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Research Article

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Two weight estimates for a class of (p, q) type sublinear operators and their commutators

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Abstract: In the present paper, the authors investigate the two weight, weak-(p, q) type norm inequalities for a class of sublinear operators $\mathcal{T}_\gamma$ and their commutators $[b, \mathcal{T}_\gamma]$ on weighted Morrey and Amalgam spaces. What should be stressed is that we introduce the new BMO type space and our results generalize known results before.

Keywords: Sublinear operator, Two weight, Morrey space, Amalgam space, BMO space

MSC: 42B25, 42B35

1 Introduction

As it is well-known, Muckenhoupt [1] characterized the weights ω by means of the Hardy-Littlewood maximal operator M. He showed that M is bounded on $L^p(\omega)$ if and only if ω satisfied the so-called A_p condition: there exists a constant C such that for all cube Q,

$$\left(\frac{1}{|Q|}\int_Q \omega(x)dx\right)\left(\frac{1}{|Q|}\int_Q \omega(x)^{-p'/p}dx\right)^{p/p'} \leq C,$$

Muckenhoupt and Wheeden [2] showed that fractional integral operator I_α is bounded from $L^p(\omega)$ to $L^q(\omega)$ if and only if ω satisfied the so-called $A_{p,q}$ condition: there exists a constant C such that for all cube Q,

$$\left(\frac{1}{|Q|}\int_Q \omega(x)dx\right)\left(\frac{1}{|Q|}\int_Q \omega(x)^{-p'/q}dx\right)^{q/p'} \leq C,$$

These estimates are of interest on their own and they also have relevance to partial differential equations and quantum mechanics.

On the other hand, Saywer [3] characterized the two weight inequality. However, Saywer's condition is often difficult to verify in practice, since it involves the maximal operator. Thus it is necessary to look for other simple sufficient conditions. The first attempt was made by Neugebauer[4] in 1983. He gave a sufficient condition closely in spirit to the classical A_p condtion: if weight (u, v) satisfied the so-called " power-bump " condition:

$$\left(\frac{1}{|Q|}\int_Q v(x)^r dx\right)^{1/(rp)}\left(\frac{1}{|Q|}\int_Q u(x)^{-rp'/p}dx\right)^{1/(rp')} \leq C,$$

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for some $r > 1$. Later, in 1995, Pérez [5] improved condition by

$$\left(\frac{1}{|Q|}\int_Q v(x)^r dx\right)^{1/(rp)}\left(\frac{1}{|Q|}\int_Q u(x)^{-p'/p} dx\right)^{1/p'} \le C,$$

A long-standing problem in harmonic analysis has been to characterize the weights governing weighted norm inequalities for classical operators. The purpose of this paper is to give (p, q)-type two weight norm inequalities for class of sublinear operators and their commutators by using the pair of weights (u, v) which satisfies a Muckenhoupt condition with a " power-bump " and " Orlicz-bump " on the weight v.

Precisely, this paper is organized as follows.

In Section 2 contains some definitions. The next Section 3, we give some basic lemmas and investigate the two weight, weak-(p, q) type norm inequalities for a class of sublinear operators $\mathcal{T}_\gamma$ on weighted Morrey and Amalgam spaces. Finally, two weight norm inequality for sublinear operators high order commutator $[b, \mathcal{T}_\gamma]_m$ is considered in Section 4.

Throughout this paper all notation is standard or will be defined as needed. We have used the notation $Q(y, r)$ to denote the cube centered at y and its side length $r > 0$. Given $\lambda > 0$,a cube $Q(y, r)$,$\lambda Q(y, r)$ stands for the cube concentric with $Q(y, r)$ and having side length $\lambda\sqrt{n}$ times as long, that is $\lambda Q(y, r) := Q(y, \lambda\sqrt{n}r)$. Given a lebesgue measurable set E, χ_E will denote the characteristic function of E, $|E|$ is the Lesbesgue measure of E and weighted mrasure of E by $\omega(E)$, where $\omega(E) := \int_E \omega(x)dx$. We also denote $E^c := \mathbb{R}^n \backslash E$ the complement of E. The class A_∞ is defined as union of the classes for $1 < p < \infty$. Given a weight ω, we say that $\omega \in \triangle_2$, if there exists a constant $C > 0$ such that for any cube $Q \subset \mathbb{R}^n$, $\omega(2Q) \le C\omega(Q)$. By the way, the letter C will be used for various constants that may vary from line to line but remains independent of the main variables.

2 Some preliminaries

2.1 Sublinear operators and their commutators

In this paper, we cosider a class of linear or sublinear operator, which satisfies that given $0 \le \gamma < n$, for any $f \in L^1(\mathbb{R}^n)$ with compact support and $x \notin \operatorname{supp} f$,

$$|\mathcal{T}_\gamma f(x)| \le C \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-\gamma}} dy, \ x \in \mathbb{R}^n \tag{2.1}$$

The condition (2.1) was first introduced by Soria and Weiss in [6] ($\gamma = 0$). It is easy to see that (2.1) is satisfied by many integral operators in harmonic analysis. When $\gamma = 0$, such as the Hardy-Littewood maximal operator, Calderón-Zygmund singular integral operators, Bochner-Riesz operators at the critical index and so on. When $0 < \gamma < n$, such as the fractional maximal operator, Riesz potential operators and fractional oscillatory singular integrals and so on.

Given $0 \le \gamma < n$. Let $m \ge 1$. b is a locally integrable function on $\mathbb{R}^n$, and suppose that the m order commutator $[b, \mathcal{T}_\gamma]_m$ stands for a linear or a sublinear operator, which satisfies that for any $f \in L^1(\mathbb{R}^n)$ with compact support and $x \notin \operatorname{supp} f$,

$$|[b, \mathcal{T}_\gamma]_m f(x)| \le C \int_{\mathbb{R}^n} \frac{|b(x) - b(y)|^m |f(y)|}{|x-y|^{n-\gamma}} dy, \ x \in \mathbb{R}^n \tag{2.2}$$

Observe that $[b, \mathcal{T}_\gamma]_0 = \mathcal{T}$, $[b, \mathcal{T}_\gamma]_1 = [b, \mathcal{T}_\gamma]$ and $[b, \mathcal{T}_\gamma]_m = [b, [b, \mathcal{T}_\gamma]_{m-1}]$.

2.2 Morrey spaces

The well-know Morrey space introduced in [7] to investigate the local behavior of solutions to second order elliptic partial differential equations, and presented in various books, see [8–11].The Morrey space is a properly wider space than the lebesgue space when $0 < q < p < \infty$ ($cf.$ [12]) and this space works well with the fractional integral operators, see [13–16]. For the maximal operator in Morrey spaces we refer to [17, 18], while the Calderon-Zygmund type singular operators is known from [19–22].

Definition 2.1. Let $1 < p < \infty$, $0 \le \lambda < 1$. The classical Morrey space $L^{p,\lambda}$ to be the subset of all $L^p(\mathbb{R}^n)$ locally integrable functions f on $\mathbb{R}^n$ so that

$$\|f\|_{L^{p,\lambda}} = \sup_{Q \ni x} \left(\frac{1}{|Q|^\lambda} \int_Q |f(x)|^p dx \right)^{1/p} < \infty.$$

In particular, $L^{p,0} = L^p$ and $L^{p,1} = L^\infty$.

In 2009, Komori[19] introduced the weighted morrey spaces, and gave the definitions as follows.

Definition 2.2. Let $1 < p < \infty$, $0 \le \lambda < 1$ and u, v be two weights. The classical Morrey space $L^{p,\lambda}(u, v)$ to be the subset of all $L^p(\mathbb{R}^n)$ locally integrable functions f on $\mathbb{R}^n$ so that

$$\|f\|_{L^{p,\lambda}(u,v)} = \sup_{Q \ni x} \left(\frac{1}{v(Q)^\lambda} \int_Q |f(x)|^p u(x) dx \right)^{1/p} < \infty.$$

We are now ready for the definition of weak Morrey spaces.

Definition 2.3. Let $1 < p < \infty$, $0 \le \lambda < 1$ and ω be a weight. The weighted weak Morrey space $WL^{p,\lambda}(\omega)$ to be the subset of all $L^p(\mathbb{R}^n)$ locally integrable functions f on $\mathbb{R}^n$ so that

$$\|f\|_{WL^{p,\lambda}(\omega)} = \sup_{Q \ni x} \sup_{\delta > 0} \frac{1}{\omega(Q)^{\lambda/p}} \delta \cdot \omega(\{x \in Q : |f(x)| > \delta\})^{1/p} < \infty.$$

2.3 BMO spaces

Definition 2.4. [23] Let $q \ge 1$, the space $BMO(\mathbb{R}^n)$ to be the subset of all locally integrable functions f on $\mathbb{R}^n$ so that

$$\|f\|_{BMO^q} := \sup_{Q} \left(\frac{1}{|Q|} \int_Q |f(x) - f_Q|^q dx \right)^{1/q} < \infty,$$

where f_Q denotes the mean value of f over Q, that is $f_Q := \frac{1}{|Q|} \int_Q f(x)dx$.

Remark 2.5. By John-Nirenberg's inequality, we have $\|f\|_{BMO} = \|f\|_{BMO^q}$ for all $q \ge 1$, so we denote by BMO simple.

