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Xue Yu and Jiangmin Pan*

2-closures of primitive permutation groups of holomorph type

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Abstract: The 2-closure $G^{(2)}$ of a permutation group G on a finite set Ω is the largest subgroup of $\mathsf{Sym}(\Omega)$ which has the same orbits as G in the induced action on $\Omega \times \Omega$. In this paper, the 2-closures of certain primitive permutation groups of holomorph simple and holomorph compound types are determined.

Keywords: 2-closure; primitive permutation group; holomorph type

MSC: 05C25, 20B15, 20B30

1 Introduction

Let G be a permutation group on a finite set Ω. Then G has a natural induced action on $\Omega^{(2)} := \Omega \times \Omega$ as follows:

$$(\alpha, \beta)^g = (\alpha^g, \beta^g), \quad where\ \alpha, \beta \in \Omega,\ g \in G.$$

Each orbit of G of the above action is called an *orbital* of G on Ω. We denote by $Orbl(G, \Omega)$ the set of all orbitals of G on Ω.

In 1969, Wielandt [1] defined the *2-closure* of G to be the group

$$G^{(2)} := \{x \in \mathsf{Sym}(\Omega) \mid O^x = O\ for\ each\ O \in Orbl(G, \Omega)\},$$

namely the largest subgroup of $\mathsf{Sym}(\Omega)$ that has the same orbitals of G. Clearly, $G^{(2)} \geq G$. If $G = G^{(2)}$, then G is called 2-*closed*. See some basic properties of 2-closures in Wielandt [1, 2].

A transitive permutation group $G \leq \mathsf{Sym}(\Omega)$ is called *primitive* if the vertex stabilizer $G_\alpha := \{g \in G \mid \alpha^g = \alpha\}$ is maximal in G. Denote by $\mathsf{soc}(G)$ the *socle* of G, namely the product of all minimal normal subgroups of G. Then either $\mathsf{soc}(G)$ is a minimal normal subgroup of G, or $\mathsf{soc}(G) = M \times N$ with $M \cong N$ being minimal normal subgroups of G. According to the structure and the action of $\mathsf{soc}(G)$, the well-known O'Nan-Scott theorem divided the finite primitive permutation groups into eight types, refer to [3–5]. The eight types are described as follows:

HA (Holomorph Affine): $\mathsf{soc}(G) \cong \mathbb{Z}_p^l$ is regular on Ω and $X \leq \mathbb{Z}_p^l : \mathsf{GL}(l, p)$, where p is a prime and $l \geq 1$;

HS (Holomorph Simple): $\mathsf{soc}(G) = M \times N$, where $M \cong N \cong T$ are nonabelian simple normal subgroups of X and regular on Ω;

HC (Holomorph Compound): $\mathsf{soc}(G) = M \times N$, where $M \cong N \cong T^k$ are minimal normal subgroups of X and regular on Ω with T nonabelian simple and $k \geq 2$;

Xue Yu: School of Mathematics and Statistics, Yunnan University, Kunming, Yunnan 650500, P. R. China,
E-mail: yuxue1212@163.com
***Corresponding Author: Jiangmin Pan:** School of Statistics and Mathematics, Yunnan University of Finance and Economics, Kunming, Yunnan 650221, P. R. China, E-mail: jmpan@ynu.edu.cn

AS (Almost simple): $\mathsf{soc}(G) = T$ is a nonabelian simple group and not regular on Ω, and $T \lhd G \leq \mathsf{Aut}(T)$;

SD (Simple Diagonal): $\mathsf{soc}(G) \cong T^l$ is a minimal normal subgroup of G, with $l \geq 2$ and T is nonabelian simple, $(\mathsf{soc}(G))_\alpha = \{(t, \cdots, t) \mid t \in T\} \cong T$ for $\alpha \in \Omega$, and $G \leq T^l.(\mathsf{Out}(T) \times \mathsf{S}_l)$;

