DE GRUYTER Open Math. 2019; 17:813–827

Open Mathematics

Research Article

Jing Zhang*

# Variation inequalities related to Schrödinger operators on weighted Morrey spaces

https://doi.org/10.1515/math-2019-0065
Received December 8, 2017; accepted May 16, 2019

**Abstract:** This paper establishes the boundedness of the variation operators associated with Riesz transforms and commutators generated by the Riesz transforms and BMO-type functions in the Schrödinger setting on the weighted Morrey spaces related to certain nonnegative potentials belonging to the reverse Hölder class.

**Keywords:** Schrödinger operators; variation operators; Riesz transforms; commutators; weighted Morrey spaces

**MSC:** 42B20, 42B35

## 1 Introduction

Given a family of bounded operators $\mathcal{T} = \{T_\epsilon\}_{\epsilon>0}$ acting between spaces of functions, one of the most significative problems in harmonic analysis is the existence of limits $\lim_{\epsilon\to0^+} T_\epsilon f$ and $\lim_{\epsilon\to\infty} T_\epsilon f$, when $f$ belongs to a certain space of functions. A question that arises naturally is what is the speed of convergence of the above limits. A classical way to measure the speed of convergence of $\{T_\epsilon\}_{\epsilon>0}$ is to study "square function" of the type $(\sum_{i=1}^{\infty} |T_{\epsilon_i} f - T_{\epsilon_{i+1}} f|^2)^{1/2}$, where $\epsilon_i \searrow 0$. Recently, other expressions have been considered, among which is the $q$ -variation operator defined by

$$\mathcal{V}_q(\mathcal{T}f)(x) := \sup_{\epsilon_i \searrow 0} \left( \sum_{i=1}^{\infty} |T_{\epsilon_{i+1}} f(x) - T_{\epsilon_i} f(x)|^q \right)^{1/q}, \tag{1.1}$$

where the supremum is taken over all sequence $\{\epsilon_i\}$ decreasing to zero. We denote $F_q$ the space that includes all the functions $\varphi : (0,\infty) \to \mathbb{R}$, such that

$$\|\varphi\|_{F_q} = \sup_{\{\epsilon_i\}_{i\in\mathbb{N}}} \left( \sum_{i=0}^{\infty} |\varphi(\epsilon_i) - \varphi(\epsilon_{i+1})|^q \right)^{1/q} < \infty. \tag{1.2}$$

Then $\|\cdot\|_{F_q}$ is a seminorm on $F_q$. It can be written that

$$\mathcal{V}_q(T_\epsilon)(f) = \|T_\epsilon f\|_{F_q}. \tag{1.3}$$

The variation for martingales and some families of operators have been studied in many recent papers on probability, ergodic theory, and harmonic analysis. We refer the readers to [1–6] and the references therein for more background information.

Recently, Betancor et al. [7] studied the bounded behaviors of variation operators for some Schrödinger type operators in Lebesgue spaces. Precisely, let $n \geq 3$ and $\mathcal{L} = -\Delta + V$ be the Schrödinger operator defined

***Corresponding Author: Jing Zhang:** School of Mathematics and Statistics, Yili Normal University, Yining Xinjiang 835000, China, E-mail: zjmath66@126.com

on $\mathbb{R}^n$ associated with a fixed non-negative potential $V \in RH_s$ (the reverse Hölder class) for $s \geq n/2$, that is, there exists $C > 0$ such that

$$\Big(\frac{1}{|B|}\int_B V(x)^s dx\Big)^{1/s} \leq \frac{C}{|B|}\int_B V(x)dx \tag{1.4}$$

for every ball B in $\mathbb{R}^n$. For $\ell = 1, \cdots n$, consider the $\ell$-th Riesz transform in the $\mathcal{L}$-context, which can be defined by

$$R_\ell^{\mathcal{L}}(f)(x) = \lim_{\varepsilon\to 0^+} R_{\ell,\varepsilon}^{\mathcal{L}}(f)(x) := \lim_{\varepsilon\to 0^+}\int_{|x-y|>\varepsilon} \mathfrak{R}_\ell^{\mathcal{L}}(x,y)f(y)dy, \quad a.e.\, x \in \mathbb{R},$$

and its adjoint operator by

$$R_\ell^{\mathcal{L},\star}(f)(x) = \lim_{\varepsilon\to 0^+} R_{\ell,\varepsilon}^{\mathcal{L},\star}(f)(x) := \lim_{\varepsilon\to 0^+}\int_{|x-y|>\varepsilon} \mathfrak{R}_\ell^{\mathcal{L}}(y,x)f(y)dy, \quad a.e.\, x \in \mathbb{R}.$$

Here, for every $x,\ y \in \mathbb{R}^n$, $x \neq y$,

$$\mathfrak{R}_\ell^{\mathcal{L}}(x,y) = -\frac{1}{2\pi}\int_{\mathbb{R}} (-i\tau)^{-1/2}\frac{\partial}{\partial x_\ell}\Gamma(x,y,\tau)d\tau,$$

where $\Gamma(x,y,\tau)$ represents the fundamental solution for the operator $\mathcal{L} + i\tau$ (see [7, 8]).

Betancor et al. [7] proved that when $n/2 \leq s < n$, for $\ell = 1,\ 2, \cdots,\ n$, the variation operators $\mathcal{V}_q(R_{\ell,\varepsilon}^{\mathcal{L}})$ (resp., $\mathcal{V}_q(R_{\ell,\varepsilon}^{\mathcal{L},\star})$) associated with the family of truncations $\{R_{l,\varepsilon}^{\mathcal{L}}\}_{\varepsilon>0}$ (resp., $\{R_{\ell,\varepsilon}^{\mathcal{L},\star}\}_{\varepsilon>0}$) are bounded from $L^p(\mathbb{R}^n)$ into itself for $1 < p < p_0$ (resp., $p_0' < p < \infty$) with $1/p_0 = 1/s - 1/n$, and $\mathcal{V}_q(R_{\ell,\varepsilon}^{\mathcal{L}})$ are of weak type $(1,\ 1)$; moreover, when $s \geq n$, both $\mathcal{V}_q(R_{\ell,\varepsilon}^{\mathcal{L}})$ and $\mathcal{V}_q(R_{\ell,\varepsilon}^{\mathcal{L},\star})$ are bounded from $L^p(\mathbb{R}^n)$ into itself for $1 < p < \infty$ and of weak type $(1,\ 1)$. More recently, Tang and Zhang [9] extend the results above to the weighted $L^p$ space. They established the weighted $L^p$ boundedness for $\mathcal{V}_q(R_{\ell,\varepsilon}^{\mathcal{L}})$ and $\mathcal{V}_q(R_{\ell,\varepsilon}^{\mathcal{L},\star})$ with the weight $A_p^{\gamma,\theta}$ class (see section 2 for the defination), which includes the Mukenhoupt weight class. By different method, Zhang and Wu in [10] also obtained $L^p$ boundedness for $\mathcal{V}_q(R_{\ell,\varepsilon}^{\mathcal{L}})$ and $\mathcal{V}_q(R_{\ell,\varepsilon}^{\mathcal{L},\star})$ and the weighted weak type (1,1) estimation $\mathcal{V}_q(R_{\ell,\varepsilon}^{\mathcal{L}})$ with the weight $A_p^{\gamma,\theta}$ class .

In addition, for every $V \in RH_{n/2}$, Shen [11] introduced the function $\gamma$, which is called as the critical radius and defined as

$$\gamma(x) = \sup\Big\{r > 0 : \frac{1}{r^{n-2}}\int_{B(x,r)} V(y)dy \leq 1\Big\}, \quad x \in \mathbb{R}^n, \tag{1.5}$$

and plays key roles in the theory of harmonic analysis operators associated to $\mathcal{L}$. In [12], Bongioann et al defined the space $BMO_\theta(\gamma)$, $\theta \geq 0$, as follows.

**Definition 1.1.** *A locally integrable function b in $\mathbb{R}^n$ is $BMO_\theta(\gamma)$ provided that*

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} |b(y) - b_{B(x,r)}|dy \leq C\Big(1 + \frac{r}{\gamma(x)}\Big)^\theta \tag{1.6}$$

*for all $x \in \mathbb{R}^n$ and $r > 0$, where $b_B = |B|^{-1}\int_B |b(x) - b_B|dx$. We denote for $b \in BMO_\theta(\gamma)$*

$$\|b\|_{BMO_\theta(\gamma)} = \sup_{x\in\mathbb{R}^n,\, r>0}\frac{1}{|B(x,r)|}\int_{B(x,r)} |b(y) - b_{B(x,r)}|dy\Big(1 + \frac{r}{\gamma(x)}\Big)^{-\theta}.$$

It is easy to check that $BMO(\mathbb{R}^n) = BMO_0(\gamma) \subset BMO_\theta(\gamma) \subset BMO_{\theta'}(\gamma)$ for $0 \leq \theta \leq \theta'$. Set $BMO_\infty(\gamma) = \cup_{\theta>0} BMO_\theta(\gamma)$. Then $BMO_\infty(\gamma)$ is larger than $BMO(\mathbb{R}^n)$ in general (see [12]).