2.4 Amalgam spaces

Let $1 \le p, q \le \infty$, a measurable functions $f \in L^q_{loc}(\mathbb{R}^n)$ is said to be in the amalgam spaces $(L^q, L^p)(\mathbb{R}^n)$ of $L^q(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n)$ and if $\|f\chi_{Q(y,r)}\|_{L^q(\mathbb{R}^n)}$ belongs to $L^p(\mathbb{R}^n)$, where $\chi_{Q(y,r)}$ is the characteristic function of the cube $Q(y, r)$.

$$\|f\|_{(L^q,L^p)(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n} \|f\chi_{Q(y,r)}\|^p_{L^q(\mathbb{R}^n)} dy \right)^{\frac{1}{p}}$$

is a norm on $(L^q, L^p)(\mathbb{R}^n)$ under which it is a Banach space with the usual modification when $p = \infty$. These spaces were first introduced by Winer [24] in 1926 and its systematic study goes back to the work of Holland [25].

In 1989, Fofana [26, 27] considered the subspace $(L^q, L^p)^\alpha(\mathbb{R}^n)$ of $(L^q, L^p)(\mathbb{R}^n)$ in connection with the study of the continuity of the fractional maximal operator of Hardy-Littlewood and of the Fourier transformation in $\mathbb{R}^n$, which consists of measurable functions f so that for $1 \le \alpha \le \infty$, $1 \le p, q < \infty$,

$$\|f\|_{(L^q,L^p)^\alpha(\mathbb{R}^n)} := \sup_{r>0} \left(\int_{\mathbb{R}^n} \left((Q(y,r))^{1/\alpha - 1/q - 1/p} \|f\chi_{Q(y,r)}\|_{L^q(\mathbb{R}^n)} \right)^p dy \right)^{\frac{1}{p}} < \infty$$

and a usual modification version for $p = \infty$ or $q = \infty$. As it was shown in [26] that the space $(L^q, L^p)^\alpha(\mathbb{R}^n)$ is non-trivial if and only if $q \le \alpha \le p$. Thus in the remaning of this paper we will always assume that this condition $q \le \alpha \le p$ is satisfied. By the definitions, it is clear (also see [27]) that $(L^q, L^q)(\mathbb{R}^n) = L^q(\mathbb{R}^n)$, $(L^q, L^\infty)^\alpha(\mathbb{R}^n) = L^{q,(nq/\alpha)}(\mathbb{R}^n)$, where $L^{q,\lambda}(\mathbb{R}^n)$ with $1 \le q < \infty$ and $0 < \lambda < n$ is the classical Morrey space.

Recently, Wang [28] studied the weighted version of these amalgam spaces.

Definition 2.6. Let u, v, ω be three weights on $\mathbb{R}^n$ and $1 \le q \le \alpha \le p \le \infty$, the weighted amalgam spaces $(L^q, L^p)^\alpha(u, v, \omega)$ as the space of all measurable functions f so that

$$\|f\|_{(L^q,L^p)^\alpha(u,v,\omega)} := \sup_{r>0} \left(\int_{\mathbb{R}^n} \left(v(Q(y,r))^{1/\alpha - 1/q - 1/p} \|f\chi_{Q(y,r)}\|_{L^q(u)} \right)^p \omega(y)dy \right)^{\frac{1}{p}} < \infty$$

and a usual modification version for $p = \infty$ or $q = \infty$, where $L^q(u)$ is the weighted Lebesgue space.

It is easy to find that when $\lambda = 1 - q/\alpha$ and $1 \le q < \alpha < \infty$, the space $(L^q(\omega), L^\infty)^\alpha(\mathbb{R}^n)$ is the weighted Morrey space $L^{q,\lambda}(\omega)$, which first introduced by Komori [19] in 2009. Next we introduce the new BMO type space, and our main results generalize the results in [28].

Definition 2.7. Let $1 \le q \le \alpha \le p \le \infty$. The space $(BMO^q, L^p)(v, \omega)$ is defined as the set of all locally integrable functions f satisfying $\|f\|_{(BMO^q,L^p)^\alpha(v,\omega)} < \infty$, where

$$\|f\|_{(BMO^q,L^p)^\alpha(v,\omega)} := \sup_{r>0} \left(\int_{\mathbb{R}^n} \left((v(Q(y,r))^{1/\alpha - 1/q - 1/p} \|(f - f_{Q(y,r)})\chi_{Q(y,r)}\|_{L^q(v)} \right)^p \omega(y)dy \right)^{1/p}.$$

where the $f_{Q(y,r)}$ denote the mean value of f on $Q(y,r)$. It is clear that the space goes back to the classical BMO space when $\alpha = \infty$.

3 Sublinear operators

To prove our main results, we need the following Lemma.

Lemma 3.1. [19] *If $\omega \in \triangle_2$, then there exists a constant $A > 1$ such that*

$$\omega(2Q) \ge A\omega(Q).$$

Remark 3.2. If $\gamma > 0$ and $\omega \in \triangle_2$, then there exists a constant C such that

$$\begin{aligned} \sum_{j=1}^{\infty} \left(\frac{\omega(Q)}{\omega(2^{j+1}Q)} \right)^\gamma &\le \sum_{j=1}^{\infty} \left(\frac{\omega(Q)}{A^{j+1}\omega(Q)} \right)^\gamma \\ &= \sum_{j=1}^{\infty} \left(\frac{1}{A^{j+1}} \right)^\gamma \\ &\le C. \end{aligned}$$

Theorem 3.3. *Let $1 < p \le q < \infty$, $0 < \lambda < p/q$ and $\mathcal{T}_\gamma$ satisfy (2.1) with $0 \le \gamma < n$. Given a pair of weights (u, v), suppose that for some $r > 1$ and for all cubes $Q \subset \mathbb{R}^n$,*

$$|Q|^{\gamma/n + 1/q - 1/p} \left(\frac{1}{|Q|} \int_Q v(x)^r dx \right)^{1/(rq)} \left(\frac{1}{|Q|} \int_Q u(x)^{-p'/p} dx \right)^{1/p'} \le C \tag{3.1}$$

Furthermore, we also suppose that $\mathcal{T}_\gamma$ satisfies the weak-(p,q) type inequality

$$\delta \cdot v(\{x \in \mathbb{R}^n : |\mathcal{T}_\gamma f(x)| > \delta\})^{1/q} \leq C\left(\int_{\mathbb{R}^n} |f(x)|^p u(x)dx\right)^{1/p}, \ \delta > 0. \tag{3.2}$$

If $v \in \triangle_2$, then the sublinear operator $\mathcal{T}_\gamma$ is bounded from $L^{p,\lambda}(u,v)$ to $WL^{q,\lambda q/p}(v)$.

Proof of Theorem 3.3. Let $f \in L^{p,\lambda}(u,v)$ with $1 < p \leq q < \infty$ and $0 < \lambda < p/q$. Fix $Q := Q(x_0, r) \subset \mathbb{R}^n$, we split $f = f_1 + f_2$ with $f_1 = f\chi_{2Q}$, where $2Q := Q(x_0, 2\sqrt{n}r)$. Then, for any given $\delta > 0$,

$$\begin{aligned}
&\frac{1}{v(Q)^{q\lambda/p\cdot 1/q}}\delta \cdot \left(v(\{x \in Q : |\mathcal{T}_\gamma(f)| > \delta\})\right)^{1/q} \\
&\leq \frac{1}{v(Q)^{\lambda/p}}\delta \cdot \left(v(\{x \in Q : |\mathcal{T}_\gamma(f_1)| > \delta/2\})\right)^{1/q} \\
&\quad + \frac{1}{v(Q)^{\lambda/p}}\delta \cdot \left(v(\{x \in Q : |\mathcal{T}_\gamma(f_2)| > \delta/2\})\right)^{1/q} \\
&=: I + II.
\end{aligned}$$

For I, we recall that $v \in \triangle_2$. By the assumption (3.2), we have

$$\begin{aligned}
I &\leq \frac{C}{v(Q)^{\lambda/p}}\left(\int_{\mathbb{R}^n} |f_1(x)|^p u(x)dx\right)^{1/p} \\
&= \frac{C}{v(Q)^{\lambda/p}}\left(\int_{2Q} |f(x)|^p u(x)dx\right)^{1/p} \\
&\leq C\left(\frac{v(2Q)}{v(Q)}\right)^{\lambda/p} \|f\|_{L^{p,\lambda}(u,v)} \\
&\leq C\|f\|_{L^{p,\lambda}(u,v)}.
\end{aligned}$$