CD (Compound Diagonal): $\mathsf{soc}(G) \cong T^l$ with $l \geq 4$ and T nonabelian simple, and $(\mathsf{soc}(G))_\alpha \cong T^k$ with $k \geq 2$ and $k \mid l$;

TW (Twisted Wreath Product): $\mathsf{soc}(G) \cong T^l$ is regular on Ω and G_α is insoluble, where T is nonabelian simple and $l \geq 6$;

PA (Product Action): $\mathsf{soc}(G) \cong T^l$ is a minimal normal subgroup of X with T nonabelian simple and $l \geq 2$, and $G \leq \mathsf{Aut}(T) \wr \mathsf{S}_l$; moreover, there exists a set Δ such that $\Omega = \Delta^l$, $\mathsf{Aut}(T)$ acts irregularly on Δ and G acts on Δ^l as follows: for $(w_1, \dots, w_l) \in \Omega^l$, $(h_1, \dots, h_l) \in \mathsf{Aut}(T)^l$ and $\sigma \in \mathsf{S}_l$,

$$(w_1, \cdots, w_l)^{(h_1,\dots,h_l)\sigma} = (w_{1^{\sigma^{-1}}}^{h_{1^{\sigma^{-1}}}}, \dots, w_{l^{\sigma^{-1}}}^{h_{l^{\sigma^{-1}}}}).$$

In 1988, Liebeck, Praeger and Saxl [6] studied the 2-closures of simply primitive (namely primitive but not 2-transitive) permutation groups of almost simple type, and proved that if G is a such group with socle T, then either $T \trianglelefteq G^{(2)} \leq \mathsf{Aut}(T)$ or $(G, G^{(2)})$ is given in an explicit list. For more results regarding primitive groups and their 2-closures, see [7–13]. The main purpose of this paper is to determine the 2-closures of certain primitive permutation groups of HS and HC types; for convenience both types are called *holomorph type*.

The notations used in this paper are standard. For example, for a positive integer n, we denote by $\mathbb{Z}_n$ the cyclic group of order n, and by S_n the symmetric group of degree n. Given two groups N and H, denote by $N \times H$ the direct product of N and H, by $N.H$ an extension of N by H, and if such an extension is split, then we write $N : H$ instead of $N.H$. Also, we denote by G^n the direct product of the group G of n terms, and by $H \wr P$ the wreath product of groups H and P.

The main results of this paper are as follows.

Theorem 1.1. *Let $G = T : \mathsf{Aut}(T)$ be a primitive permutation group of type* HS *on Ω, where T is a nonabelian simple group. Then the following statements hold:*
(i) If there exists $t \in T$ which is not conjugate to t^{-1} in $\mathsf{Aut}(T)$, then $G^{(2)} = G$.
(ii) If each $t \in T$ is conjugate to t^{-1} in $\mathsf{Aut}(T)$, then $G^{(2)} = G.\mathbb{Z}_2$ is primitive of type SD.

Theorem 1.2. *Let $G = N : \mathsf{Aut}(N)$ be a primitive permutation group of type* HC *on Ω, where $N \cong T^d$ with T a nonabelian simple group and $d \geq 2$. Then the following statements hold:*
(i) If there exists $t \in T$ which is not conjugate to t^{-1} in $\mathsf{Aut}(T)$, then $G^{(2)} = G$.
(ii) If each $t \in T$ is conjugate to t^{-1} in $\mathsf{Aut}(T)$, then

$$G^{(2)} = [T^2.(\mathsf{Out}(T) \times \mathbb{Z}_2)]^d.\mathsf{S}_d$$

is primitive of type CD.

After this introductory section, we explain the actions of certain primitive permutation groups in Section 2, and complete the proofs of Theorems 1.1 and 1.2 in Section 3.

2 Actions of primitive permutation groups

In this section, we explain the actions of primitive permutation groups of several types that will be used in Section 3. The main contents of this section can be found in Giudici, Li and Praeger [14, Section 2] and Liebeck, Praeger and Saxl [15].