For $b \in BMO_\infty(\gamma)$ and $\ell = 1, \cdots, n$, the commutators $R_{b,\ell}^{\mathcal{L}}$ and $R_{b,\ell}^{\mathcal{L},\star}$ are defined by

$$R_{b,\ell}^{\mathcal{L}}(f) = bR_\ell^{\mathcal{L}} - R_\ell^{\mathcal{L}}(bf), \quad \text{and} \quad R_{b,\ell}^{\mathcal{L},\star}(f) = bR_\ell^{\mathcal{L},\star} - R_\ell^{\mathcal{L},\star}(bf) \quad \text{for } f \in C_c^\infty(\mathbb{R}^n).$$

It was shown in [12] that, for every $b \in BMO_\infty(\gamma)$ and $\ell = 1, \cdots, n$, the operators $R^{\mathcal{L}}_{b,\ell}$ (resp., $R^{\mathcal{L},\star}_{b,\ell}$) are bounded on $L^p(\mathbb{R}^n)$, provided that $1 < p < p_0$ (resp., $p_0' < p < \infty$) with $1/p_0 = (1/s - 1/n)_+$ for $V \in RH_s$, $s \geq n/2$. In [7], Betancor et al obtained the following point-wise representations of the commutator operators by a principal value integral:

$$R^{\mathcal{L}}_{b,\ell}(f)(x) = \lim_{\varepsilon\to 0} R^{\mathcal{L}}_{b,\ell,\varepsilon}(f)(x) := \lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} (b(x) - b(y))\mathfrak{R}^{\mathcal{L}}_{\ell}(x, y)f(y)dy, \quad a.e.\ x \in \mathbb{R}^n,$$

and

$$R^{\mathcal{L},\star}_{b,\ell}(f)(x) = \lim_{\varepsilon\to 0} R^{\mathcal{L},\star}_{b,\ell,\varepsilon}(f)(x) := \lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} (b(x) - b(y))\mathfrak{R}^{\mathcal{L}}_{\ell}(y, x)f(y)dy, \quad a.e.\ x \in \mathbb{R}^n.$$

Moreover, the authors in [7] proved that for every $b \in BMO_\infty(\gamma)$ and $\ell = 1, \cdots, n$, the variation operators $\mathcal{V}_q(R^{\mathcal{L}}_{b,\ell,\varepsilon})$ (resp., $\mathcal{V}_q(R^{\mathcal{L},\star}_{b,\ell,\varepsilon})$) associated with the family of truncations $\{R^{\mathcal{L}}_{b,\ell,\varepsilon}\}_{\varepsilon>0}$ (resp., $\{R^{\mathcal{L},\star}_{b,\ell,\varepsilon}\}_{\varepsilon>0}$) are bounded from $L^p(\mathbb{R}^n)$ into itself, provided that $1 < p < p_0$ (resp., $p_0' < p < \infty$) with $1/p_0 = (1/s - 1/n)_+$ for $V \in RH_s$, $s \geq n/2$. In [9], the above results have been extend to the weighted Lebesgue space.

On the other hand, in order to extend the boundedness of Schrödinger type operators in Lebesgue spaces, Pan and Tang [13] introduced the following weighted Morrey spaces related to the non-negative potential $V$, denoted by $L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)$.

**Definition 1.2.** *Let $k > 1$, $p \in [1, \infty)$, $\alpha \in (-\infty, +\infty)$ and $\lambda \in [0, 1)$. For $f \in L^p_{loc}(\mathbb{R}^n)$ and $V \in RH_q$ $(q > 1)$, we say $f \in L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)$ provided that*

$$\|f\|^p_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)} = \sup_{B(x_0,r)\subset\mathbb{R}^n} \left(1 + \frac{r}{\gamma(x_0)}\right)^{\alpha} \omega(B(x_0, kr))^{-\lambda} \int_{B(x_0,r)} |f(x)|^p \omega(x)dx < \infty, \tag{1.7}$$

*where $B = B(x_0, r)$ denote a ball with centered at $x_0$ and radius $r$, $\gamma(x_0)$ is the critical radius at $x_0$ defined as in* (1.5) *and the weight function $\omega \in A^{\gamma,\infty}_p$.*

Clearly, when $\alpha = 0$ or $V = 0$, $\omega = 1$ and $0 < \lambda < 1$, the spaces $L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)$ are the classical Morrey spaces $L^{p,\lambda}(\mathbb{R}^n)$, which were introduced by Morrey [14] in 1938 and were subsequently found to have many important applications to the elliptic equations (see [15–20]), the Navier-Stokes equations (see [21–23]) and the Schrödinger equations (see [24–27]) etc. When $\alpha = 0$ or $V = 0$ and $0 < \lambda < 1$, the spaces $L^{p,\lambda}_{\omega}(\mathbb{R}^n)$ is first introduced in [28] , where $\omega \in A_p(\mathbb{R}^n)$(Muckenhoupt weights class). It is easy to see that $L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n) \subset L^{p,\lambda}_{\omega}(\mathbb{R}^n)$ for $\alpha > 0$ and $L^{p,\lambda}_{\omega}(\mathbb{R}^n) \subset L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)$ for $\alpha < 0$. In [29], the authors established the $L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)$-boundedness of the Riesz transforms $\mathcal{V}_q(R^{\mathcal{L}}_{\ell})$, $\mathcal{V}_q(R^{\mathcal{L},\star}_{\ell})$, and the corresponding commutators with $V \in B_{n/2}$.

Based on the above, it is a natural and interesting question whether we can establish the $L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)$-boundedness of the variation operators aforementioned in Schrödinger setting. The main purpose of this paper is to answer this question. Our results can be formulated as follows:

**Theorem 1.3.** *Let $\ell = 1, \cdots n$, $q > 2$ and $V \in RH_s$. Assume $\alpha \in (-\infty, +\infty)$ and $\lambda \in (0, 1)$. Then*
(i) *If $n/2 \leq s < n$ and $p_0$ is such that $1/p_0 = 1/s - 1/n$, then the variation operator $\mathcal{V}_q(R^{\mathcal{L},\star}_{l,\varepsilon})$ is bounded on $L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)$ for $p_0' \leq p < \infty$ and $\omega \in A^{\gamma,\infty}_{p/p_0'}$.*
(ii) *If $n/2 \leq s < n$ and $p_0$ is such that $1/p_0 = 1/s - 1/n$, then the variation operator $\mathcal{V}_q(R^{\mathcal{L}}_{l,\varepsilon})$ is bounded on $L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)$ for $1 < p \leq p_0$ and $\omega^{-\frac{1}{p-1}} \in A^{\gamma,\infty}_{p'/p_0'}$.*
(iii) *If $s \geq n$, then the variation operators $\mathcal{V}_q(R^{\mathcal{L},\star}_{l,\varepsilon})$ and $\mathcal{V}_q(R^{\mathcal{L}}_{l,\varepsilon})$ are bounded on $L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)$ for $1 < p < \infty$ and $\omega \in A^{\gamma,\infty}_p$.*

**Theorem 1.4.** *Let $\ell = 1, \cdots, n$. Assume that $q > 2$ and $V \in RH_s$.* (i) *If $n/2 \leq s < n$ then for $p = 1$, $\eta > 0$ and $\omega^{p_0'} \in A^{\gamma,\infty}_1$,*

$$\eta\left(1 + \frac{r}{\gamma(x)}\right)^{\alpha} \omega(B(x, 2r))^{-\lambda} \omega\{y \in B(x, r) : |\mathcal{V}_q(R^{\mathcal{L}}_{\ell,\varepsilon}f)(y)| > \eta\} \leq C\|f\|_{L^{1,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)},$$

(ii) *If* $s \geq n$ *, then for* $p = 1$, $\eta > 0$ *and* $\omega \in A_1^{\gamma,\infty}$,

$$\eta\Big(1+\frac{r}{\gamma(x)}\Big)^{\alpha}\omega(B(x,2r))^{-\lambda}\omega\{y \in B(x,r) : |\mathcal{V}_q(R^{\mathcal{L}}_{\ell,\varepsilon}f)(y)| > \eta\} \leq C\|f\|_{L^{1,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)},$$

*and*

$$\eta\Big(1+\frac{r}{\gamma(x)}\Big)^{\alpha}\omega(B(x,2r))^{-\lambda}\omega\{y \in B(x,r) : |\mathcal{V}_q(R^{\mathcal{L},\star}_{\ell,\varepsilon}f)(y)| > \eta\} \leq C\|f\|_{L^{1,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)},$$

*holds for all balls B, where C is independent of* $x, r, \eta$ *and* $f$.