For the term II, observe that for $x, x_0 \in Q$ and $y \in (2Q)^c$ we have $|x-y| \approx |x_0 - y|$. Thus, from Chebyshev's inequality and Hölder's inequality, we can obtain

$$\begin{aligned}
II &\leq \frac{2}{v(Q)^{\lambda/p}}\left(\int_Q |\mathcal{T}_\gamma(f_2)(x)|^q v(x)dx\right)^{1/q} \\
&\leq \frac{2}{v(Q)^{\lambda/p}}\left(\int_Q \left|\int_{(2Q)^c} \frac{|f(y)|}{|x-y|^{n-\gamma}}dy\right|^q v(x)dx\right)^{1/q} \\
&= \frac{2}{v(Q)^{\lambda/p}}\left(\int_Q \left|\sum_{j=1}^{\infty}\int_{2^{j+1}Q\setminus 2^jQ} \frac{|f(y)|}{|x-y|^{n-\gamma}}dy\right|^q v(x)dx\right)^{1/q} \\
&\leq \frac{2}{v(Q)^{\lambda/p}}\left(\int_Q \left|\sum_{j=1}^{\infty}\int_{2^{j+1}Q\setminus 2^jQ} \frac{|f(y)|}{|x_0-y|^{n-\gamma}}dy\right|^q v(x)dx\right)^{1/q} \\
&\leq Cv(Q)^{1/q-\lambda/p}\sum_{j=1}^{\infty}\frac{1}{|2^{j+1}Q|^{1-\gamma/n}}\int_{2^{j+1}Q}|f(y)|dy \\
&\leq Cv(Q)^{1/q-\lambda/p}\sum_{j=1}^{\infty}\frac{1}{|2^{j+1}Q|^{1-\gamma/n}}\left(\int_{2^{j+1}Q}u(y)^{-p'/p}dy\right)^{1/p'}\left(\int_{2^{j+1}Q}|f(y)|^p u(y)dy\right)^{1/p} \\
&\leq C\|f\|_{L^{p,\lambda}(u,v)}v(Q)^{1/q-\lambda/p}\sum_{j=1}^{\infty}\frac{v(2^{j+1}Q)^{\lambda/p}}{|2^{j+1}Q|^{1-\gamma/n}}\left(\int_{2^{j+1}Q}u(y)^{-p'/p}dy\right)^{1/p'} \\
&= C\|f\|_{L^{p,\lambda}(u,v)}\sum_{j=1}^{\infty}\frac{v(Q)^{1/q-\lambda/p}}{v(2^{j+1}Q)^{1/q-\lambda/p}}\frac{v(2^{j+1}Q)^{1/q}}{|2^{j+1}Q|^{1-\gamma/n}}\left(\int_{2^{j+1}Q}u(y)^{-p'/p}dy\right)^{1/p'}.
\end{aligned}$$

For any positive integer j, we apply Hölder's inequality, (3.1) and Lemma 3.1 to get

$$\begin{aligned}
&\le C\|f\|_{L^{p,\lambda}(u,v)}\sum_{j=1}^{\infty}\frac{v(Q)^{1/q-\lambda/p}}{v(2^{j+1}Q)^{1/q-\lambda/p}}\\
&\quad\times\frac{|2^{j+1}Q|^{1/r'q}}{|2^{j+1}Q|^{1-\gamma/n}}\left(\int_{2^{j+1}Q}v(y)^r dy\right)^{1/(rq)}\left(\int_{2^{j+1}Q}u(y)^{-p'/p}dy\right)^{1/p'}\\
&\le C\|f\|_{L^{p,\lambda}(u,v)}\sum_{j=1}^{\infty}\frac{v(Q)^{1/q-\lambda/p}}{v(2^{j+1}Q)^{1/q-\gamma/p}}\\
&= C\|f\|_{L^{p,\lambda}(u,v)}.
\end{aligned}$$

Combining the above estimates for I and II, and then taking the supremum over all cubes $Q\subset\mathbb{R}^n$ and all $\delta>0$, we finish the proof of Theorem. □

Theorem 3.4. *Let $1<p\le q<\infty$ and $\mathcal{T}_\gamma$ satisfy (2.1) with $0\le\gamma<n$. Given a pair of weights (u,v), suppose that for some $r>1$ and for all cubes $Q\subset\mathbb{R}^n$,*

$$|Q|^{\gamma/n+1/q-1/p}\left(\frac{1}{|Q|}\int_Q v(x)^r dx\right)^{1/(rq)}\left(\frac{1}{|Q|}\int_Q u(x)^{-p'/p}dx\right)^{1/p'}\le C \tag{3.3}$$

Furthermore, we also suppose that $\mathcal{T}_\gamma$ satisfies the weak-(p,q) type inequality

$$\delta\cdot v(\{x\in\mathbb{R}^n:|\mathcal{T}_\gamma f(x)|>\delta\})^{1/q}\le C\left(\int_{\mathbb{R}^n}|f(x)|^p u(x)dx\right)^{1/p},\ \delta>0. \tag{3.4}$$

If $p\le\alpha\le\beta<s<\infty$ and $v,\omega\in\triangle_2$, then the sublinear operator $\mathcal{T}_\gamma$ is bounded from $(L^p,L^s)^\alpha(u,v,\omega)$ to $(L^{q,\infty},L^s)^\beta(v,\omega)$ with $1/\beta=1/\alpha-(1/p-1/q)$.

Proof of Theorem 3.4. Let $f\in(L^p,L^s)^\alpha(u,v,\omega)$ with $1<p\le\alpha\le\beta<s<\infty$ and $v,\omega\in\triangle_2$. Fix $Q:=Q(y,r)\subset\mathbb{R}^n$, we split $f=f_1+f_2$ with $f_1=f\chi_{2Q}$, where $2Q:=Q(y,2\sqrt{n}r)$. Then, for any $y\in\mathbb{R}^n$,

$$\begin{aligned}
&v(Q(y,r))^{1/\beta-1/q-1/s}\|\mathcal{T}_\gamma(f)\chi_{Q(y,r)}\|_{L^{q,\infty}(v)}\\
&\le v(Q(y,r))^{1/\beta-1/q-1/s}\|\mathcal{T}_\gamma(f_1)\chi_{Q(y,r)}\|_{L^{q,\infty}(v)}\\
&\quad+v(Q(y,r))^{1/\beta-1/q-1/s}\|\mathcal{T}_\gamma(f_2)\chi_{Q(y,r)}\|_{L^{q,\infty}(v)}\\
&=: I+II.
\end{aligned}$$

For I, according to assumption (3.4), we obtain

$$\begin{aligned}
I&\le v(Q(y,r))^{1/\beta-1/q-1/s}\|\mathcal{T}_\gamma(f_1)\|_{L^{q,\infty}(v)}\\
&\le Cv(Q(y,r))^{1/\beta-1/q-1/s}\left(\int_{\mathbb{R}^n}|f_1(x)|^p u(x)dx\right)^{1/p}\\
&= Cv(Q(y,r))^{1/\beta-1/q-1/s}\left(\int_{Q(y,2\sqrt{n}r)}|f(x)|^p u(x)dx\right)^{1/p}\\
&= Cv(Q(y,r))^{1/\alpha-1/p-1/s}\|f\chi_{Q(y,2\sqrt{n}r)}\|_{L^p(u)}\\
&= C\left(\frac{v(Q(y,r))}{v(Q(y,2\sqrt{n}r))}\right)^{1/\alpha-1/p-1/s}v(Q(y,2\sqrt{n}r))^{1/\alpha-1/p-1/s}\|f\chi_{Q(y,2\sqrt{n}r)}\|_{L^p(u)}\\
&\le Cv(Q(y,2\sqrt{n}r))^{1/\alpha-1/p-1/s}\|f\chi_{Q(y,2\sqrt{n}r)}\|_{L^p(u)}.
\end{aligned}$$

where we have used $1/\beta-1/q-1/s=1/\alpha-1/p-1/s$, $1/\alpha-1/p-1/s<0$ and $v\in\triangle_2$. For the term II, observe that for $x,x_0\in Q$ and $y\in(2Q)^c$ we have $|x-y|\approx|x_0-y|$. Thus, by Chebyshev's inequality and Hölder's inequality yields

$$II\le 2v(Q(y,r))^{1/\beta-1/q-1/s}\left(\int_{Q(y,r)}|T_\gamma(f_2)(x)|^q v(x)dx\right)^{1/q}$$

$$
\begin{aligned}
&\leq 2v(Q(y,r))^{1/\beta-1/q-1/s}\left(\int_{Q(y,r)}\Big|\int_{(Q(y,2\sqrt{n}r))^c}\frac{|f(y)|}{|x-y|^{n-\alpha}}dy\Big|^q v(x)dx\right)^{1/q}\\
&= 2v(Q(y,r))^{1/\beta-1/q-1/s}\left(\int_{Q(y,r)}\Big|\sum_{j=1}^{\infty}\int_{Q(y,2^{j+1}\sqrt{n}r)\setminus Q(y,2^j\sqrt{n}r)}\frac{|f(y)|}{|x-y|^{n-\gamma}}dy\Big|^q v(x)dx\right)^{1/q}\\
&\leq 2v(Q(y,r))^{1/\beta-1/q-1/s}\left(\int_{Q(y,r)}\Big|\sum_{j=1}^{\infty}\int_{Q(y,2^{j+1}\sqrt{n}r)\setminus Q(y,2^j\sqrt{n}r)}\frac{|f(y)|}{|x_0-y|^{n-\gamma}}dy\Big|^q v(x)dx\right)^{1/q}\\
&\leq Cv(Q(y,r))^{1/\beta-1/s}\sum_{j=1}^{\infty}\frac{1}{|Q(y,2^{j+1}\sqrt{n}r)|^{1-\gamma/n}}\int_{Q(y,2^{j+1}\sqrt{n}r)}|f(z)|dz\\
&\leq Cv(Q(y,r))^{1/\beta-1/s}\sum_{j=1}^{\infty}\frac{1}{|Q(y,2^{j+1}\sqrt{n}r)|^{1-\gamma/n}}\left(\int_{Q(y,2^{j+1}\sqrt{n}r)}|f(z)|^p u(z)dz\right)^{1/p}\\
&\quad\times\left(\int_{Q(y,2^{j+1}\sqrt{n}r)}u(z)^{-p'/p}dz\right)^{1/p'}\\
&\leq C\sum_{j=1}^{\infty}\left(\frac{v(Q(y,r))}{v(Q(y,2^{j+1}\sqrt{n}r))}\right)^{1/\beta-1/s}v(Q(y,2^{j+1}\sqrt{n}r))^{1/\beta-1/q-1/s}\|f\chi_{Q(y,2^{j+1}\sqrt{n}r)}\|_{L^p(u)}\\
&\quad\times\frac{v(Q(y,2^{j+1}\sqrt{n}r))^{1/q}}{|Q(y,2^{j+1}\sqrt{n}r)|^{1-\gamma/n}}\left(\int_{Q(y,2^{j+1}\sqrt{n}r)}u(z)^{-p'/p}dz\right)^{1/p'}.
\end{aligned}
$$