For a group G, denote by

$$\hat{G} = \{\hat{g} \mid \hat{g} : x \mapsto xg, \text{ for all } g, x \in G\},$$

$$\check{G} = \{\hat{g} \mid \hat{g}: \ x \mapsto g^{-1}x, \ \text{for all } g, x \in G\}$$

the *right regular representation* and the *left regular representation* of G, respectively. Clearly, both $\hat{G}$ and $\check{G}$ can be viewed as regular (permutation) subgroups on G, and centralize other.

Let H be a subgroup of G, and denote by $[G:H]$ the set of cosets of H in G. The *coset action* of G on $[G:H]$ is defined as follows:

$$(Hg)^x = Hgx, \quad for\ g, x \in G.$$

It is well known each transitive action of a group X on a set Ω can be identified (permutation equivalent) with the coset action of X on $[X : X_\alpha]$ with $\alpha \in \Omega$. In particular, if X is regular on Ω, then $X_\alpha = 1$, and the action of X on Ω can be identified with the right regular representation $\hat{X}$ (and so Ω can be identified with X).

In the following, we give several remarks on the actions of primitive permutation groups of several types. Let $G \leq \mathsf{Sym}(\Omega)$ be a primitive permutation group, and let $N = \mathsf{soc}(G)$.

Remark 1. Assume that G is of type HS or HC. Then $N = M_1 \times M_2$, where $M_1 \cong M_2 \cong T^d$ with $d \geq 1$ are nonabelian minimal normal subgroups of G and regular on Ω, and $G \leq N_{\mathsf{Sym}(\Omega)}(N) = N : \mathsf{Aut}(N)$. Moreover, the action of G on Ω as follows:

$$x^{n\sigma} = (xn)^\sigma = x^\sigma n^\sigma,$$

where $x \in \Omega$, $n \in N$ and $\sigma \in \mathsf{Aut}(N)$.

On the other hand, since $M_1 \cong T^d$ is regular on Ω, we may identify $\Omega = T^d$ and identify the action of M_1 on Ω with the right regular representation $\hat{T^d}$, hence $M_2 = C_G(M_1)$ can be identified with $\check{T^d}$. Consequently, we obtain

$$G \leq (\hat{T^d} \times \check{T^d}).(\mathsf{Out}(T)^d \wr \mathsf{S}_d),$$

and the action of G on $\Omega = T^d$ is as follows:

$$(t_1, t_2, \ldots, t_d)^{\widehat{(g_1, g_2, \ldots, g_d)}} = (t_1 g_1, t_2 g_2, \ldots, t_d g_d),$$

$$(t_1, t_2, \ldots, t_d)^{\widecheck{(g_1, g_2, \ldots, g_d)}} = (g_1^{-1} t_1, g_2^{-1} t_2, \ldots, g_d^{-1} t_d),$$

$$(t_1, t_2, \ldots, t_d)^{(\sigma_1, \sigma_2, \ldots, \sigma_d)} = (t_1^{\sigma_1}, t_2^{\sigma_2}, \ldots, t_d^{\sigma_d}),$$

$$(t_1, t_2, \ldots, t_d)^{\tau} = (t_{1^{\tau^{-1}}}, t_{2^{\tau^{-1}}}, \ldots, t_{d^{\tau^{-1}}}),$$

where $(t_1, t_2, \ldots, t_d), (g_1, g_2, \ldots, g_d) \in T^d$, $(\sigma_1, \sigma_2, \ldots, \sigma_d) \in \mathsf{Out}(T)^d$ and $\tau \in \mathsf{S}_d$.