**Theorem 1.5.** *Let* $\ell = 1, \cdots, n$, $q > 2$, $b \in BMO^{\gamma}_{\theta}$ *with* $\theta > 0$ *and* $V \in RH_s$, *Assume that* $\alpha \in (-\infty, +\infty)$ *and* $\lambda \in (0,1)$.
(i) *If* $n/2 \leq s < n$ *and* $p_0$ *is such that* $1/p_0 = 1/s - 1/n$, *then the variation operator* $\mathcal{V}_q(R^{\mathcal{L},\star}_{b,l,\varepsilon})$ *is bounded on* $L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)$ *for* $p_0' \leq p < \infty$ *and* $\omega \in A^{\gamma,\infty}_{p/p_0'}$.
(ii) *If* $n/2 \leq s < n$ *and* $p_0$ *is such that* $1/p_0 = 1/s - 1/n$, *then the variation operator* $\mathcal{V}_q(R^{\mathcal{L}}_{b,l,\varepsilon})$ *is bounded on* $L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)$ *for* $1 < p \leq p_0$, *with* $\omega^{-\frac{1}{p-1}} \in A^{\gamma,\infty}_{p'/p_0'}$.
(iii) *If* $s \geq n$, *then the variation operator* $\mathcal{V}_q(R^{\mathcal{L},\star}_{b,l,\varepsilon})$ *and* $\mathcal{V}_q(R^{\mathcal{L}}_{b,l,\varepsilon})$ *are bounded on* $L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)$ *for* $1 < p < \infty$ *and* $\omega \in A^{\gamma,\infty}_p$.

**Remark 1.6.** In [8], it was proved that if $V$ is a nonnegative polynomial, then $V \in RH_s$ for any $1 < s < \infty$. Therefore, as special cases of our results, the corresponding ones to the Hermite operator: $H = -\Delta + |x|^2$ hold. This can be regarded as the generalization of the corresponding results in [2, 30].
The rest of this paper is organized as follows. In Section 2, we will recall some properties of the function $\gamma$ and some basic facts concerning weights $A^{\gamma}_p$ , which will play a crucial roles in our arguments. In Section 3, we will prove Theorem 1.3 and 1.4, the proof of Theorem 1.5 will be given in Section 4. Throughout this paper, the letter $C$ always denotes a positive constant that is independent of main parameters involved but whose value may differ from line to line. We use $f \lesssim g$ to denote $f \leq Cg$. If $f \lesssim g \lesssim f$, we write $f \sim g$. For any $t \in (0,\infty)$, we denote the ball $B(x,tr)$ by $tB$. Given any $p \in [1,\infty]$, $p' = \frac{p}{p-1}$ denote its conjugate index. In particular, it should be pointed out that these weighted Morrey spaces in Definition 1.2 are equivalent for different $k > 1$.

## 2 Preliminaries

In this section, we recall some known results, which will be used in our next proofs. We first recall some properties of the auxiliary function $\gamma(x)$, which will be used below.

**Lemma 2.1.** *(cf. [8]) If* $V \in RH_{n/2}$, *then there exist* $c_0$ *and* $l_0 \geq 1$ *such that for all* $x, y \in \mathbb{R}^n$,

$$\frac{1}{c_0}\gamma(x)\Big(1+\frac{|x-y|}{\gamma(x)}\Big)^{-l_0} \leq \gamma(y) \leq c_0\gamma(x)\Big(1+\frac{|x-y|}{\gamma(x)}\Big)^{l_0/(l_0+1)}. \tag{2.1}$$

*In particular,* $\gamma(x) \sim \gamma(y)$ *if* $|x-y| < C\gamma(x)$, *and the ball* $B(x,\gamma(x))$ *is called critical.*

According to [31], we recall a new class of weights $A^{\gamma,\infty}_p = \bigcup_{\theta \geq 0} A^{\gamma,\theta}_p$ for $p \geq 1$, where $A^{\gamma,\theta}_p (p > 1)$ is the set of those weights satisfying

$$\Big(\int_B \omega(y)dy\Big)^{1/p}\Big(\int_B \omega^{-\frac{1}{p-1}}(y)dy\Big)^{1/p'} \leq C|B|\Big(1+\frac{r}{\gamma(x)}\Big)^{\theta}, \tag{2.2}$$

and $A^{\gamma,\theta}_1$ is the set of those weights $\omega$ such that

$$\frac{1}{|B|}\int_B \omega(y)dy \leq C\Big(1+\frac{r}{\gamma(x)}\Big)^{\theta} \inf_{B \ni y} \omega(y) \tag{2.3}$$

for every ball $B = B(x, r)$.

Clearly, the classes $A_p^{\gamma,\theta}$ are increasing with $\theta$ and for $\theta = 0$, they are just the Munkenhoupt classes $A_p$. From [32], we know that the following properties for $A_p^{\gamma,\theta}$ hold.

**Lemma 2.2.** ([32]) *Let* $1 < p < \infty$, $0 < \theta < \infty$. *Then*

(*i*) $A_{p_1}^{\gamma,\theta} \subset A_{p_2}^{\gamma,\theta}$ *for* $1 \le p_1 < p_2 < \infty$;

(*ii*) $\omega \in A_p^{\gamma,\theta}$ *if and only if* $\omega^{-\frac{1}{p-1}} \in A_{p'}^{\gamma,\theta}$, *where* $1/p + 1/p' = 1$;

(*iii*) *if* $\omega \in A_p^{\gamma,\theta}$ *for* $1 \le p < \infty$, *then there exists a constant such that for any* $\kappa > 1$,

$$\omega(\kappa B(x_0, r)) \le C\left(1 + \frac{\kappa r}{\gamma(x_0)}\right)^{(l_0+1)\theta} \omega(B(x_0, r)). \tag{2.4}$$

**Lemma 2.3.** ([32]) *Let* $0 < \theta < \infty$, $1 \le p < \infty$. *If* $\omega \in A_p^{\gamma,\theta}$, *then there exist positive constant* $\delta > 1$, $\sigma$ *and* $C$ *such that*

$$\frac{\omega(E)}{\omega(B)} \le C\left(\frac{|E|}{|B|}\right)^{1/\delta}\left(1 + \frac{r}{\gamma(x_0)}\right)^{\sigma} \tag{2.5}$$

*for any measurable subset* $E$ *of a ball* $B(x_0, r)$.

**Lemma 2.4.** ([32]) *If* $\omega \in A_p^{\gamma,\infty}$, $1 \le p < \infty$, *then there exist positive constant* $\beta$, $\tau$ *and* $C$ *such that*

$$\left(\frac{1}{|B|}\int_B \omega^{1+\beta}\right)^{\frac{1}{1+\beta}} \le C\left(\frac{1}{|B|}\int_B \omega\right)\left(1 + \frac{r}{\gamma(x)}\right)^{\tau}, \tag{2.6}$$

*for every ball* $B = B(x, r)$.

**Lemma 2.5.** ([32]) *If* $\omega \in A_p^{\gamma,\infty}$, $1 \le p < \infty$, *then there exist positive constant* $\epsilon > 0$ *such that* $\omega \in A_{p-\epsilon}^{\gamma,\infty}$ *for every ball* $B = B(x, r)$.

# 3 Proof of Theorems 1.3 and 1.4

This section is devoted to the proof of Theorem 1.3 and 1.4. We first recall several auxiliary results. Let $\mathfrak{R}_\ell$ and $\mathfrak{R}_\ell^\star$ be the kernel function of $R_\ell^{\mathcal{L}}$ and $R_\ell^{\mathcal{L},\star}$, respectively. The following estimates for the kernel functions were established in [8] and will be very useful in the sequel.

**Lemma 3.1.** [8] *Let* $l = 1, \cdots, n$, $V \in RH_s$ *with* $s > n/2$. *Then:*

*(i) For every* $N \in \mathbb{N}$, *there exist* $C > 0$ *such that*

$$|\mathfrak{R}_\ell^\star(x, z)| \le C\frac{(1 + |x - z|/\gamma(x))^{-N}}{|x - z|^{n-1}}\Big(\int_{B(z, \frac{|x-z|}{4})} \frac{V(u)}{|u - z|^{n-1}} du + \frac{1}{|x - z|}\Big). \tag{3.1}$$

*Moreover, the last inequality also holds with* $\gamma(x)$ *replaced by* $\gamma(z)$.

*(ii) When* $s > n$, *the term involving* $V$ *can be dropped from inequalities* (3.1).