A further application of Hölder's inequality, (3.3) and Lemma 3.1, we have

$$
\begin{aligned}
&\leq C\sum_{j=1}^{\infty}\left(\frac{v(Q(y,r))}{v(Q(y,2^{j+1}\sqrt{n}r))}\right)^{1/\beta-1/s}v(Q(y,2^{j+1}\sqrt{n}r))^{1/\beta-1/q-1/s}\|f\chi_{Q(y,2^{j+1}\sqrt{n}r)}\|_{L^p(u)}\\
&\quad\times\frac{|Q(y,2^{j+1}\sqrt{n}r)|^{1/r'q}}{|Q(y,2^{j+1}\sqrt{n}r)|^{1-\gamma/n}}\left(\int_{Q(y,2^{j+1}\sqrt{n}r)}v(z)^r dz\right)^{1/(rq)}\left(\int_{Q(y,2^{j+1}\sqrt{n}r)}u(z)^{-p'/p}dz\right)^{1/p'}\\
&\leq C\sum_{j=1}^{\infty}\left(\frac{v(Q(y,r))}{v(Q(y,2^{j+1}\sqrt{n}r))}\right)^{1/\beta-1/s}v(Q(y,2^{j+1}\sqrt{n}r))^{1/\alpha-1/p-1/s}\|f\chi_{Q(y,2^{j+1}\sqrt{n}r)}\|_{L^p(u)}\\
&\leq C\sum_{j=1}^{\infty}v(Q(y,2^{j+1}\sqrt{n}r))^{1/\alpha-1/p-1/s}\|f\chi_{Q(y,2^{j+1}\sqrt{n}r)}\|_{L^p(u)}.
\end{aligned}
$$

which is desired inequality. Combining the above estimates for I and II, and taking the $L^s(\omega)$-norm of both sides of with respect to the variable y, we finish the proof of Theorem. □

Theorem 3.5. *Let $1 < p \leq q < \infty$ and $\mathcal{T}_\gamma$ satisfy (2.1) with $0 \leq \gamma < n$. Given a pair of weights (u, v), suppose that for some $r > 1$ and for all cubes $Q \subset \mathbb{R}^n$,*

$$
|Q|^{\gamma/n+1/q-1/p}\left(\frac{1}{|Q|}\int_Q v(x)^r dx\right)^{1/(rq)}\left(\frac{1}{|Q|}\int_Q u(x)^{-p'/p}dx\right)^{1/p'}\leq C \tag{3.5}
$$

Furthermore, if $\lambda = p/q$, then the sublinear operator $\mathcal{T}_\gamma$ is bounded from $L^{p,\lambda}(u, v)$ to BMO.

Proof of Theorem 3.5. Let $f \in L^{p,\lambda}(u, v)$ with $1 < p \leq q < \infty$ and $\lambda = p/q$. Fix $Q := Q(x_0, r) \subset \mathbb{R}^n$, we split $f = f_1 + f_2$ with $f_1 = f\chi_{4Q}$, where $4Q := Q(x_0, 4\sqrt{n}r)$. Then,

$$
\begin{aligned}
&\frac{1}{|Q|}\int_Q|\mathcal{T}_\gamma f(x)-(\mathcal{T}_\gamma f)_Q|dx\\
&\leq\frac{1}{|Q|}\int_Q|\mathcal{T}_\gamma f_1(x)-(\mathcal{T}_\gamma f_1)_Q|dx+\frac{1}{|Q|}\int_Q|\mathcal{T}_\gamma f_2(x)-(\mathcal{T}_\gamma f_2)_Q|dx
\end{aligned}
$$

$$=: I + II.$$

For I, it follows directly from Fubini's theorem that

$$\begin{aligned} I &\le \frac{2}{|Q|}\int_Q |\mathcal{T}_\gamma f_1(x)|dx \\ &\le \frac{C}{|Q|}\int_Q\int_{4Q}\frac{1}{|x-y|^{n-\gamma}}|f(y)|dydx \\ &= \frac{C}{|Q|}\int_{4Q}\int_Q\frac{1}{|x-y|^{n-\gamma}}dx|f(y)|dy. \end{aligned}$$

By simple geometric observation, we have for any $x \in Q$ and $y \in 4Q$,

$$|x-y| \le |x-x_0| + |x_0-y| \le 3nr.$$

Using the transform $x - y \mapsto z$ and polar coordinates, we have

$$\begin{aligned} \int_Q \frac{1}{|x-y|^{n-\gamma}}dx &\le \int_{|z|\le 3nr}\frac{1}{|z|^{n-\gamma}}dz \\ &= \omega_{n-1}\int_0^{3nr}\frac{1}{\rho^{n-\gamma}}\rho^{n-1}d\rho \\ &= \omega_{n-1}\frac{1}{\gamma}(3nr)^{\gamma}. \end{aligned} \tag{3.6}$$

where ω_{n-1} denote the measure of the unit sphere. Therefore,

$$I \le \frac{C}{|Q|^{1-\gamma/n}}\int_{4Q}|f(y)|dy.$$

Notice that $\lambda/p = 1/q$. It follows from Hölder's inequality and (3.5) that

$$\begin{aligned} I &\le \frac{C}{|Q|^{1-\gamma/n}}\left(\int_{4Q}|f(y)|^p u(y)dy\right)^{1/p}\left(\int_{4Q}u(y)^{-p'/p}dy\right)^{1/p'} \\ &\le C\|f\|_{L^{p,\lambda}(u,v)}\frac{v(4Q)^{\lambda/p}}{|Q|^{1-\gamma/n}}\left(\int_{4Q}u(y)^{-p'/p}dy\right)^{1/p'} \\ &\le C\|f\|_{L^{p,\lambda}(u,v)}\frac{|4Q|^{1/(r'q)}}{|4Q|^{1-\gamma/n}}\left(\int_{4Q}v(y)^r dy\right)^{1/(rq)}\left(\int_{4Q}u(y)^{-p'/p}dy\right)^{1/p'} \\ &\le C\|f\|_{L^{p,\lambda}(u,v)}. \end{aligned}$$

For the term II, observe that for $x, y \in Q$ and $z \in (4Q)^c$, we have $|x-z| \ge 2|x-y|$ and $|x-z| \approx |z-x_0|$. Thus, we have that for any $x \in Q$,

$$\begin{aligned} |\mathcal{T}_\gamma f_2(x) - (\mathcal{T}_\gamma f_2)_Q| &= \left|\frac{1}{|Q|}\int_Q(\mathcal{T}_\gamma f_2(x) - \mathcal{T}_\gamma f_2(y))dy\right| \\ &\le \frac{1}{|Q|}\int_Q\int_{(4Q)^c}\left|\frac{1}{|x-z|^{n-\gamma}} - \frac{1}{|y-z|^{n-\gamma}}\right||f(z)|dzdy \\ &\le \frac{C}{|Q|}\int_Q\int_{(4Q)^c}\frac{|x-y|}{|x-z|^{n-\gamma+1}}|f(z)|dzdy \\ &\le \frac{C}{|Q|}\int_Q\int_{(4Q)^c}\frac{r}{|x_0-z|^{n-\gamma+1}}|f(z)|dzdy \\ &\le C\int_{(4Q)^c}\frac{r}{|x_0-z|^{n-\gamma+1}}|f(z)|dz \\ &\le C\sum_{j=2}^{\infty}\frac{1}{2^j}\frac{1}{|2^{j+1}Q|^{1-\gamma/n}}\int_{2^{j+1}Q}|f(z)|dz. \end{aligned}$$