Remark 2. Assume that G is of type SD and $N = \mathsf{soc}(G) = T^2$, with T a nonabelian simple group. Then $N_\alpha = \{(t, t, \ldots, t) \mid t \in T\} \cong T$ for $\alpha \in \Omega$, and

$$G \leq T^2.(\mathsf{Out}(T) \times \mathsf{S}_2) \leq \mathsf{Aut}(T^2).$$

Since there is an one to one correspondence between $[N : N_\alpha]$ to T via $N_\alpha(t_1, t_2) \to t_1^{-1} t_2$, we may identify Ω with T. Now, for $t \in \Omega$, we easily conclude that the action of G on Ω is as following (or see [15, P. 307]):

$$t^{(t_1, t_2)} = t_1^{-1} t t_2 \ for\ (t_1, t_2) \in T^2,$$

$$t^{(\alpha, \alpha)} = t^\alpha \ for\ (\alpha, \alpha) \in \mathsf{Out}(T),$$

$$t^{(12)} = t^{-1} \ for\ (12) \in \mathsf{S}_2.$$

Remark 3. Assume that G is of type HC or CD and $N = \mathsf{soc}(G) = T^d$, with T a nonabelian simple group and $d \geq 4$. Then G can be built from a primitive permutation group of type HS or SD respectively by product action, see [14, P. 296]. Hence there exists a set Δ, and a primitive permutation group H of type HS or SD on Δ, such that $\Omega = \Delta^k$ for some $k \mid d$ and $G \leq H \wr \mathsf{S}_k = H^k : \mathsf{S}_k$, and G acts on Δ^k in product action.

3 Proofs of Theorems 1.1 and 1.2

In this section, we prove Theorem 1.1 and Theorem 1.2. We first give an observation.

Lemma 3.1. *Let $G = T^d : \mathrm{Aut}(T^d)$ be a primitive permutation group of type* HC *on Ω. We may identify $\Delta = T^d$ (see Remark 1 in Section* 2*). Then the vertex stabilizer $G_1 := \{g \in G \mid 1^g = 1\} = \mathrm{Aut}(T^d)$, where* 1 *is the identity element of T^d.*

Proof. Let $g\sigma \in G$, where $g \in T^d$ and $\sigma \in \mathrm{Aut}(T^d)$. By Remark 1, we have $1^{g\sigma} = g^\sigma$. Hence $g\sigma \in G_1$ if and only $g^\sigma = 1$, or equivalently $g = 1$. The lemma follows. □

Proof of Theorem 1.1. Suppose that $G = T : \mathrm{Aut}(T)$ is a primitive permutation group of type HS on Ω, where T is a nonabelian simple group. Since $G^{(2)}$ is an overgroup of G, by Praeger [3, Proposition 8.1], either $G^{(2)} = \mathrm{A}_n$ or S_n with $n = |\Omega|$, or $\mathrm{soc}(G^{(2)}) = \mathrm{soc}(G) = T^2$ and $G^{(2)}$ is of type HS or SD.

If $G^{(2)} = \mathrm{A}_n$ or S_n, then $G^{(2)}$ is 2-transitive on Ω, hence $G^{(2)}$ and so G has exactly two orbitals, namely G is 2-transitive, which is a contradiction as a well known theorem of Burnside (see [16, Theorem 4.1B]) states that a 2-transitive permutation group is either of type HA or AS.

Therefore, $G^{(2)}$ is of type HS or of type SD and $\mathrm{soc}(G^{(2)}) = \mathrm{soc}(G) = T^2$. Consequently, we always have $G \le G^{(2)} \le G : \mathrm{S}_2$. Since T is regular on Ω, we may identify $\Omega = T$.

Assume first that there exists an element $t \in T$ which is not conjugate to t^{-1} in $\mathrm{Aut}(T)$. Let $O_1 = (1, t)^G$ and $O_2 = (1, t^{-1})^G$ be two orbitals of G on Ω. If $O_1 = O_2$, then $(1, t) = (1, t^{-1})^x = (1^x, (t^{-1})^x)$ for some $x \in G$. It follows that $t = (t^{-1})^x$ and $1^x = 1$. Thus $x \in G_1 = \mathrm{Aut}(T)$ by Lemma 3.1, and t is conjugate to t^{-1} in $\mathrm{Aut}(T)$, a contradiction.