*Proof of Theorem 1.3.* To prove (i), without loss of generality, we may assume $\alpha < 0$ and $\omega \in A_{p/p_0'}^{\gamma,\infty}$. Pick any ball $B := B(x_0, r)$ and write $f(x) = f_0(x) + \sum_{i=1}^{\infty} f_i(x)$, where $f_0 = f\chi_{2B}$, $f_i = f\chi_{2^{i+1}B\setminus 2^i B}$ for $i \ge 1$. By the weighted $L^p$-boundedness of $\mathcal{V}_q(R_{\ell,\varepsilon}^{\mathcal{L}})(f)$ (see [9]). Hence, we have

$$\int_B |\mathcal{V}_q(R_{\ell,\varepsilon}^{\mathcal{L},\star})(f_0)(x)|^p \omega(x)dx \lesssim \int_{2B} |f(x)|^p \omega(x)dx \lesssim \left(1 + \frac{r}{\gamma(x_0)}\right)^{-\alpha} \omega(4B)^{\lambda} \|f\|_{L_{\alpha,V,\omega}^{p,\lambda}(\mathbb{R}^n)}^p. \tag{3.2}$$

From (1.3), we have

$$\begin{aligned}\mathcal{V}_q(R^{\mathcal{L},\star}_{\ell,\varepsilon})(f_i)(x) &= \Big\| \int\limits_{\varepsilon_{i+1}<|x-y|<\varepsilon_i} \mathfrak{R}^{\mathcal{L},\star}_{\ell}(x,y) f_i(y)\,dy \Big\|_{F_q} \\ &\le \int\limits_{\mathbb{R}^n} \|\chi_{\{\varepsilon_{i+1}<|x-y|<\varepsilon_i\}}(y)\|_{F_q} |\mathfrak{R}^{\mathcal{L},\star}_{\ell}(x,y)||f_i(y)|\,dy \\ &\le \int\limits_{2^{i+1}B\backslash 2^iB} |\mathfrak{R}^{\mathcal{L},\star}_{\ell}(x,y)||f(y)|\,dy. \end{aligned} \tag{3.3}$$

In the term last but one, we used that $\|\chi_{\varepsilon_{i+1}<|x-y|<\varepsilon_i}\|_{F_q} \le 1$.
Now it follows from Lemma 3.1 that

$$\begin{aligned}&\int\limits_B |\mathcal{V}_q(R^{\mathcal{L},\star}_{\ell,\varepsilon})(f_i)(x)|^p \omega(x)\,dx \\ &\lesssim \int\limits_B \Big( \int\limits_{2^{i+1}B\backslash 2^iB} |\mathfrak{R}^{\mathcal{L},\star}_{\ell}(x,y)||f(y)|\,dy \Big)^p \omega(x)\,dx \\ &\lesssim \int\limits_B \Big( \int\limits_{2^{i+1}B\backslash 2^iB} \Big(1+\frac{|x-y|}{\gamma(x)}\Big)^{-N} |x-y|^{-n}|f(y)|\,dy \Big)^p \omega(x)\,dx \\ &\quad + \int\limits_B \Big( \int\limits_{2^{i+1}B\backslash 2^iB} \Big(1+\frac{|x-y|}{\gamma(x)}\Big)^{-N} |x-y|^{1-n}|f(y)|\Big( \int\limits_{B(y,|x-y|/4)} \frac{V(z)\,dz}{|z-y|^{n-1}} \Big)dy \Big)^p \omega(x)\,dx \\ &=: \mathrm{A}_1 + \mathrm{A}_2. \end{aligned}$$

For term $\mathrm{A}_1$, using Lemma 2.1, we have

$$\begin{aligned}\mathrm{A}_1 &\lesssim \int\limits_B \Big(1+\frac{2^ir}{\gamma(x_0)}\Big)^{-Np/(l_0+1)} (2^ir)^{-np} \Big( \int\limits_{2^{i+1}B} |f(y)|\,dy \Big)^p \omega(x)\,dx \\ &\lesssim \Big(1+\frac{2^ir}{\gamma(x_0)}\Big)^{-Np/(l_0+1)} (2^ir)^{-np}\omega(B) \Big( \int\limits_{2^{i+1}B} |f(y)|\,dy \Big)^p \\ &\lesssim \Big(1+\frac{2^ir}{\gamma(x_0)}\Big)^{-Np/(l_0+1)} (2^ir)^{-np}\omega(B) \\ &\qquad \times \Big( \int\limits_{2^{i+1}B} |f(y)|^p \omega(y)\,dy \Big) \Big( \int\limits_{2^{i+1}B} \omega(y)^{-p'/p}\,dy \Big)^{p/p'} \\ &\lesssim \|f\|^p_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)} \Big(1+\frac{2^ir}{\gamma(x_0)}\Big)^{-Np/(l_0+1)-\alpha+\theta p} \\ &\qquad \times \omega(2B)\omega(2^{i+1}B)^{\lambda}\omega(2^{i+1}B)^{-1} \\ &\lesssim \|f\|^p_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)} \Big(1+\frac{r}{\gamma(x_0)}\Big)^{-\alpha} \omega(2B)^{\lambda} \\ &\qquad \times \Big(1+\frac{2^ir}{\gamma(x_0)}\Big)^{-Np/(l_0+1)-\alpha+\theta p+(l_0+1)\theta} \Big(\frac{\omega(2B)}{\omega(2^{i+1}B)}\Big)^{1-\lambda} \\ &\lesssim \|f\|^p_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)} \Big(1+\frac{r}{\gamma(x_0)}\Big)^{-\alpha} \omega(2B)^{\lambda} \Big(\frac{|2B|}{|2^{i+1}B|}\Big)^{(1-\lambda)/\delta} \\ &\qquad \times \Big(1+\frac{2^ir}{\gamma(x_0)}\Big)^{-Np/(l_0+1)-\alpha+\theta p+(l_0+1)\theta+\sigma(1-\lambda)} \\ &\lesssim \|f\|^p_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)} \Big(1+\frac{r}{\gamma(x_0)}\Big)^{-\alpha} \omega(2B)^{\lambda} 2^{-in(1-\lambda)/\delta} \end{aligned}$$

$$\times\left(1+\frac{2^i r}{\gamma(x_0)}\right)^{-Np/(l_0+1)-\alpha+\theta p+(l_0+1)\theta+\sigma(1-\lambda)}. \tag{3.4}$$

Now, we will estimate the term $A_2$. Using (2.1) and Hölder's inequality, we can write

$$\begin{aligned}
\mathrm{A}_2 &\lesssim \int_B (2^i r)^{(1-n)p}\left(1+\frac{2^i r}{\gamma(x_0)}\right)^{-Np/(l_0+1)}\Big(\int_{2^{i+1}B}|f(y)| \\
&\quad\times\Big(\int_{2^{i+1}B}\frac{V(z)dz}{|z-y|^{n-1}}\Big)dy\Big)^p\omega(x)dx \\
&\lesssim (2^i r)^{(1-n)p}\left(1+\frac{2^i r}{\gamma(x_0)}\right)^{-Np/(l_0+1)}\omega(B)\Big(\int_{2^{i+1}B}|f(y)| \\
&\quad\times\mathcal{I}_1(V\chi_{2^{i+1}B})(y)dy\Big)^p.
\end{aligned} \tag{3.5}$$

If we choose $\varrho$ such that $\frac{1}{\varrho}+\frac{1}{p_0}+\frac{1}{p}=1$. Then by Hölder's inequality, we have

$$\begin{aligned}
&\left(\int_{2^{i+1}B}|f(y)||\mathcal{I}_1(V\chi_{2^{i+2}B})(y)|dy\right)^p \\
&\le \|\mathcal{I}_1(V\chi_{2^{i+2}B})(y)\|_{L^{p_0}}^p\|f\chi_{2^{i+1}B}\|_{L^p(\omega)}^p\left(\int_{2^{i+1}B}\omega(y)^{-\varrho/p}dy\right)^{p/\varrho}.
\end{aligned} \tag{3.6}$$

Using the boundedness of the 1-th Euclidean fractional integral $\mathcal{I}: L^s\to L^{p_0}$ with $1/p_0=1/s-1/n$, we obtain that

$$\|\mathcal{I}_1(V\chi_{2^{i+2}B})\|_{L^{p_0}}^p \lesssim \|V\chi_{2^{i+2}B}\|_{L^s}^p. \tag{3.7}$$

Recall that $V\in B_s$ for some $s>1$ implies that $V$ satisfies the doubling condition,i.e., there exist constants $\mu\ge 1$ and $C$ such that,

$$\int_{tB}V(x)dx\le Ct^{n\mu}\int_B V(x)dx$$

holds for every ball $B$ and $t>1$. Therefore

$$\begin{aligned}
\|V\chi_{2^{i+2}B}\|_{L^s}^p &\lesssim (2^i r)^{np/s-np}\left(\int_{2^{i+1}B}V(x)dx\right)^p \\
&\lesssim (2^i r)^{np/s-np}\Big(\frac{2^i r}{\gamma(x_0)}\Big)^{np\mu}\gamma(x_0)^{np-2p}.
\end{aligned} \tag{3.8}$$