Futher, notice that $\lambda/p = 1/q$, by Hölder's inequality and (3.5),

$$\begin{aligned}
&\le C\sum_{j=1}^{\infty}\frac{1}{2^j}\frac{1}{|2^{j+1}Q|^{1-\gamma/n}}\left(\int_{2^{j+1}Q}u(z)^{-p'/p}dz\right)^{1/p'}\left(\int_{2^{j+1}Q}|f(y)|^p u(z)dz\right)^{1/p}\\
&\le C\|f\|_{L^{p,\lambda}(u,v)}\sum_{j=1}^{\infty}\frac{1}{2^j}\frac{v(2^{j+1}Q)^{\lambda/p}}{|2^{j+1}Q|^{1-\gamma/n}}\left(\int_{2^{j+1}Q}u(z)^{-p'/p}dz\right)^{1/p'}\\
&= C\|f\|_{L^{p,\lambda}(u,v)}\sum_{j=1}^{\infty}\frac{1}{2^j}\frac{v(2^{j+1}Q)^{1/q}}{|2^{j+1}Q|^{1-\gamma/n}}\left(\int_{2^{j+1}Q}u(z)^{-p'/p}dz\right)^{1/p'}\\
&\le C\|f\|_{L^{p,\lambda}(u,v)}\sum_{j=1}^{\infty}\frac{1}{2^j}\frac{|2^{j+1}Q|^{1/(r'q)}}{|2^{j+1}Q|^{1-\gamma/n}}\left(\int_{2^{j+1}Q}v(z)^r dz\right)^{1/(rq)}\left(\int_{2^{j+1}Q}u(z)^{-p'/p}dz\right)^{1/p'}\\
&\le C\|f\|_{L^{p,\lambda}(u,v)}.
\end{aligned}$$

Therefore,

$$II \le C\|f\|_{L^{p,\lambda}(u,v)}.$$

Combining the above estimates for I and II, and then taking the supremum over all cubes $Q \subset \mathbb{R}^n$, we finish the proof of Theorem. □

Theorem 3.6. *Let* $1 < p \le q < \infty$ *and* $\mathcal{T}_\gamma$ *satisfy (2.1) with* $0 \le \gamma < n$. *Given a pair of weights* (u, v), *suppose that for some* $r > 1$ *and for all cubes* $Q \subset \mathbb{R}^n$,

$$|Q|^{\gamma/n+1/q-1/p}\left(\frac{1}{|Q|}\int_Q v(x)^r dx\right)^{1/(rq)}\left(\frac{1}{|Q|}\int_Q u(x)^{-p'/p}dx\right)^{1/p'} \le C \tag{3.7}$$

Furthermore, we also suppose that $\mathcal{T}_\gamma$ *satisfies the* (p, q) *type inequality*

$$\left(\int_{\mathbb{R}^n}|\mathcal{T}_\gamma f(x)|^q v(x)dx\right)^{1/q} \le C\left(\int_{\mathbb{R}^n}|f(x)|^p u(x)dx\right)^{1/p}. \tag{3.8}$$

If $p \le \alpha \le \beta < s < \infty$, $1/s = 1/\alpha - (1/p - 1/q)$ *and* $v, \omega \in \triangle_2$, *then the sublinear operator* $\mathcal{T}_\gamma$ *is bounded from* $(L^p, L^s)^\alpha(u, v, \omega)$ *to* $(BMO^q, L^s)^\beta(v, \omega)$ *with* $1/\beta = 1/\alpha - (1/p - 1/q)$.

Proof of Theorem 3.6. Let $f \in (L^p, L^s)^\alpha(u, v, \omega)$ with $1 < p \le \alpha \le \beta < s < \infty$ and $v, \omega \in \triangle_2$. Fix $Q := Q(y, r) \subset \mathbb{R}^n$, we split $f = f_1 + f_2$ with $f_1 = f\chi_{4Q}$, where $4Q := Q(y, 4\sqrt{n}r)$. Then, for any given $y \in \mathbb{R}^n$ and $r > 0$,

$$\begin{aligned}
&v(Q(y,r))^{1/\beta-1/q-1/s}\|(\mathcal{T}_\gamma f(x) - (\mathcal{T}_\gamma f)_{Q(y,r)})\chi_{Q(y,r)}\|_{L^q(v)}\\
&\le v(Q(y,r))^{1/\beta-1/q-1/s}\|(\mathcal{T}_\gamma f_1(x) - (\mathcal{T}_\gamma f_1)_{Q(y,r)})\chi_{Q(y,r)}\|_{L^q(v)}\\
&\quad + v(Q(y,r))^{1/\beta-1/q-1/s}\|(\mathcal{T}_\gamma f_2(x) - (\mathcal{T}_\gamma f_2)_{Q(y,r)})\chi_{Q(y,r)}\|_{L^q(v)}\\
&=: I + II.
\end{aligned}$$

For I, according to assumption (3.8), we obtain

$$\begin{aligned}
I &\le 2v(Q(y,r))^{1/\beta-1/q-1/s}\left(\int_{Q(y,r)}|\mathcal{T}_\gamma f_1(x)|^q v(x)dx\right)^{1/q}\\
&\le Cv(Q(y,r))^{1/\beta-1/q-1/s}\left(\int_{\mathbb{R}^n}|f_1(x)|^p u(x)dx\right)^{1/p}\\
&= Cv(Q(y,r))^{1/\beta-1/q-1/s}\left(\int_{Q(y,4\sqrt{n}r)}|f(x)|^p u(x)dx\right)^{1/p}\\
&= Cv(Q(y,r))^{1/\alpha-1/p-1/s}\|f\chi_{Q(y,4\sqrt{n}r)}\|_{L^p(u)}\\
&= C\left(\frac{v(Q(y,r))}{v(Q(y,4\sqrt{n}r))}\right)^{1/\alpha-1/p-1/s} v(Q(y,4\sqrt{n}r))^{1/\alpha-1/p-1/s}\|f\chi_{Q(y,4\sqrt{n}r)}\|_{L^p(u)}
\end{aligned}$$

$$\le Cv(Q(y, 4\sqrt{n}r))^{1/\alpha-1/p-1/s}\|f\chi_{Q(y,4\sqrt{n}r)}\|_{L^p(u)}.$$

where we have used $1/\beta - 1/q - 1/s = 1/\alpha - 1/p - 1/s$, $1/\alpha - 1/p - 1/s < 0$ and $v \in \triangle_2$. For the term II, observe that for $x, z \in Q(y, r)$ and $\eta \in Q(y, 4\sqrt{n}r)^c$, we have $|x-\eta| \ge 2|x-z|$ and $|x-\eta| \approx |\eta - y|$. Thus, we have that for any $x \in Q(y, r)$,

$$\begin{aligned}
|\mathcal{T}_\gamma f_2(x) - (\mathcal{T}_\gamma f_2)_{Q(y,r)}| &= \left|\frac{1}{|Q(y,r)|}\int_{Q(y,r)}(\mathcal{T}_\gamma f_2(x) - \mathcal{T}_\gamma f_2(z))dz\right| \\
&\le \frac{C}{|Q(y,r)|}\int_{Q(y,r)}\int_{(Q(y,4\sqrt{n}r))^c}\left|\frac{1}{|x-\eta|^{n-\gamma}} - \frac{1}{|z-\eta|^{n-\gamma}}\right||f(\eta)|d\eta dz \\
&\le \frac{C}{|Q(y,r)|}\int_{Q(y,r)}\int_{(Q(y,4\sqrt{n}r))^c}\frac{|x-z|}{|x-\eta|^{n-\gamma+1}}|f(\eta)|d\eta dz \\
&\le \frac{C}{|Q(y,r)|}\int_{Q(y,r)}\int_{(Q(y,4\sqrt{n}r))^c}\frac{|x-z|}{|y-\eta|^{n-\gamma+1}}|f(\eta)|d\eta dz \\
&= C\int_{(Q(y,4\sqrt{n}r))^c}\frac{r}{|y-\eta|^{n-\gamma+1}}|f(\eta)|d\eta \\
&\le C\sum_{j=2}^{\infty}\frac{1}{2^j}\frac{1}{|Q(y,2^{j+1}\sqrt{n}r)|^{1-\gamma/n}}\int_{(Q(y,2^{j+1}\sqrt{n}r))}|f(\eta)|d\eta.
\end{aligned}$$

Therefore,

$$II \le Cv(Q(y,r))^{1/\beta-1/s}\sum_{j=2}^{\infty}\frac{1}{2^j}\frac{1}{|Q(y,2^{j+1}\sqrt{n}r)|^{1-\gamma/n}}\int_{(Q(y,2^{j+1}\sqrt{n}r))}|f(\eta)|d\eta.$$