Therefore, $O_1 \ne O_2$. For each $t \in T$ and $(12) \in \mathrm{S}_2$, we have $t^{(12)} = t^{-1}$ by Remark 2 in Section 2, it follows

$$O_1^{(12)} = (1, t)^{G(12)} = (1, t)^{(12)G} = (1, t^{-1})^G = O_2,$$

that is, $(12) \notin G^{(2)}$. Hence $G^{(2)} = G$, part (i) of Theorem 1.1 holds.

Now assume that each element $t \in T$ is conjugate to t^{-1} in $\mathrm{Aut}(T)$. By the transitivity of G on Ω, each orbital of G on Ω can be expressed as $(1, t)^G$ for some $t \in T$. By assumption, there is $\alpha \in \mathrm{Aut}(T) \le G$ such that $t^\alpha = t^{-1}$. By Remark 2, it follows that

$$((1, t)^G)^{(12)} = (1, t)^{G(12)} = (1, t)^{(12)G} = (1, t^{-1})^G = (1, t)^{\alpha G} = (1, t)^G,$$

that is, $(12) \in G^{(2)}$. Hence $G^{(2)} = G.2$, part (ii) of Theorem 1.1 holds. This completes the proof of Theorem 1.1. □

Now, we complete the proof of Theorem 1.2.

Proof of Theorem 1.2. Suppose that $G = N : \mathrm{Aut}(N)$ is a primitive permutation group of type HC on Ω, where $N = T^d$ with T a nonabelian simple group and $d \ge 2$. Since $G^{(2)}$ is an overgroup of G and $\mathrm{soc}(G) = T^{2d}$ with $2d \ge 4$, by Praeger [3, Proposition 8.1], either $G^{(2)} = \mathrm{A}_n$ or S_n with $n = |\Omega|$, or $\mathrm{soc}(G^{(2)}) = \mathrm{soc}(G)$ and $G^{(2)}$ is a primitive permutation group of type HC or CD.

If $G^{(2)} = \mathrm{A}_n$ or S_n, then G is 2-transitive on Ω, and so is G since G and $G^{(2)}$ have the same orbitals, by a theorem of Burnside (see [16, Theorem 4.1B]), G is of type HA or AS, a contradiction.

Thus $G^{(2)}$ is a primitive permutation group of type HC or CD, and $\mathrm{soc}(G^{(2)}) = \mathrm{soc}(G) = T^{2d}$. By Remark 3, it easily follows that there exists a set Δ and a primitive permutation group H of type SD on Δ with $\mathrm{soc}(H) = T^2$, such that $\Omega = \Delta^d$, and $G \le G^{(2)} \le H \wr \mathrm{S}_d = H^d : \mathrm{S}_d$, and $G^{(2)}$ acts on Δ^d in product action. Notice that $T^2 \lhd H \le T^2.(\mathrm{Out}(T) \times \mathrm{S}_2)$.

Notice that H is a primitive permutation group of type SD on Δ with socle T^2, $(\mathrm{soc}(H))_\delta = \{(t, t) \mid t \in T\} \cong T$ for each $\delta \in \Delta$, $\mathrm{soc}(H)$ has a normal subgroup T which is regular on Δ. Thus we may identify $\Delta = T$, and hence identify $\Omega = T^d$. Let O be an orbital of G, and let $\mathbf{1} = (1, 1, \ldots, 1) \in T^d = \Omega$. By the vertex transitivity of G, we may assume $O = (1, \gamma)^G$, where $\gamma = (t_1, t_2, \ldots, t_d) \in T^d = \Omega$.

Now, we prove that part (i) and part (ii) of Theorem 1.2 are true respectively.