Since $\omega\in A^{\gamma,\theta}_{p/p_0'}$ and $p/\varrho=p/p_0'-1$, we get

$$\begin{aligned}
&\left(\int_{2^{i+1}B}\omega(y)dy\right)\left(\int_{2^{i+1}B}\omega(y)^{-\varrho/p}dy\right)^{p/\varrho} \\
&\lesssim |2^{i+1}B|^{p/p_0'}\left(1+\frac{2^i r}{\gamma(x_0)}\right)^{\theta p/p_0'}.
\end{aligned} \tag{3.9}$$

Go back to (3.10), we have

$$\mathrm{A}_2\lesssim (2^i r)^{(1-n)p+\frac{pn}{p_0'}+\frac{np}{q}-np+np-2p}\left(1+\frac{2^i r}{\gamma(x_0)}\right)^{-Np/(l_0+1)}\frac{\omega(2B)}{\omega(2^{i+1}B)}$$

$$\times \left(\frac{2^i r}{\gamma(x_0)}\right)^{np\mu+2p-np} \int_{2^{i+1}B} |f(y)|^p \omega(y)dy$$
$$\lesssim \|f\|^p_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)} \left(1+\frac{r}{\gamma(x_0)}\right)^{-\alpha} \omega(2B)^\lambda \left(\frac{\omega(2B)}{\omega(2^{i+1}B)}\right)^{1-\lambda}$$
$$\times \left(1+\frac{2^i r}{\gamma(x_0)}\right)^{-Np/(l_0+1)+\theta p/p_0'+(l_0+1)\theta\lambda+np\mu+p(2-n)+(1-\lambda)\sigma}$$
$$\lesssim \|f\|^p_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)} \left(1+\frac{r}{\gamma(x_0)}\right)^{-\alpha} \omega(2B)^\lambda 2^{-in(1-\lambda)/\delta}$$
$$\times \left(1+\frac{2^i r}{\gamma(x_0)}\right)^{-Np/(l_0+1)+\theta p/p_0'+(l_0+1)\theta\lambda+np\mu+p(2-n)+(1-\lambda)\sigma}. \tag{3.10}$$

This together with (3.4) by choosing $N$ which is big enough, we get

$$\|\mathcal{V}_q(R^{\mathcal{L}}_{\ell,\varepsilon})(f)\|_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)} \le C\|f\|_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)}.$$

For part (ii), we note that the adjoint $R^{\mathcal{L}}_{\ell,\varepsilon}$ of $R^{\mathcal{L},*}_{\ell,\varepsilon}$, when $V \in B_s$, with $n/2 \le s < n$. We write

$$\mathcal{V}_q(R^{\mathcal{L}}_{\ell,\varepsilon}f)(x) = \left\| \int_{\varepsilon_{i+1}<|x-y|<\varepsilon_i} \mathfrak{R}_\ell(x,y)f(y)dy \right\|_{F_q}$$
$$= \left\| \int_{\varepsilon_{i+1}<|x-y|<\varepsilon_i} \mathfrak{R}_\ell{}^*(y,x)f(y)dy \right\|_{F_q}.$$

By proceeding as the previous proofs (i), we can prove that $\mathcal{V}_q(R^{\mathcal{L}}_{\ell,\varepsilon})$ is bounded on $L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)$ for $1 < p \le p_0$, with $\omega^{-\frac{1}{p-1}} \in A^\gamma_{p'/p_0'}$. We omit the details.

The proof of part (iii) can be given analogously as in (i) and (ii), we leave the part to the part to the interested readers and complete the proof of therorem 1.3. □

*Proof of Theorem 1.4.* As for the case $p = 1$, by replacing (3.2) with the corresponding weak estimate. By the $\mathcal{V}_q(R^{\mathcal{L}}_{l,\varepsilon})(f)$ is bounded of weighted weak type (1,1) (see [10]), we have

$$\omega\left(\left\{y \in B : |\mathcal{V}_q(R^{\mathcal{L}}_{\ell,\varepsilon})(f_0)(y)| > \frac{\eta}{2}\right\}\right)$$
$$\lesssim \frac{1}{\eta}\|f\|_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)}\left(1+\frac{r}{\gamma(x_0)}\right)^{-\alpha}\omega(4B)^\lambda. \tag{3.11}$$

According (3.3) and (3.1), we get

$$\omega\left(\left\{y \in B : \left|\mathcal{V}_q(R^{\mathcal{L}}_{\ell,\varepsilon})\Big(\sum_{i=1}^\infty f_i\Big)(y)\right| > \frac{\eta}{2}\right\}\right)$$
$$\lesssim \frac{1}{\eta}\sum_{i=1}^\infty \int_B |\mathcal{V}_q(R^{\mathcal{L}}_{\ell,\varepsilon})(f_i)(y)|\omega(y)dy$$
$$\lesssim \frac{1}{\eta}\sum_{i=1}^\infty \int_B \left(\int_{2^{i+1}B\setminus 2^iB} |R^{\mathcal{L}}_\ell(y,z)||f(z)|dz\right)\omega(y)dy$$
$$\lesssim \frac{1}{\eta}\sum_{i=1}^\infty \int_B \left(\int_{2^{i+1}B\setminus 2^iB} \frac{|f(z)|}{(1+|y-z|/\gamma(z))^N|y-z|^n}dz\right)\omega(y)dy$$
$$+\frac{1}{\eta}\sum_{i=1}^\infty \int_B \left(\int_{2^{i+1}B\setminus 2^iB} \frac{|f(z)|dz}{(1+|y-z|/\gamma(z))^N|y-z|^{n-1}}\right.$$

$$\times\left(\int_{B(y,|y-z|/4)}\frac{V(u)du}{|u-y|^{n-1}}\right)\Bigg)\omega(y)dy$$
$$=: \mathrm{D}_1+\mathrm{D}_2. \tag{3.12}$$

Note that for $\omega^{p_0'}\in A_1^{\gamma,\infty}$, we have $\omega^{p_0'}\in A_1^{\gamma,\theta_0}$ for some $\theta_0\geq 0$. In this case it is true that $\omega\in A_1^{\gamma,\theta}$ with $\theta=\theta_0/p_0'$. Therefore, we obtain

$$\begin{aligned}
\mathrm{D}_1 &\lesssim \frac{1}{\eta}\sum_{i=1}^{\infty}\left(1+\frac{2^ir}{\gamma(x_0)}\right)^{-\frac{N}{l_0+1}}(2^ir)^{-n}\omega(B)\int_{2^{i+1}B}|f(z)|dz\\
&\lesssim \frac{1}{\eta}\sum_{i=1}^{\infty}\left(1+\frac{2^ir}{\gamma(x_0)}\right)^{-\frac{N}{l_0+1}+\theta_0/p_0'}(2^ir)^{-n}|2^{i+1}B|\\
&\quad\times\frac{\omega(B)}{\omega(2^{i+1}B)}\int_{2^{i+1}B}|f(z)|\omega(z)dz\\
&\lesssim \frac{1}{\eta}\|f\|_{L^{1,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)}\left(1+\frac{r}{\gamma(x_0)}\right)^{-\alpha}\omega(2B)^{\lambda}\\
&\quad\times\sum_{i=1}^{\infty}\left(1+\frac{2^ir}{\gamma(x_0)}\right)^{-N/l_0+1+\theta_0/p_0'-\alpha}\left(\frac{\omega(2B)}{\omega(2^{i+1}B)}\right)^{1-\lambda}\\
&\lesssim \frac{1}{\eta}\|f\|_{L^{1,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)}\left(1+\frac{r}{\gamma(x_0)}\right)^{-\alpha}\omega(2B)^{\lambda}(2^i)^{n(\lambda-1)/\delta}\\
&\quad\times\sum_{i=1}^{\infty}\left(1+\frac{2^ir}{\gamma(x_0)}\right)^{-N/l_0+1+\theta_0/p_0'-\alpha(1-\lambda)\delta}.
\end{aligned}$$