A further application of Hölder's inequality, (3.7) and Lemma 3.1, we have

$$\begin{aligned}
&\le Cv(Q(y,r))^{1/\beta-1/s}\sum_{j=2}^{\infty}\frac{1}{2^j}\frac{1}{|Q(y,2^{j+1}\sqrt{n}r)|^{1-\gamma/n}}\left(\int_{Q(y,2^{j+1}\sqrt{n}r)}|f(\eta)|^p u(\eta)d\eta\right)^{1/p} \\
&\quad\times\left(\int_{Q(y,2^{j+1}\sqrt{n}r)}u(\eta)^{-p'/p}d\eta\right)^{1/p'} \\
&\le C\sum_{j=2}^{\infty}\frac{1}{2^j}\left(\frac{v(Q(y,r))}{v(Q(y,2^{j+1}\sqrt{n}r))}\right)^{1/\beta-1/s}v(Q(y,2^{j+1}\sqrt{n}r))^{1/\beta-1/q-1/s}\|f\chi_{Q(y,2^{j+1}\sqrt{n}r)}\|_{L^p(u)} \\
&\quad\times\frac{v(Q(y,2^{j+1}\sqrt{n}r))^{1/q}}{|Q(y,2^{j+1}\sqrt{n}r)|^{1-\gamma/n}}\left(\int_{Q(y,2^{j+1}\sqrt{n}r)}u(\eta)^{-p'/p}d\eta\right)^{1/p'} \\
&\le C\sum_{j=2}^{\infty}\frac{1}{2^j}\left(\frac{v(Q(y,r))}{v(Q(y,2^{j+1}\sqrt{n}r))}\right)^{1/\beta-1/s}v(Q(y,2^{j+1}\sqrt{n}r))^{1/\beta-1/q-1/s}\|f\chi_{Q(y,2^{j+1}\sqrt{n}r)}\|_{L^p(u)} \\
&\quad\times\frac{|Q(y,2^{j+1}\sqrt{n}r)|^{1/r'q}}{|Q(y,2^{j+1}\sqrt{n}r)|^{1-\gamma/n}}\left(\int_{Q(y,2^{j+1}\sqrt{n}r)}v(\eta)^r d\eta\right)^{1/(rq)}\left(\int_{Q(y,2^{j+1}\sqrt{n}r)}u(\eta)^{-p'/p}d\eta\right)^{1/p'} \\
&\le C\sum_{j=2}^{\infty}\frac{1}{2^j}\left(\frac{v(Q(y,r))}{v(Q(y,2^{j+1}\sqrt{n}r))}\right)^{1/\beta-1/s}v(Q(y,2^{j+1}\sqrt{n}r))^{1/\alpha-1/p-1/s}\|f\chi_{Q(y,2^{j+1}\sqrt{n}r)}\|_{L^p(u)} \\
&\le C\sum_{j=2}^{\infty}\frac{1}{2^j}v(Q(y,2^{j+1}\sqrt{n}r))^{1/\alpha-1/p-1/s}\|f\chi_{Q(y,2^{j+1}\sqrt{n}r)}\|_{L^p(u)}.
\end{aligned}$$

which is desired inequality. Combining the above estimates for I and II, and taking the $L^s(\omega)$-norm of both sides of with respect to the variable y, we finish the proof of Theorem. □

4 High order commutators

4.1 Orlicz spaces

Since commutators have a greater degree of " singularity ", we need a slightly stronger condition. Roughly, we need to " bump " the right hand term as well, but it suffices to do so in the scale of Orlicz spaces, so it is called as " Orlicz bump ".

Next, we recall some basic facts about Orlicz spaces. Let Φ be a Young function, that is to say, $\Phi : [0,+\infty) \to [0,+\infty)$ is a continuous, convex and increasing function and satisfies $\Phi(0) = 0$ and $\Phi(t) \to +\infty$ as $t \to +\infty$. Given a Young function Φ, let E be a measurable set with $|E| < \infty$, define the Luxemburg norm of f over E as

$$\|f\|_{\Phi,E} := inf \left\{ \lambda > 0 : \frac{1}{|E|} \int_E \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \leq 1 \right\}.$$

In particular, when $\Phi = t^p$, $1 < p < \infty$, it is easy to check that

$$\|f\|_{\Phi,Q} = \left(\frac{1}{|Q|}\int_Q |f(x)|^p dx\right)^{1/p}.$$

that is, the Luxemburg norm coincides with the normalized L^p norm. For further details, we refer the reader to [29].

4.2 Boundedness of sublinear operator commutators

To prove our theorem we need the following Lemmas.

Lemma 4.1. [30] *Let b be a function in $BMO(\mathbb{R}^n)$. Then*

(i) For every cube $Q \subset \mathbb{R}^n$ and for any positive integer j, then

$$|b_{2^{j+1}Q} - b_Q| \leq C(j+1)\|b\|_{BMO}.$$

(ii) Let $1 < p < \infty$. For every cube $Q \subset \mathbb{R}^n$ and for any $\omega \in A_\infty$. Then

$$\left(\int_Q |b(x) - b_Q|^p \omega(x) dx\right)^{1/p} \leq C\omega(Q)^{1/p}\|b\|_{BMO}.$$

Lemma 4.2. [29] *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be Young functions such that for all $t > 0$,*

$$\mathcal{A}^{-1}(t)\mathcal{B}^{-1}(t) \leq \mathcal{C}^{-1}(t),$$

where $\mathcal{A}^{-1}(t)$ is the inverse function of $\mathcal{A}(t)$. Then for all functions f and g and all cubes $Q \subset \mathbb{R}^n$, the generalized Hölder's inequality

$$\|fg\|_{\mathcal{C},Q} \leq 2\|f\|_{\mathcal{A},Q}\|g\|_{\mathcal{B},Q}.$$

Lemma 4.3. [31] *If $\omega \in A_\infty$, then there exist $\delta > 0$, $C > 0$ such that a measurable set E contained in a cube Q, the following inequality holds:*

$$\frac{\omega(E)}{\omega(Q)} \leq C\left(\frac{|E|}{|Q|}\right)^\delta.$$

Now let us state our main results.

Theorem 4.4. *Let $m \geq 1$, $1 < p \leq q < \infty$, $0 < \lambda < p/q$, $b \in BMO$ and $[b, \mathcal{T}_\gamma]_m$ satisfy (2.1) with $0 \leq \gamma < n$. Given a pair of weights (u, v), suppose that for some $r > 1$ and for all cubes Q in $\mathbb{R}^n$,*

$$|Q|^{\gamma/n+1/q-1/p}\left(\frac{1}{|Q|}\int_Q v(x)^r dx\right)^{1/rq}\|u^{-1/p}\|_{\Phi_m,Q} \leq C, \tag{4.1}$$

where $\Phi_1(t) = t^{p'}\left(1 + log(e + t)\right)^{p'}$, $\Phi_m(t) = t^{p'}\left(1 + log^+ t\right)^{mp'}$ $(m = 2, 3, \dots)$.

Furthermore, we also suppose that $[b, \mathcal{T}_\gamma]_m$ satisfies the weak-(p, q) type inequality

$$\delta \cdot v(\{x \in \mathbb{R}^n : |[b, \mathcal{T}_\alpha]_m(f)(x)| > \delta\})^{1/q} \leq C\left(\int_{\mathbb{R}^n}|f(x)|^p u(x)dx\right)^{1/p}, \ \delta > 0. \tag{4.2}$$

If $v \in A_\infty$, then the sublinear operator higher order commutators $[b, \mathcal{T}_\gamma]_m$ is bounded from $L^{p,\lambda}(u, v)$ to $WL^{q,\lambda q/p}(v)$.

Proof of Theorem 4.4. Because the method is similar, we only need to prove the case of $m = 1$. Let $f \in L^{p,\lambda}(u, v)$ with $1 < p \leq q < \infty$ and $0 < \lambda < p/q$. Fix $Q := Q(x_0, r) \subset \mathbb{R}^n$, we split $f = f_1 + f_2$ with $f_1 = f\chi_{2Q}$, where $2Q := Q(x_0, 2\sqrt{n}r)$. Then, for any given $\delta > 0$,

$$\begin{aligned}
&\frac{1}{v(Q)^{q\lambda/p\cdot 1/q}}\delta \cdot \left(v(\{x \in Q : |[b, \mathcal{T}_\gamma](f)(x)| > \delta\})\right)^{1/q}\\
&\leq \frac{1}{v(Q)^{\lambda/p}}\delta \cdot \left(v(\{x \in Q : |[b, \mathcal{T}_\gamma](f_1)(x)| > \delta/2\})\right)^{1/q}\\
&\quad + \frac{1}{v(Q)^{\lambda/p}}\delta \cdot \left(v(\{x \in Q : |[b, \mathcal{T}_\gamma](f_2)(x)| > \delta/2\})\right)^{1/q}\\
&=: I + II.
\end{aligned}$$

For I, we notice that $v \in \triangle_2$ $(cf.$ [32]). according to assumption (4.2), we have

$$\begin{aligned}
I &\leq \frac{C}{v(Q)^{\lambda/p}}\left(\int_{\mathbb{R}^n}|f_1(x)|^p u(x)dx\right)^{1/p}\\
&= \frac{C}{v(Q)^{\lambda/p}}\left(\int_{2Q}|f(x)|^p u(x)dx\right)^{1/p}\\
&\leq C\left(\frac{v(2Q)}{v(Q)}\right)^{\lambda/p}\|f\|_{L^{p,\lambda}(u,v)}\\
&\leq C\|f\|_{L^{p,\lambda}(u,v)}.
\end{aligned}$$

For the term II, notice that

$$\begin{aligned}
|[b, \mathcal{T}_\gamma](f_2)(x)| &\leq C\int_{\mathbb{R}^n}\frac{|b(x) - b(y)||f_2(y)|}{|x - y|^{n-\gamma}}dy\\
&\leq C|b(x) - b_Q|\int_{\mathbb{R}^n}\frac{|f_2(y)|}{|x - y|^{n-\gamma}}dy\\
&\quad + \leq C\int_{\mathbb{R}^n}\frac{|b(y) - b_Q||f_2(y)|}{|x - y|^{n-\gamma}}dy\\
&=: \xi(x) + \eta(x).
\end{aligned}$$