(i) Suppose $G^{(2)} > G$. Then there is $x \in G^{(2)} \setminus G$. Since $G^{(2)} \leq H \wr \mathsf{S}_d = H^d : \mathsf{S}_d$, we may set $x = (h_1, h_2, \ldots, h_d)\tau$, where $h_i \in H_i$ with $1 \leq i \leq d$ and $\tau \in \mathsf{S}_d$. Since H is primitive of type SD with socle T^2, we may write $h_i = \xi_i(u_{i1}, u_{i2})\sigma_i$, where $(u_{i1}, u_{i2}) \in T^2$, $\sigma_i \in \mathsf{Out}(T)$ and $\xi_i \in \mathsf{S}_2$. By Remarks 2 and 3, we conclude

$$\begin{aligned}(\mathbf{1}, \gamma)^x &= (\mathbf{1}^x, \gamma^x)\\ &= (\mathbf{1}^{(h_1,h_2,\ldots,h_d)\tau}, (t_1, t_2, \ldots, t_d)^{(h_1,h_2,\ldots,h_d)\tau})\\ &= (\mathbf{1}, (t_1, t_2, \ldots, t_d))^{(\xi_1(u_{11,12})\sigma_1, \xi_2(u_{21,22})\sigma_2, \ldots, \xi_d(u_{d1,d2})\sigma_d)\tau}.\end{aligned}$$

If $\xi_i = 1$ for each $i \in \{1, 2, \ldots, d\}$, then $x \in G$, a contradiction. Hence without loss of the generality, we may assume $\xi_1 \neq 1$. Then

$$\begin{aligned}(\mathbf{1}, \gamma)^x &= (\mathbf{1}, (t_1^{\xi_1}, t_2^{\xi_2}, \ldots, t_d^{\xi_d}))^{((u_{11},u_{12})\sigma_1,(u_{21},u_{22})\sigma_2,\ldots,(u_{d1},u_{d2})\sigma_d)\tau}\\ &= (\mathbf{1}, (t_1^{-1}, t_2^{\epsilon_2}, \ldots, t_d^{\epsilon_d}))^{((u_{11},u_{12})\sigma_1,(u_{21},u_{22})\sigma_2,\ldots,(u_{d1},u_{d2})\sigma_d)\tau},\end{aligned}$$

where $\epsilon_i = 1$ if $\xi_i = 1$, and $\epsilon_i = -1$ if $\xi_i \neq 1$ with $1 \leq i \leq d$.

Let $g_0 := ((u_{11}, u_{12})\sigma_1, (u_{21}, u_{22})\sigma_2, \ldots, (u_{d1}, u_{d2})\sigma_d)\tau \in G$. Then

$$(\mathbf{1}, (t_1^{-1}, t_2^{\epsilon_2}, \ldots, t_d^{\epsilon_d})) = (\mathbf{1}, (t_1^{-1}, t_2^{\epsilon_2}, \ldots, t_d^{\epsilon_d}))^{g_0 g_0^{-1}} \in (\mathbf{1}, \gamma)^G$$

and $(\mathbf{1}, \gamma)^x \in (\mathbf{1}, \gamma)^G$. Hence

$$(\mathbf{1}, (t_1^{-1}, t_2^{\epsilon_2}, \ldots, t_d^{\epsilon_d})) = (\mathbf{1}, (t_1, t_2, \ldots, t_d))^y = (\mathbf{1}^y, (t_1, t_2, \ldots, t_d)^y)$$

for some $y \in G$. So $\mathbf{1}^y = \mathbf{1}$, and hence $y \in G_{\mathbf{1}} = \mathsf{Aut}(T)^d$ by lemma 3.1. It follows that $y = (\eta_1, \eta_2, \ldots, \eta_d)$, where $\eta_i \in \mathsf{Aut}(T)^i$ with $1 \leq i \leq d$. Thus $(t_1^{-1}, t_2^{\epsilon_2}, \ldots, t_d^{\epsilon_d}) = (t_1, t_2, \ldots, t_d)^y = (t_1^{\eta_1}, t_2^{\eta_2}, \ldots, t_d^{\eta_d})$. Hence $t_1^{-1} = t_1^{\eta_1}$. That is, t_1 is conjugate to t_1^{-1} in $\mathsf{Aut}(T)$, part (i) of Theorem 1.2 holds.