By Hölder's inequality and $\omega^{p_0'}\in A_1^{\gamma,\theta_0}$, we have

$$\begin{aligned}
&\int_B \mathcal{I}_1(V\chi_{2^{i+2}B})(y)\omega(y)dy\\
&\lesssim \|\mathcal{I}_1(V\chi_{2^{i+2}B})(y)\|_{L^{p_0}}\left(\int_B\omega(y)^{p_0'}dy\right)^{1/p_0'}\\
&\lesssim \|\mathcal{I}_1(V\chi_{2^{i+2}B})(y)\|_{L^{p_0}}|B|^{1/p_0'}\left(1+\frac{r}{\gamma(x_0)}\right)^{\theta_0/p_0'}\left(\inf_{B\ni y}\omega(y)^{p_0'}\right)^{1/p_0'}\\
&\lesssim \|\mathcal{I}_1(V\chi_{2^{i+2}B})(y)\|_{L^{p_0}}|B|^{1/p_0'}\left(1+\frac{r}{\gamma(x_0)}\right)^{\theta_0/p_0'}\inf_{B\ni y}\omega(y).
\end{aligned}$$

From (3.7) and (3.8), we obtain

$$\begin{aligned}
\mathrm{D}_2 &\lesssim \frac{1}{\eta}\sum_{i=1}^{\infty}\left(1+\frac{2^ir}{\gamma(x_0)}\right)^{-\frac{N}{l_0+1}}(2^ir)^{1-n}\int_{2^{i+1}B}|f(z)|dz\\
&\quad\times\int_B \mathcal{I}_1(V\chi_{2^{i+2}B})(y)\omega(y)dy\\
&\lesssim \frac{1}{\eta}\sum_{i=1}^{\infty}\left(1+\frac{2^ir}{\gamma(x_0)}\right)^{\frac{-N}{l_0+1}+\frac{\theta_0}{p_0'}+n\mu+2-n}(2^ir)^{\frac{n}{q}-n-1}\\
&\quad\times r^{\frac{n}{p_0'}}\int_{2^{i+1}B}|f(z)|\omega(z)dz
\end{aligned}$$

$$\begin{aligned}&\lesssim \frac{1}{\eta}\|f\|_{L^{1,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)}\left(1+\frac{r}{\gamma(x_0)}\right)^{-\alpha}\omega(2B)^{\lambda}\\&\quad\times\sum_{i=1}^{\infty}\left(1+\frac{2^i r}{\gamma(x_0)}\right)^{\frac{-N}{l_0+1}+\frac{\theta_0}{p_0'}+n\mu+2-n-\alpha}(2^i)^{-\frac{n}{p_0'}}\left(\frac{\omega(2B)}{\omega(2^{i+1}B)}\right)^{-\lambda}\\&\lesssim \frac{1}{\eta}\|f\|_{L^{1,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)}\left(1+\frac{r}{\gamma(x_0)}\right)^{-\alpha}\omega(2B)^{\lambda}\\&\quad\times\sum_{i=1}^{\infty}\left(1+\frac{2^i r}{\gamma(x_0)}\right)^{\frac{-N}{l_0+1}+\frac{\theta_0}{p_0'}+n\mu+2-n-\alpha}(2^i)^{n(\lambda/\delta-1/p_0')}\end{aligned}$$

where we choose $N$ which is big enough and we have

$$\mathrm{D}_1\lesssim \frac{1}{\eta}\|f\|_{L^{1,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)}\left(1+\frac{r}{\gamma(x_0)}\right)^{-\alpha}\omega(2B)^{\lambda},$$

and

$$\mathrm{D}_2\lesssim \frac{1}{\eta}\|f\|_{L^{1,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)}\left(1+\frac{r}{\gamma(x_0)}\right)^{-\alpha}\omega(2B)^{\lambda}.$$

These together with (3.11) imply that

$$\eta\left(1+\frac{r}{\gamma(x)}\right)^{\alpha}\omega(2B)^{-\lambda}\omega\{y\in B(x,r):|\mathcal{V}_q(R^{\mathcal{L}}_{\ell,\varepsilon}f)(y)|>\eta\}\le C\|f\|_{L^{1,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)},$$

which complete the proof of Theorem 1.4. □

## 4 Proof of Theorem 1.5

In what follows, we will prove Theorem 1.5. The following property of $BMO_\infty(\gamma)$ functions.

**Lemma 4.1.** *(cf. [12]) Let $\theta>0$ and $1\le s<\infty$. If $b\in BMO_\theta(\gamma)$, then*

$$\left(\frac{1}{|B|}\int_B |b-b_B|^s\right)^{1/s}\le C\|b\|_{BMO_\theta(\gamma)}(1+\frac{r}{\gamma(x_0)})^{\theta'} \tag{4.1}$$

*for all $B=B$, with $x_0\in\mathbb{R}^n$ and $r>0$, where $\theta'=(l_0+1)\theta$ and $l_0$ the constant appearing in* (2.1).

*Proof of Theorem 1.5.* We will only prove the results of $\mathcal{V}_q(R^{\mathcal{L},\star}_{b,\ell,\varepsilon})(f)$ in part (i). Without loss of generality, we may assume that $\alpha<0$. Pick any $x_0\in\mathbb{R}^n$ and $r>0$, and write

$$f(x)=f_0(x)+\sum_{i=1}^{\infty}f_i(x),$$

where $f_0=f\chi_{B(x_0,2r)}$, $f_i=f\chi_{2^{i+1}B)\setminus 2^iB}$ for $i\ge1$. By the weighted $L^p$-boundedness of $\mathcal{V}_q(\mathcal{R}^{\mathcal{L},\star}_{b,\ell,\varepsilon})(f)$ (see [9]), we have

$$\begin{aligned}&\int_B|\mathcal{V}_q(R^{\mathcal{L},\star}_{b,\ell,\varepsilon})(f_0)(x)|^p\omega(x)dx\\&\lesssim\|b\|^p_{BMO_\theta(\gamma)}\int_{2B}|f(x)|^p\omega(x)dx\\&\lesssim C\|b\|^p_{BMO_\theta(\gamma)}\left(1+\frac{r}{\gamma(x_0)}\right)^{-\alpha}\omega(4B)^{\lambda}\|f\|^p_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)}.\end{aligned} \tag{4.2}$$

set $b_B = |B|^{-1}\int_B b(x)dx$. For $i \geq 1$, according to (1.3), we have

$$\begin{aligned}
\mathcal{V}_q(R^{\mathcal{L},\star}_{b,\ell,\varepsilon})(f_i)(x) &= \|R^{\mathcal{L},\star}_{b,\ell,\varepsilon}(f_i)(x))\|_{F_q} \\
&= \Big\| \int\limits_{\varepsilon_{i+1}<|x-y|<\varepsilon_i} (b(x)-b(y))\mathfrak{R}^{\mathcal{L},\star}_{\ell}(x,y)f_i(y)dy \Big\|_{F_q} \\
&\leq \int\limits_{\mathbb{R}^n} \|\chi_{\{\varepsilon_{i+1}<|x-y|<\varepsilon_i\}(y)}\|_{F_q} |b(x)-b(y)||\mathfrak{R}^{\mathcal{L},\star}_{\ell}(x,y)||f_i(y)|dy \\
&\leq \int\limits_{\mathbb{R}^n} |b(x)-b(y)||\mathfrak{R}^{\mathcal{L},\star}_{\ell}(x,y)||f_i(y)|dy \\
&\leq |b(x)-b_B| \int\limits_{2^{i+1}B\setminus 2^iB} |\mathfrak{R}^{\mathcal{L},\star}_{\ell}(x,y)||f(y)|dy \\
&\quad + \int\limits_{2^{i+1}B\setminus 2^iB} |b(y)-b_B||\mathfrak{R}^{\mathcal{L},\star}_{\ell}(x,y)||f(y)|dy.
\end{aligned}$$

Applying Lemma 3.1, we can write

$$\begin{aligned}
&\int\limits_B |\mathcal{V}_q(R^{\mathcal{L},\star}_{b,\ell,\varepsilon})(f_i)(x)|^p\omega(x)dx \\
&\lesssim \int\limits_B |b(x)-b_B|^p\Big(\int\limits_{2^{i+1}B\setminus 2^iB} \Big(1+\frac{|x-y|}{\gamma(y)}\Big)^{-N} \\
&\qquad \times|x-y|^{-n}|f(y)|dy\Big)^p\omega(x)dx \\
&\quad + \int\limits_B |b(x)-b_B|^p\Big(\int\limits_{2^{i+1}B\setminus 2^iB} \Big(1+\frac{|x-y|}{\gamma(y)}\Big)^{-N} \\
&\qquad \times|x-y|^{1-n}|f(y)|\Big(\int\limits_{B(y,|x-y|/4)} \frac{V(z)dz}{|z-y|^{n-1}}\Big)dy\Big)^p\omega(x)dx \\
&\quad + \int\limits_B \Big(\int\limits_{2^{i+1}B\setminus 2^iB} |b(y)-b_B|\Big(1+\frac{|x-y|}{\gamma(y)}\Big)^{-N} \\
&\qquad \times|x-y|^{-n}|f(y)|dy\Big)^p\omega(x)dx \\
&\quad + \int\limits_B \Big(\int\limits_{2^{i+1}B\setminus 2^iB} |b(y)-b_B|\Big(1+\frac{|x-y|}{\gamma(y)}\Big)^{-N} \\
&\qquad \times|x-y|^{1-n}|f(y)|\Big(\int\limits_{B(y,|x-y|/4)} \frac{V(z)dz}{|z-y|^{n-1}}\Big)dy\Big)^p\omega(x)dx \\
&=: \mathrm{B}_1+\mathrm{B}_2+\mathrm{B}_3+\mathrm{B}_4.
\end{aligned}$$

