < \frac{2 + 0.6}{2} < \frac{3}{2} = \frac{b_1^L e^{-r_1^M \tau_1}}{b_2^M e^{-r_2^L \tau_2}}. \tag{4.2}$$

(4.1) together with (4.2) shows that all the conditions of Corollary 3.2 are hold, and so, the second species will be driven to extinction.

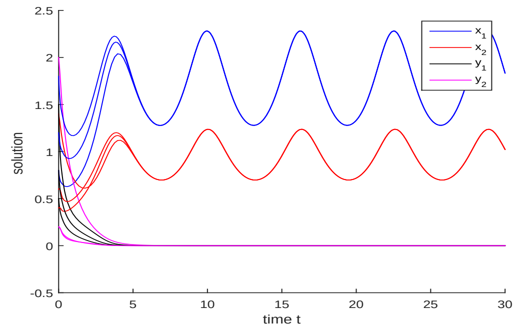

Figure 1: Dynamics behaviors of the solution $(x_1(t), y_1(t), x_2(t), y_2(t))^T$ of system (4.1) with initial condition $(\varphi_1(\theta), \varphi_2(\theta), \psi_1(\theta), \psi_2(\theta))^T$ =$(1.2, 0.4, 0.8, 0.2)^T$, $(1.8, 0.6, 0.5, 0.2)$ and $(0.8, 1.4, 1.5, 2)^T$, $\theta \in [-0.2, 0]$, respectively.

Example 4.2. Now let's further incorporate the harvesting effort to system (1.11), this leads to the following system

$$\begin{aligned}
\dot{x}_1(t) &= 3e^{-0.2}x_1(t-0.2) - (1.5+0.5\cos(t))x_1^2(t) \\
&\quad -(2+\sin(t))x_1(t)x_2(t) - 0.2x_1^2(t)x_2(t) - q_1(t)E_1(t)x_1, \\
\dot{y}_1(t) &= 3x_1(t) - y_1(t) - 3e^{-0.2}x_1(t-0.2), \\
\dot{x}_2(t) &= 2e^{-0.2}x_2(t-0.2) - (3.5+0.5\cos(t))x_2^2(t) \\
&\quad -2x_1(t)x_2(t) - 0.1x_1(t)(x_2(t))^2 - q_2(t)E_2(t)x_2, \\
\dot{y}_2(t) &= 2x_2(t) - y_2(t) - 2e^{-0.2}x_2(t-0.2),
\end{aligned} \tag{4.3}$$

where $\tau_1 = 0.2$, $\tau_2 = 0.2$, $b_1(t) = 4$, $r_1(t) = 1$, $a_{11}(t) = 1.5+0.5\cos(t)$, $a_{12}(t) = 2+\sin(t)$, $d_1(t) = 0.2$, $d_2(t) = 0.1$, $b_2(t) = 2$, $r_2(t) = 1$, $a_{21}(t) = 2$, $a_{22}(t) = 3.5+0.5\cos(t)$.
(1) Take $q_1(t)E_1(t) = 3$, $q_2(t)E_2(t) = 2$, in this case, $b_i^M e^{-r_i^L\tau_i} < q_i^L E_i^L$, $i = 1, 2$ holds, and so, from Remark 2.1, this is overfishing case, and all the species will be driven to extinction. Fig 2. support this assertion.
(2) Take $q_1(t)E_1(t) = 0.2e^{-0.2}$, $q_2(t)E_2(t) = 0$, in this case, there are no harvest on the second species, also, the harvesting of the first species is restrict to a limited case. $\dfrac{b_1^L e^{-r_1^M\tau_1} - q_1^M E_1^M}{b_2^M e^{-r_2^L\tau_2}} = \dfrac{2.8}{2}$,

$$M_{11} = \frac{b_1^M e^{-r_1^L\tau_1} - q_1^M E_1^M}{a_{11}^L} = \frac{2.8e^{-0.2}}{1}.$$

Thus,

$$\frac{a_{11}^M + d_1^M M_1}{a_{21}^L} = \frac{2+0.2\times 2.8e^{-0.2}}{2} < \frac{2+0.6}{2} < \frac{2.8}{2} = \frac{b_1^L e^{-r_1^M\tau_1} - q_1^M E_1^M}{b_2^M e^{-r_2^L\tau_2}}. \tag{4.4}$$

$$\frac{a_{12}^M}{a_{22}^L} = 1 < \frac{2.8}{2} = \frac{b_1^L e^{-r_1^M\tau_1} - q_1^M E_1^M}{b_2^M e^{-r_2^L\tau_2}}. \tag{4.5}$$