(ii) Suppose that each $t \in T$ is conjugate to t^{-1} in $\mathsf{Aut}(T)$. Note that $G^{(2)} \leq H \wr S_d \leq (T^2(\mathsf{Out}(T) \times \mathsf{S}_2)) \wr \mathsf{S}_d$, we only need to prove $T^2(\mathsf{Out}(T) \times \mathsf{S}_2)] \wr \mathsf{S}_d \leq G$. For any

$$x = (\xi_1(u_{11}, u_{12})\sigma_1, \xi_2(u_{21}, u_{22})\sigma_2, \ldots, \xi_d(u_{d1}, u_{d2})\sigma_d) \in [T^2.(\mathsf{Out}(T) \times \mathsf{S}_2)]^d,$$

where $\xi_i \in S_2$, $(u_{i1}, u_{i2}) \in T^2$ and $\sigma_i \in \mathsf{Out}(T)$ with $1 \leq i \leq d$, it is easy to see that $O^x = O$ if and only if $(\mathbf{1}, \gamma)^{gx} \in O$ for each $g \in G$. Notice that

$$\begin{aligned}gx &= g(\xi_1(u_{11}, u_{12})\sigma_1, \xi_2(u_{21}, u_{22})\sigma_2, \ldots, \xi_d(u_{d1}, u_{d2})\sigma_d)\\ &= g(\xi_1, \xi_2, \ldots, \xi_d)((u_{11}, u_{12})\sigma_1, (u_{21}, u_{22})\sigma_2, \ldots, (u_{d1}, u_{d2})\sigma_d)\\ &= g((u_{11}'', u_{12}'')\sigma_1'', (u_{21}'', u_{22}'')\sigma_2'', \ldots, (u_{d1}'', u_{d2}'')\sigma_d'')(\xi_1, \xi_2, \ldots, \xi_d)\end{aligned}$$

for some $(u_{i1}'', u_{i2}'')\sigma_i'' \in T^2.\mathsf{Out}(T)$ with $1 \leq i \leq d$. Set

$$g' := g((u_{11}'', u_{12}'')\sigma_1'', (u_{21}'', u_{22}'')\sigma_2'', \ldots, (u_{d1}'', u_{d2}'')\sigma_d'').$$

Then $g' \in G$. It follows that $gx = g'(\xi_1, \xi_2, \ldots, \xi_d)$. Note that $g' \in G = K^d.S_d = (T^2.\mathsf{Out}(T))^d.\mathsf{S}_d$, then $g = (\tau_1, \tau_2, \ldots, \tau_d)\pi$, where $\tau_i \in K$, $\pi \in P$ with $1 \leq i \leq d$. Thus

$$\begin{aligned}(\mathbf{1}, \gamma)^{gx} &= (\mathbf{1}, \gamma)^{g'(\xi_1,\xi_2,\ldots,\xi_d)}\\ &= (\mathbf{1}, (t_1, t_2, \ldots, t_d))^{g'(\xi_1,\xi_2,\ldots,\xi_d)}\\ &= (\mathbf{1}, (t_1^{\tau_1}, t_2^{\tau_2}, \ldots, t_d^{\tau_d})^{\pi})^{(\xi_1,\xi_2,\ldots,\xi_d)}\\ &= (\mathbf{1}, (t_{1^{\pi^{-1}}}^{\tau_{1^{\pi^{-1}}}}, t_{2^{\pi^{-1}}}^{\tau_{2^{\pi^{-1}}}}, \ldots, t_{d^{\pi^{-1}}}^{\tau_{d^{\pi^{-1}}}}))^{(\xi_1,\xi_2,\ldots,\xi_d)}\end{aligned}$$