Using Lemma 2.1, Hölder's inequality, Lemma 4.1 and Lemma 2.2-2.4, we have

$$\begin{aligned}
\mathrm{B}_1 &\leq \int\limits_B |b(x)-b_B|^p\Big(1+\frac{2^ir}{\gamma(x_0)}\Big)^{-\frac{Np}{l_0+1}}(2^ir)^{-np}\Big(\int\limits_{2^{i+1}B}|f(y)|dy\Big)^p\omega(x)dx \\
&\lesssim \|f\|^p_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)}\Big(1+\frac{2^ir}{\gamma(x_0)}\Big)^{-\alpha-\frac{Np}{l_0+1}+(l_0+1)\theta\lambda+\theta p}\omega(2^{i+1}B)^{\lambda-1}\int\limits_B|b(x)-b_B|^p\omega(x)dx \\
&\lesssim \|b\|^p_\theta\|f\|^p_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)}\omega(2B)^\lambda\Big(1+\frac{2^ir}{\gamma(x_0)}\Big)^{-\alpha-\frac{Np}{l_0+1}+(l_0+1)\theta\lambda+\theta p+\tau}\Big(\frac{\omega(2B)}{\omega(2^{i+1}B)}\Big)^{1-\lambda}
\end{aligned}$$

$$\lesssim \|b\|_{\theta}^{p}\|f\|^{p}_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)}\omega(2B)^{\lambda}\Big(1+\frac{2^i r}{\gamma(x_0)}\Big)^{-\alpha-\frac{Np}{l_0+1}+(l_0+1)\theta\lambda+\theta p+\tau+\sigma(1-\lambda)}(2^i)^{(\lambda-1)n/\delta}. \tag{4.3}$$

Similarly, we can get

$$\begin{aligned} \mathrm{B}_3 &\lesssim \int_B (2^i r)^{-np}\Big(1+\frac{2^i r}{\gamma(x_0)}\Big)^{-\frac{Np}{l_0+1}}\Bigg(\int_{2^{i+1}B}|b(y)-b_B||f(y)|dy\Bigg)^p \omega(x)dx \\ &\lesssim (2^i r)^{-np}\Big(1+\frac{2^i r}{\gamma(x_0)}\Big)^{-\frac{Np}{l_0+1}}\omega(2B)\Bigg(\int_{2^{i+1}B}|b(y)-b_B||f(y)|dy\Bigg)^p \end{aligned}$$

we choose $\varrho$ satisfies $\frac{1}{\varrho}+\frac{1}{p_0}+\frac{1}{p}=1$. According to Hölder's inequality, $\omega\in A_{p/p_0'}$ and (4.1), we have

$$\begin{aligned} &\Bigg(\int_{2^{i+1}B}|b(y)-b_B||f(y)|dy\Bigg)^p \\ &\le \Bigg(\int_{2^{i+1}B}|b(y)-b_B|^{p_0}dy\Bigg)^{p/p_0}\Bigg(\int_{2^{i+1}B}|f(y)|^p\omega(y)dy\Bigg)\Bigg(\int_{2^{i+1}B}\omega(y)^{-\varrho/p}\Bigg)^{p/\varrho} \\ &\lesssim \|b\|_{\theta}^{p}\|f\|^{p}_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)}(2^i r)^{pn}\Bigg(1+\frac{2^i r}{\gamma(x_0)}\Bigg)^{(\theta p)/p_0'-\alpha}\omega(2^{i+1}B)^{\lambda-1} \end{aligned}$$

and we obtain

$$\begin{aligned} \mathrm{B}_3 &\lesssim \|b\|_{\theta}^{p}\|f\|^{p}_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)}\omega(B)^{\lambda}\Bigg(\frac{\omega(2B)}{\omega(2^{i+1}B)}\Bigg)^{1-\lambda}\Bigg(1+\frac{2^i r}{\gamma(x_0)}\Bigg)^{-\frac{Np}{l_0+1}+(\theta p)/p_0'-\alpha} \\ &\lesssim \|b\|_{\theta}^{p}\|f\|^{p}_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)}\omega(B)^{\lambda}(2^i)^{(\lambda-1)n/\delta}\Bigg(1+\frac{2^i r}{\gamma(x_0)}\Bigg)^{-\frac{Np}{l_0+1}+(\theta p)/p_0'-\alpha+\sigma(1-\lambda)+\theta' p}. \end{aligned} \tag{4.4}$$

Imitating the estimation of term $A_2$, using Lemma 2.1, Hölder's inequality, Lemma 4.1 and Lemma 2.4 , we obtain that

$$\begin{aligned} \mathrm{B}_2 &\lesssim \int_B |b(x)-b_B|^p(2^i r)^{(1-n)p}\Big(1+\frac{2^i r}{\gamma(x_0)}\Big)^{-\frac{Np}{l_0+1}} \\ &\quad\times\Big(\int_{2^{i+1}B}|f(y)|\mathfrak{I}_1\big(V\chi_{2^{i+2}B}\big)(y)dy\Big)^p\omega(x)dx \\ &\lesssim (2^i r)^{(1-n)p}\Big(1+\frac{2^i r}{\gamma(x_0)}\Big)^{-\frac{Np}{l_0+1}+\tau}\|b\|_{\theta}^{p}\omega(B) \\ &\quad\times\Big(\int_{2^{i+1}B}|f(y)|\mathfrak{I}_1\big(V\chi_{2^{i+2}B}\big)(y)dy\Big)^p. \end{aligned}$$

According to (3.6)-(3.9), we have

$$\begin{aligned} &\Big(\int_{2^{i+1}B}|f(y)|\mathfrak{I}_1\big(V\chi_{2^{i+2}B}\big)(y)dy\Big)^p \\ &\lesssim \|f\|^{p}_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)}\Bigg(1+\frac{2^i r}{\gamma(x_0)}\Bigg)^{\theta p/p_0'+np\mu-2p-np-\alpha} \\ &\quad\times\omega(2^{i+1}B)^{\lambda-1}(2^i r)^{p(n-1)}. \end{aligned}$$

Then

$$\mathrm{B}_2 \lesssim \|b\|_{\theta}^{p}\|f\|^{p}_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)}\omega(B)^{\lambda}\Bigg(\frac{\omega(2B)}{\omega(2^{i+1}B)}\Bigg)^{1-\lambda}$$

$$\times\left(1+\frac{2^i r}{\gamma(x_0)}\right)^{-\frac{Np}{l_0+1}+\tau+\frac{\theta p}{p_0'}+np\mu-2p-np-\alpha}$$
$$\lesssim \|b\|_\theta^p \|f\|^p_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)} \omega(B)^\lambda (2^i)^{(\lambda-1)n/\delta}$$
$$\times\left(1+\frac{2^i r}{\gamma(x_0)}\right)^{-\frac{Np}{l_0+1}+\tau+\frac{\theta p}{p_0'}+np\mu-2p-np-\alpha+\sigma(1-\lambda)}.$$

Similarly, we can estimate $B_4$ as follows,

$$\mathrm{B}_4 \lesssim (2^i r)^{(1-n)p}\left(1+\frac{2^i r}{\gamma(x_0)}\right)^{-\frac{Np}{l_0+1}}\omega(B)$$
$$\times\left(\int_{2^{i+1}B} \mathcal{I}_1(V\chi_{2^{i+2}B})(y)||b(y)-b_B||f(y)|dy\right)^p dy.$$

Now, Imitating the estimation of $B_3$, but we use Hölder's inequality with $v$ and $t$ that $\frac{1}{p}+\frac{1}{p_0}+\frac{1}{v}+\frac{1}{t}=1$. Furthermore, From lemma 2.5, we need choose $v$ that $\omega\in A^{\gamma,\infty}_{1+p/t}$.