Therefore,

$$\begin{aligned}
II &\leq \frac{1}{v(Q)^{\lambda/p}}\delta \cdot \left(v(\{x \in Q : \xi(x) > \delta/4\})\right)^{1/q}\\
&\quad + \frac{1}{v(Q)^{\lambda/p}}\delta \cdot \left(v(\{x \in Q : \eta(x) > \delta/4\})\right)^{1/q}\\
&=: II_1 + II_2.
\end{aligned}$$

Since the condition (4.1) is stronger than the condition (3.1). By Chebyshev's inequality, we obtain

$$
\begin{aligned}
II_1 &\leq \frac{4}{v(Q)^{\lambda/p}} \left(\int_Q |\xi(x)|^q v(x)dx \right)^{1/q} \\
&\leq \frac{C}{v(Q)^{\lambda/p}} \left(\int_Q |b(x) - b_Q|^q v(x)dx \right)^{1/q} \sum_{j=1}^{\infty} \frac{1}{|2^{j+1}Q|^{1-\gamma/n}} \int_{2^{j+1}Q} |f(y)|dy \\
&\leq C\|b\|_{BMO} v(Q)^{1/q-\lambda/p} \sum_{j=1}^{\infty} \frac{1}{|2^{j+1}Q|^{1-\gamma/n}} \int_{2^{j+1}Q} |f(y)|dy \\
&\leq Cv(Q)^{1/q-\lambda/p} \sum_{j=1}^{\infty} \frac{1}{|2^{j+1}Q|^{1-\gamma/n}} \left(\int_{2^{j+1}Q} |f(y)|^p u(y)dy \right)^{1/p} \left(\int_{2^{j+1}Q} u(y)^{-p'/p} dy \right)^{1/p'} \\
&\leq C\|f\|_{L^{p,\lambda}(u,v)} \sum_{j=1}^{\infty} \left(\frac{v(Q)}{v(2^{j+1}Q)} \right)^{1/q-\lambda/p} \frac{v(2^{j+1}Q)^{1/q}}{|2^{j+1}Q|^{1-\gamma/n}} \left(\int_{2^{j+1}Q} u(y)^{-p'/p} dy \right)^{1/p'} \\
&\leq C\|f\|_{L^{p,\lambda}(u,v)} \sum_{j=1}^{\infty} \left(\frac{v(Q)}{v(2^{j+1}Q)} \right)^{1/q-\lambda/p} \\
&\quad \times \frac{|2^{j+1}Q|^{1/(r'q)}}{|2^{j+1}Q|^{1-\gamma/n}} \left(\int_{2^{j+1}Q} v(y)^r dy \right)^{1/(rq)} \left(\int_{2^{j+1}Q} u(y)^{-p'/p} dy \right)^{1/p'} \\
&\leq C\|f\|_{L^{p,\lambda}(u,v)}.
\end{aligned}
$$

where we have used Lemma 3.1. For the term II_2, observe that for $x, x_0 \in Q$ and $y \in (2Q)^c$ we have $|x - y| \approx |x_0 - y|$. Thus, from Chebyshev's inequality, we can obtain

$$
\begin{aligned}
II_2 &\leq \frac{4}{v(Q)^{\lambda/p}} \left(\int_Q |\eta(x)|^q v(x)dx \right)^{1/q} \\
&\leq \frac{C}{v(Q)^{\lambda/p}} \left(\int_Q \left| \int_{(2Q)^c} \frac{|b(y) - b_Q||f(y)|}{|x - y|^{n-\gamma}} dy \right|^q v(x)dx \right)^{1/q} \\
&\leq \frac{C}{v(Q)^{\lambda/p}} \left(\int_Q \left| \sum_{j=1}^{\infty} \int_{2^{j+1}Q \setminus 2^jQ} \frac{|b(y) - b_Q||f(y)|}{|x - y|^{n-\gamma}} dy \right|^q v(x)dx \right)^{1/q} \\
&\leq \frac{C}{v(Q)^{\lambda/p}} \left(\int_Q \left| \sum_{j=1}^{\infty} \int_{2^{j+1}Q \setminus 2^jQ} \frac{|b(y) - b_Q||f(y)|}{|x_0 - y|^{n-\gamma}} dy \right|^q v(x)dx \right)^{1/q} \\
&\leq Cv(Q)^{1/q-\lambda/p} \sum_{j=1}^{\infty} \frac{1}{|2^{j+1}Q|^{1-\gamma/n}} \int_{2^{j+1}Q} |b(y) - b_Q||f(y)|dy \\
&\leq Cv(Q)^{1/q-\lambda/p} \sum_{j=1}^{\infty} \frac{1}{|2^{j+1}Q|^{1-\gamma/n}} \int_{2^{j+1}Q} |b(y) - b_{2^{j+1}Q}||f(y)|dy \\
&\quad + Cv(Q)^{1/q-\lambda/p} \sum_{j=1}^{\infty} \frac{1}{|2^{j+1}Q|^{1-\gamma/n}} \int_{2^{j+1}Q} |b_{2^{j+1}Q} - b_Q||f(y)|dy \\
&=: II_{21} + II_{22}.
\end{aligned}
$$

For II_{21}, putting $\mathcal{C} = t^{p'}$ is a Young function. We use Hölder's inequality to get,

$$
\begin{aligned}
II_{21} &\leq Cv(Q)^{1/q-\lambda/p} \sum_{j=1}^{\infty} \frac{1}{|2^{j+1}Q|^{1-\gamma/n}} \left(\int_{2^{j+1}Q} |b(y) - b_{2^{j+1}Q}|^{p'} u(y)^{-p'/p} dy \right)^{1/p'} \\
&\quad \times \left(\int_{2^{j+1}Q} |f(y)|^p u(y)dy \right)^{1/p}
\end{aligned}
$$

$$\le C\|f\|_{L^{p,\lambda}(u,v)} v(Q)^{1/q-\lambda/p} \sum_{j=1}^{\infty} \frac{v(2^{j+1}Q)^{\lambda/p}}{|2^{j+1}Q|^{1-\gamma/n}} |2^{j+1}Q|^{1/p'} \|(b - b_{2^{j+1}Q})u^{-1/p}\|_{\mathcal{C},2^{j+1}Q}.$$

For $1 < p < \infty$, we have

$$\begin{aligned}\mathcal{C}^{-1}(t) = t^{1/p'} &= \frac{t^{1/p'}}{1+log^+ t}(1+log^+ t)\\ &=: \mathcal{A}^{-1}(t)\mathcal{B}^{-1}(t),\end{aligned}$$

and it is easy to see that $\mathcal{A} \approx t^{p'}(1+log^+ t)^{p'}$, $\mathcal{B} \approx e^t - 1$. $\|f\|_{expL,Q}$ denotes the mean Luxemburg norm of f on cube Q with Young function $\mathcal{B} \approx e^t - 1$. By Lemma 4.2

$$\begin{aligned}\|(b - b_{2^{j+1}Q})u^{-1/p}\|_{\mathcal{C},2^{j+1}Q} &\le C\|b - b_{2^{j+1}Q}\|_{expL,2^{j+1}Q}\|u^{-1/p}\|_{\mathcal{A},2^{j+1}Q}\\ &\le C\|b\|_{BMO}\|u^{-1/p}\|_{\mathcal{A},2^{j+1}Q},\end{aligned}$$

where we have used the well-known fact that for any cube $Q \subset \mathbb{R}^n$ ($cf.$ [30]),

$$\|b - b_Q\|_{expL,Q} \le C\|b\|_{BMO}.$$

Therefore, we apply Hölder's inequality, (4.1) and Lemma 3.1 to get

$$\begin{aligned}II_{21} &\le C\|f\|_{L^{p,\lambda}(u,v)} \sum_{j=1}^{\infty} \left(\frac{v(Q)}{v(2^{j+1}Q)}\right)^{1/q-\alpha/p} \frac{v(2^{j+1}Q)^{1/q}}{|2^{j+1}Q|^{1/p-\gamma/n}} \|b\|_{BMO}\|u^{-1/p}\|_{\mathcal{A},2^{j+1}Q}\\ &\le C\|f\|_{L^{p,\lambda}(u,v)} \sum_{j=1}^{\infty} \left(\frac{v(Q)}{v(2^{j+1}Q)}\right)^{1/q-\alpha/p}\\ &\quad \times |2^{j+1}Q|^{\gamma/n+1/q-1/p} \left(\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q} v(y)^r dy\right)^{1/rq} \|u^{-1/p}\|_{\mathcal{A},2^{j+1}Q}\\ &\le C\|f\|_{L^{p,\lambda}(u,v)}.\end{aligned}$$