$$
\begin{aligned}
&= \{\mathbf{1}, ((t_{1^{\pi^{-1}}}^{\tau_{1^{\pi^{-1}}}})^{\xi_1}, (t_{2^{\pi^{-1}}}^{\tau_{2^{\pi^{-1}}}})^{\xi_2}, \dots, (t_{d^{\pi^{-1}}}^{\tau_{d^{\pi^{-1}}}})^{\xi_d})\} \\
&= (\mathbf{1}, ((t_{1^{\pi^{-1}}}^{\xi_1})^{\tau_{1^{\pi^{-1}}}}, (t_{2^{\pi^{-1}}}^{\xi_2})^{\tau_{2^{\pi^{-1}}}}, \dots, (t_{d^{\pi^{-1}}}^{\xi_d})^{\tau_{d^{\pi^{-1}}}})) \\
&= (\mathbf{1}, ((t_{1^{\pi^{-1}}}^{\epsilon_1})^{\tau_{1^{\pi^{-1}}}}, (t_{2^{\pi^{-1}}}^{\epsilon_2})^{\tau_{2^{\pi^{-1}}}}, \dots, (t_{d^{\pi^{-1}}}^{\epsilon_d})^{\tau_{d^{\pi^{-1}}}})) \\
&= (\mathbf{1}, ((t_1^{\epsilon_1})^{\tau_1}, (t_2^{\epsilon_2})^{\tau_2}, \dots, (t_d^{\epsilon_d})^{\tau_d})^{\pi}) \\
&= (\mathbf{1}, ((t_1^{\epsilon_1})^{\tau_1}, (t_2^{\epsilon_2})^{\tau_2}, \dots, (t_d^{\epsilon_d})^{\tau_d}))^{\pi} \\
&= (\mathbf{1}, (t_1^{\epsilon_1}, t_2^{\epsilon_2}, \dots, t_d^{\epsilon_d}))^{(\tau_1, \tau_2, \dots, \tau_d)\pi},
\end{aligned}
$$

where $(t_{i^{\pi^{-1}}}^{\tau_{i^{\pi^{-1}}}})^{\xi_i} = t_{i^{\pi^{-1}}}^{\tau_{i^{\pi^{-1}}}}$ if $\xi_i = 1$, and $(t_{i^{\pi^{-1}}}^{\tau_{i^{\pi^{-1}}}})^{\xi_i} = (t_{i^{\pi^{-1}}}^{\tau_{i^{\pi^{-1}}}})^{-1} = (t_{i^{\pi^{-1}}}^{-1})^{\tau_{i^{\pi^{-1}}}}$ if $\xi_i \neq 1$. Since t is conjugate to t^{-1} in $\mathrm{Aut}(T)$ for each $t \in T$, it follows that $t_i^{\eta_i} = t_i^{\epsilon_i}$ for some $\eta_i \in \mathrm{Aut}(T)$ with $1 \le i \le d$. Then

$$
\begin{aligned}
(\mathbf{1}, \gamma)^{gx} &= [\mathbf{1}, (t_1^{\eta_1}, t_2^{\eta_2}, \dots, t_d^{\eta_d})]^{g'} \\
&= [\mathbf{1}, (t_1, t_2, \dots, t_d)]^{(\eta_1, \eta_2, \dots, \eta_d)g'} \\
&= (\mathbf{1}, \gamma)^{(\eta_1, \eta_2, \dots, \eta_d)g'}.
\end{aligned}
$$

Note that $(\eta_1, \eta_2, \dots, \eta_d)g' \in G_{\mathbf{1}} = \mathrm{Aut}(T)^d \le G$, it follows that $(\mathbf{1}, \gamma)^{gx} \in (\mathbf{1}, \gamma)^G$. Thus $x \in G$ and hence $G^{(2)} = [T^2.(\mathrm{Out}(T) \times S_2)]^d.S_d$. This completes the proof of Theorem 1.2. □

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