(4.4) and (4.5) show that all the conditions of Corollary 3.2 are hold, then second species will be driven to extinction. Fig. 3 also support this assertion.
(3) Take $q_1(t)E_1(t) = 2.6e^{-0.2}$, $q_2(t)E_2(t) = 0$, in this case

$$\frac{b_2^L e^{-r_2^M\tau_2}}{b_1^M e^{-r_1^L\tau_1} - q_1^L E_1^L} = \frac{2e^{-0.2}}{3e^{-0.2} - 2.6e^{-0.2}} = 5.$$

$$\frac{a_{21}^M}{a_{11}^L} = 2, \quad \frac{a_{22}^M}{a_{12}^L} = 4, \quad \frac{d_2^M}{d_1^L} = \frac{1}{2}.$$

Hence

$$\frac{b_2^L e^{-r_2^M \tau_2}}{b_1^M e^{-r_1^L \tau_1} - q_1^L E_1^L} > \max\{\frac{a_{21}^M}{a_{11}^L}, \frac{a_{22}^M}{a_{12}^L}, \frac{d_2^M}{d_1^L}\}.$$

That is, inequality (3.5) holds, from Theorem 2.4, the second species will be driven to extinction. Fig. 4 also support this assertion.

(4) Take $q_1(t)E_1(t) = 3, q_2(t)E_2(t) = 0$, in this case, the first species is overfishing, while the second one is free of harvesting. From Remark 2.1, the first species will be driven to extinction. Due to the extinction of the first species, the second one will be permanent. Fig.5 also support this assertion.

(5) Take $q_1(t)E_1(t) = 1.5, q_2(t)E_2(t) = 0$. Numeric simulation (Fig. 6) shows that in this case, two species could be coexist in a stable state.

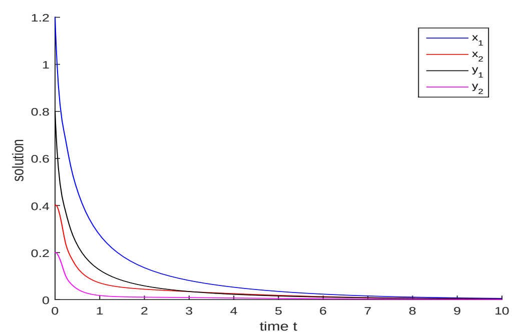

Figure 2: Dynamics behaviors of the solution $(x_1(t), y_1(t), x_2(t), y_2(t))^T$ of system (4.3) with initial condition $(\varphi_1(\theta), \varphi_2(\theta), \psi_1(\theta), \psi_2(\theta))^T$ =$(1.2, 0.4, 0.8, 0.2)^T$,$\theta \in [-0.2, 0]$, where $q_1E_1 = 3, q_2E_2 = 2$. respectively.

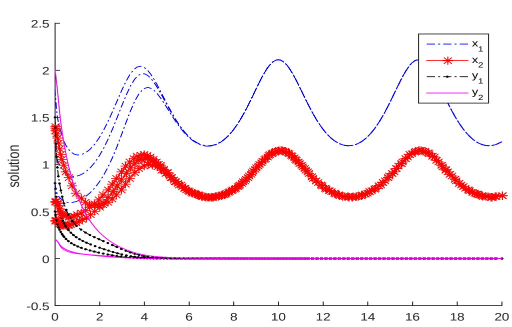

Figure 3: Dynamics behaviors of the solution $(x_1(t), y_1(t), x_2(t), y_2(t))^T$ of system (4.3) with initial condition $(\varphi_1(\theta), \varphi_2(\theta), \psi_1(\theta), \psi_2(\theta))^T$ =$(1.2, 0.4, 0.8, 0.2)^T$, $(1.8, 0.6, 0.5, 0.2)$ and $(0.8, 1.4, 1.5, 2)^T$, $\theta \in [-0.2, 0]$, respectively. Here we take $q_1E_1 = 0.2e^{-0.2}, q_2E_2 = 0$.