$$\left(\int_{2^{i+1}B} \mathcal{I}_1(V\chi_{2^{i+2}B})(y)||b(y)-b_B||f(y)|dy\right)^p$$
$$\lesssim \|\mathcal{I}_1(V\chi_{2^{i+2}B})(y)\|^p_{L^{p_0}}\|f\chi_{2^{i+1}B}\|^p_{L^p(\omega)}$$
$$\times\left(\int_{2^{i+1}B}|b(y)-b_B|^\mu dy\right)^{p/v}\left(\int_{2^{i+1}B}\omega(y)^{-t/p}dy\right)^{p/t}$$
$$\lesssim \|b\|_\theta^p\|f\|^p_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)}(2^i r)^{p(n-1)}$$
$$\times\left(1+\frac{2^i r}{\gamma(x_0)}\right)^{np/s-\alpha-2p+n+pn/t+pn/\mu}\omega(2^{i+1}B)^{\lambda-1}.$$

this yields that

$$\mathrm{B}_4 \lesssim \|b\|_\theta^p\|f\|^p_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)}\omega(2B)^\lambda\left(\frac{\omega(2B)}{\omega(2^{i+1}B)}\right)^{1-\lambda}$$
$$\times\left(1+\frac{2^i r}{\gamma(x_0)}\right)^{-\frac{Np}{l_0+1}+np/s-\alpha-2p+n+pn/t+pn/\mu}$$
$$\lesssim \|b\|_\theta^p\|f\|^p_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)}\omega(2B)^\lambda(2^i)^{n(\lambda-1)/\delta}$$
$$\times\left(1+\frac{2^i r}{\gamma(x_0)}\right)^{-\frac{Np}{l_0+1}+np/s-\alpha-2p+n+pn/t+pn/\mu+\sigma(1-\lambda)}. \tag{4.5}$$

Then (4.3)-(4.5) by choosing $N$ which is big enough, we can obtain that

$$\|\mathcal{V}_q(R^{\mathcal{L}}_{b,\ell,\varepsilon})(f)\|_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)}\le C\|b\|_{BMO_\theta}(\gamma)\|f\|_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)},$$

which completes the proof of Theorem 1.5. □

**Remark 4.2.** *The fluctuations of a family $\mathcal{T}=\{T_t\}_{t>0}$ of operators when $t\to 0^+$ can also be analyzed by using oscillation operators (see, for instance, [6, 33] etc.). If $\{t_j\}_{j\in\mathbb{N}}$ is a real decreasing sequence that converges to zero, the oscillation operator $\mathcal{O}(\mathcal{T}f)$ is defined by*

$$\mathcal{O}(\mathcal{T}f)(x):=\left(\sum_{i=1}^\infty \sup_{t_{i+1}\le\varepsilon_{i+1}<\varepsilon_i\le t_i}|T_{\varepsilon_{i+1}}f(x)-T_{\varepsilon_i}f(x)|^2\right)^{1/2}.$$

*Then, by using the procedures developed in this paper, we can establish the corresponding conclusions in Theorem 1.3 for $\mathcal{O}(R^{\mathcal{L}}_{\ell,\varepsilon})$ and $\mathcal{O}(R^{\mathcal{L},\star}_{\ell,\varepsilon})$, Theorem 1.5 for $\mathcal{O}(R^{\mathcal{L}}_{b,\ell,\varepsilon})$ and $\mathcal{O}(R^{\mathcal{L},\star}_{b,\ell,\varepsilon})$. The details are omitted.*

**Acknowledgement:** Supported by the National Natural Science Foundation of xinjiang Province of China (No. 2016D01C381) and the Doctoral Scientific Research Foundation of Yili Normal University(No. 2017YSBS09).

## References

[1] Akcoglu M., Jones R., Schwartz P., Variation in probability, ergodic theory and analysis, Illinois J. Math., 1998, 42(1), 154-177
[2] Crescimbeni R., Martin-Reyes F., Torre A., Torrea J., The $\rho$-variation of the Hermitian Riesz transform, Acta Math. Sin. (Engl. Ser.), 2010, 26, 1827-1838
[3] Gillespie T., Torrea J., Dimension free estimates for the oscillation of Riesz transforms, Israel J. Math., 2004, 141, 125-144
[4] Jones R., Ergodic theory and connections with analysis and probability, New York J. Math., 1997, 3A, 31-67
[5] Jones R., Variation inequalities for singular integrals and related operators, Contemp. Math., 2006, 411, 89-121
[6] Jones R., Reinhold K., Oscillation and variation inequalities for convolution powers, Ergodic Theory Dynam. Systems, 2001, 21, 1809-1829
[7] Betancor J., Fariña J., Harboure E., Rodríguez-Mesa L., $L^p$-boundedness properties of variation operators in the Schrödinger setting, Rev. Mat. Complut., 2013, 26, 485-534
[8] Shen Z., $L^p$ estimates for Schrödinger operators with certain potentials, Ann. Inst. Fourier (Grenoble),1995, 45(2), 513-546
[9] Tang L., Zhang Q., Variation operators for semigroups and Riesz transforms acting on weighted $L^p$ and BMO spaces in the Schrödinger setting, Rev Mat Complut, 2016, 29, 559-621
[10] Zhang J., Wu H., weighted estimates for variation operators in Schrödinger setting. submitted
[11] Shen Z., On the Neumann problem for Schrödinger operators in Lipschitz domains, Indiana Univ. Math. J., 1994, 43(1), 143-176
[12] Bongioanni B., Harboure E., Salinas O., Commutators of Riesz transform related to Schrödinger operators, J. Fourier Anal. Appl., 2011, 17, 115-134
[13] Pan G., Tang L., Boundedness for some Schrödingder type operators on weighted Morrey spaces, J. Funct. Spaces Art.ID 878629, 2014
[14] Morrey C., On the solutions of quasi-linear elliptic partial differential equations, Trans. Amer. Math. Soc., 1938, 43, 126-166
[15] Caffarelli L., Elliptic second order equations, Rend. Sem. Mat. Fis. Milano, 1988, 58, 253-284
[16] Chiarenza F., Frasca M., Longo P., Interior $W^{2,p}$-estimates for nondivergence elliptic equations with discontinuous coefficients, Ric. Mat., 1992, 40, 149-168
[17] Di Fazio G., Palagachev D., Ragusa M., Global Morrey regularity of strong solutions to the Dirichlet problem for elliptic equations with discontinuous coefficients, J. Funct. Anal., 1999, 166(2), 179-196
[18] Di Fazio G., Ragusa M., Interior estimates in Morrey spaces for strong solutions to nondivergence form equations with discontinuous coefficients, J. Funct. Anal., 1993, 112(2), 241-256
[19] Huang Q., Estimates on the generalized Morrey spaces $L_{\varphi}^{2,\lambda}$ and $BMO_{\psi}$ for linear elliptic systems, Indiana Univ. Math. J., 1996, 45(2), 397-439
[20] Palagachev D., Softova L., Singular integral operators, Morrey spaces and fine regularity of solutions to PDE's, Potential Anal., 2004, 20(3), 237-263
[21] Kato T., Strong solutions of the Navier-Stokes equation in Morrey spaces, Bol. Soc. Brasil. Mat., 1992, 22(2), 127-155
[22] Mazzucato A., Besov-Morrey spaces: functions space theory and applications to non-linear PDE, Trans. Amer. Math. Soc., 2003, 355(4), 1297-1364
[23] Taylor M., Analysis on Morrey spaces and applications to Navier-Stokes and other evolution equations, Comm. Partial Differential Equations., 1992, 17, 287-308
[24] Pérez C., Two weighted norm inequalities for Riesz potentials and uniform $L^p$-weighted Sobolev inequalities, Indiana Univ. Math. J., 1990, 39(1), 31-44
[25] Ruiz A., Vega L., Unique continuation for Schrödinger operators with potential in Morrey spaces, Publ. Mat., 1991, 35(1), 291-298
[26] Shen Z., The periodic Schrödinger operators with potentials in the Morrey class, J. Funct. Anal., 2002, 193(2), 314-345
[27] Shen Z., Boundary value problems in Morrey spaces for elliptic systems on Lipschitz domains, Amer. Math. J., 2003, 125(5), 1079-1115
[28] Komori Y., Shirai S., Weighted Morrey spaces and a singular integral operator, Math. Nachr., 2009, 282(2), 219-231
[29] Tang L., Dong J., Boundedness for some Schrödinger type operators on Morrey spaces related to certain nonnegative potentials, J. Math. Anal. Appl., 2009, 355, 101-109
[30] Crescimbeni R., Macías R., Menárguez T., Torrea J., Viviani B., The $q$-variation as an operator between maximal operators and singular integrals, J. Evol. Equ., 2009, 9(1), 81-102
[31] Bongioanni B., Harboure E., Salinas O., Classes of weights related to Schrödinger operators, J. Math. Anal. Appl., 2011, 373, 563-579

[32] Bongioanni B., Harboure E., Salinas O., Weighted inequalities for commutators of Schrödinger-Riesz transforms, J. Math. Anal. Appl., 2012, 392, 6-22
[33] Bougain J., Pointwise ergodic theorem for arithmatic sets, Inst. Hautes Études Sci. Publ. Math., 1989, 69, 5-45