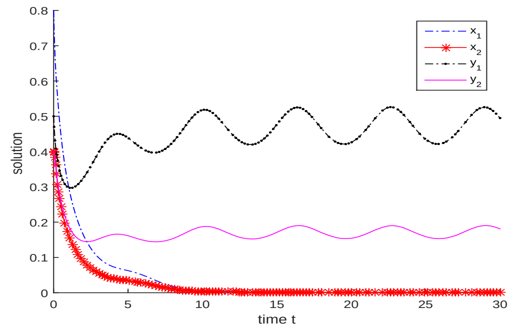

Figure 4: Dynamics behaviors of the solution $(x_1(t), y_1(t), x_2(t), y_2(t))^T$ of system (4.3) with initial condition $(\varphi_1(\theta), \varphi_2(\theta), \psi_1(\theta), \psi_2(\theta))^T = (0.8, 0.4, 0.5, 0.4)$, $\theta \in [-0.2, 0]$, respectively. Here we take $q_1E_1 = 2.6e^{-0.2}$, $q_2E_2 = 0$.

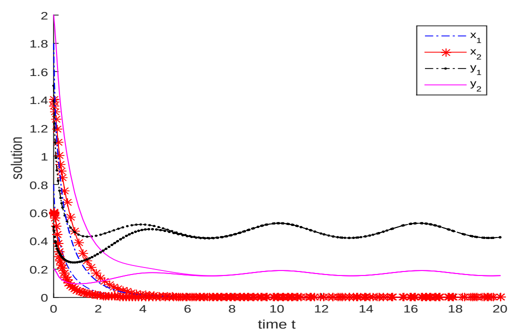

Figure 5: Dynamics behaviors of the solution $(x_1(t), y_1(t), x_2(t), y_2(t))^T$ of system (4.3) with initial condition $(\varphi_1(\theta), \varphi_2(\theta), \psi_1(\theta), \psi_2(\theta))^T = (1.8, 0.6, 0.5, 0.2)$ and $(0.8, 1.4, 1.5, 2)^T$, $\theta \in [-0.2, 0]$, respectively. Here we take $q_1E_1 = 3$, $q_2E_2 = 0$.

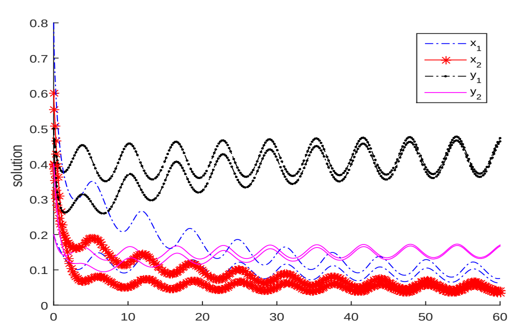

Figure 6: Dynamics behaviors of the solution $(x_1(t), y_1(t), x_2(t), y_2(t))^T$ of system (4.3) with initial condition $(\varphi_1(\theta), \varphi_2(\theta), \psi_1(\theta), \psi_2(\theta))^T = (0.2, 0.6, 0.5, 0.2)$ and $(0.8, 0.4, 0.5, 0.4)^T$, $\theta \in [-0.2, 0]$, respectively. Here we take $q_1E_1 = 1.5$, $q_2E_2 = 0$.

5 Discusion

Li and Chen [10] proposed a two species periodic competitive stage-structured Lotka-Volterra model with the effects of toxic substances, they studied the extinction property of the system. It is naturally to investigate the dynamic behaviours of system (1.9) if the conditions in [10] no longer hold, Example 1.1 in the introduction Section shows that some of the species still could be driven to extinction, this motivated us to revisit the extinction property of the system (1.9). On the other hand, Kar and Chaudhuri [36] and Gupta, Banerjee and Chandra [37] studied the influence of harvesting effect on the competition system with toxic substance. Their success motivated us to propose a two species competitive stage-structured system with the effect of toxic substance and harvesting (system (1.15)). We first show that due to the overfishing, two of the species will be driven to extinction (Remark 2.1). After that, for the appropriate harvesting case, by applying the fluctuation theorem, we are able to establish sufficient conditions which ensure one of the components be driven to extinction.

Theorem 3.1 can be seen as the generalization of Theorem 3.1 in [10], thus, we generalize the main result of [10] to the harvesting case. Theorem 3.2-3.4 are new results, which supplement and complement the main results of [6] and [10].

To show the feasibility of our main results, we study a numeric example (Example 4.2), here we make an assumption that we only harvest the first species, and if $q_1(t)E_1(t) = 0$, that is, without the capture of the first species, the second species will be driven to extinction. Then, depending on the harvesting effect $q_1(t)E_1(t)$, the system may have the following dynamic behaviors: (1) the second species still be driven to extinction (case (2)); (2) the first species will be driven to extinction (cases (3) and (4)); (3) two species could be coexist in a stable state (case (5)).

Our results and numeric examples show that harvesting is one of the most important factors to influence the dynamic behaviours of the system.

6 Declarations

Competing interests

The authors declare that there is no conflict of interests.

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Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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