For the term II_{22}, we make use of Lemma 4.1 and Hölder's inequality, then

$$\begin{aligned}II_{22} &\le Cv(Q)^{1/q-\lambda/p} \sum_{j=1}^{\infty} (j+1) \frac{\|b\|_{BMO}}{|2^{j+1}Q|^{1-\gamma/n}} \int_{2^{j+1}Q} |f(y)|dy\\ &\le Cv(Q)^{1/q-\lambda/p} \sum_{j=1}^{\infty} (j+1) \frac{1}{|2^{j+1}Q|^{1-\gamma/n}} \left(\int_{2^{j+1}Q} |f(y)|^p u(y)dy\right)^{1/p} \left(\int_{2^{j+1}Q} u(y)^{-p'/p}dy\right)^{1/p'}\\ &= C\|f\|_{L^{p,\lambda}(u,v)} \sum_{j=1}^{\infty} (j+1) \left(\frac{v(Q)}{v(2^{j+1}Q)}\right)^{1/q-\lambda/p} \frac{v(2^{j+1}Q)^{1/q}}{|2^{j+1}Q|^{1-\gamma/n}} \left(\int_{2^{j+1}Q} u(y)^{-p'/p}dy\right)^{1/p'}.\end{aligned}$$

Put $\mathcal{C}(t)$ and $\mathcal{A}(t)$ be the same as before. Obviously, $\mathcal{C}(t) \le \mathcal{A}(t)$ for all $t > 0$, then for any cube $Q \subset \mathbb{R}^n$, $\|f\|_{\mathcal{C},Q} \le \|f\|_{\mathcal{A},Q}$ by definition, which implies that the condition (4.1) is stronger than the condition (3.1). From this and Hölder's inequality yields

$$\begin{aligned}&\le C\|f\|_{L^{p,\lambda}(u,v)} \sum_{j=1}^{\infty} (j+1) \left(\frac{v(Q)}{v(2^{j+1}Q)}\right)^{1/q-\alpha/p}\\ &\quad \times \frac{|2^{j+1}Q|^{1/r'q}}{|2^{j+1}Q|^{1-\gamma/n}} \left(\int_{2^{j+1}Q} v(y)^r dy\right)^{1/rq} \left(\int_{2^{j+1}Q} u(y)^{-p'/p}dy\right)^{1/p'}\\ &\le C\|f\|_{L^{p,\lambda}(u,v)} \sum_{j=1}^{\infty} (j+1) \left(\frac{v(Q)}{v(2^{j+1}Q)}\right)^{1/q-\alpha/p}\\ &\le C\|f\|_{L^{p,\lambda}(u,v)}.\end{aligned}$$

where in the last inequality we have used Lemma 4.3.

Combining the above estimates for I and II, and then taking the supremum over all cubes $Q \subset \mathbb{R}^n$ and all $\delta > 0$, we finish the proof of Theorem. □

Theorem 4.5. *Let $m \geq 1$, $1 < p \leq q < \infty$, $0 < \lambda < p/q$, $b \in BMO$ and $[b, \mathcal{T}_\gamma]_m$ satisfy (2.1) with $0 \leq \gamma < n$. Given a pair of weights (u, v), suppose that for some $r > 1$ and for all cubes $Q \subset \mathbb{R}^n$,*

$$|Q|^{\gamma/n+1/q-1/p}\left(\frac{1}{|Q|}\int_Q \omega(x)^r dx\right)^{1/rq}\|v^{-1/p}\|_{\mathcal{A}_m,Q} \leq C, \tag{4.3}$$

where $\mathcal{A}_1(t) = t^{p'}\left(1+log(e+t)\right)^{p'}$, $\mathcal{A}_m(t) = t^{p'}\left(1+log^+ t\right)^{mp'}$ $(m = 2, 3, \ldots)$.

Furthermore, we also suppose that $[b, \mathcal{T}_\gamma]_m$ satisfies the weak-(p, q) type inequality

$$\delta \cdot v(\{x \in \mathbb{R}^n : |\mathcal{T}_\gamma f(x)| > \beta\})^{1/q} \leq C\left(\int_{\mathbb{R}^n}|f(x)|^p u(x)dx\right)^{1/p}, \ \delta > 0. \tag{4.4}$$

If $v \in \triangle_2$, $\omega \in A_\infty$, then the sublinear operator higher order commutators $[b, \mathcal{T}_\gamma]_m$ is bounded from $(L^p, L^s)^\alpha(u, v, \omega)$ to $(L^{q,\infty}, L^s)^\beta(v, \omega)$ with $1/\beta = 1/\alpha - (1/p - 1/q)$.

Proof of Theorem 4.5. Because the method is similar, we only need to prove the case of $m = 1$. Let $f \in (L^p, L^s)^\alpha(u, v, \omega)$ with $1 < p \leq \alpha < s \leq \infty$ and $v \in \triangle_2$, $\omega \in A_\infty$. Fix $Q := Q(y, r) \subset \mathbb{R}^n$, we split $f = f_1 + f_2$ with $f_1 = f\chi_{4Q}$, where $4Q := Q(y, 4\sqrt{n}r)$. Then, for any $y \in \mathbb{R}^n$,

$$\begin{aligned}
&v(Q(y,r))^{1/\beta-1/q-1/s}\|[b,\mathcal{T}_\gamma](f)\chi_{Q(y,r)}\|_{L^{q,\infty}(v)}\\
&\leq v(Q(y,r))^{1/\beta-1/q-1/s}\|[b,\mathcal{T}_\gamma](f_1)\chi_{Q(y,r)}\|_{L^{q,\infty}(v)}\\
&\quad + v(Q(y,r))^{1/\beta-1/q-1/s}\|[b,\mathcal{T}_\gamma](f_2)\chi_{Q(y,r)}\|_{L^{q,\infty}(v)}\\
&=: I + II.
\end{aligned}$$

For I, notice that $1/\beta - 1/q - 1/s = 1/\alpha - 1/p - 1/s$, $1/\alpha - 1/p - 1/s < 0$ and $v \in \triangle_2$. According to assumption (3.4), we have

$$\begin{aligned}
I &\leq v(Q(y,r))^{1/\beta-1/q-1/s}\|[b,\mathcal{T}_\gamma](f_1)\|_{L^{q,\infty}(v)}\\
&\leq Cv(Q(y,r))^{1/\beta-1/q-1/s}\left(\int_{\mathbb{R}^n}|f_1(x)|^p u(x)dx\right)^{1/p}\\
&= Cv(Q(y,r))^{1/\beta-1/q-1/s}\left(\int_{Q(y,2\sqrt{n}r)}|f(x)|^p u(x)dx\right)^{1/p}\\
&\leq Cv(Q(y,2\sqrt{n}r))^{1/\beta-1/q-1/s}\|f\chi_{Q(y,2\sqrt{n}r)}\|_{L^p(u)}\\
&\leq v(Q(y,r))^{1/\alpha-1/p-1/s}\|f\chi_{Q(y,2\sqrt{n}r)}\|_{L^p(u)}\\
&= \left(\frac{v(Q(y,r))}{v(Q(y,2\sqrt{n}r))}\right)^{1/\alpha-1/p-1/s} v(Q(y,2\sqrt{n}r))^{1/\alpha-1/p-1/s}\|f\chi_{Q(y,2\sqrt{n}r)}\|_{L^p(u)}\\
&\leq Cv(Q(y,2\sqrt{n}r))^{1/\alpha-1/p-1/s}\|f\chi_{Q(y,2\sqrt{n}r)}\|_{L^p(u)}.
\end{aligned}$$

For the term II, notice that

$$\begin{aligned}
|[b,\mathcal{T}_\gamma](f_2)(x)| &\leq |b(x)-b_{Q(y,r)}||\mathcal{T}_\gamma(f_2)(x)| + |\mathcal{T}_\gamma((b_{Q(y,r)}-b)f_2)(x)|\\
&=: \xi(x) + \eta(x).
\end{aligned}$$

Therefore,

$$\begin{aligned}
II &\leq v(Q(y,r))^{1/\beta-1/q-1/s}\|\xi(\cdot)\chi_{Q(y,r)}\|_{L^{q,\infty}(v)}\\
&\quad + v(Q(y,r))^{1/\beta-1/q-1/s}\|\eta(\cdot)\chi_{Q(y,r)}\|_{L^{q,\infty}(v)}\\
&=: II_1 + II_2.
\end{aligned}$$

For the term II_1, By Chebyshev's inequality ,Hölder's inequality and $v \in \triangle_2$ yields

$$II_1 \leq 4v(Q(y,r))^{1/\beta-1/q-1/s}\left(\int_{Q(y,r)}|\xi(x)|^q v(x)dx\right)^{1/q}$$

$$
\begin{aligned}
&\le Cv(Q(y,r))^{1/\beta-1/q-1/s}\left(\int_{Q(y,r)}|b(x)-b_{Q(y,r)}|^q v(x)dx\right)^{1/q}\\
&\quad\times\sum_{j=1}^{\infty}\frac{1}{|Q(y,2^{j+1}\sqrt{n}r)|^{1-\gamma/n}}\int_{Q(y,2^{j+1}\sqrt{n}r)}|f(z)|dz\\
&\le Cv(Q(y,r))^{1/\beta-1/s}\|b\|_{BMO}\sum_{j=1}^{\infty}\frac{1}{|Q(y,2^{j+1}\sqrt{n}r)|^{1-\gamma/n}}\left(\int_{Q(y,2^{j+1}\sqrt{n}r)}|f(z)|^p v(z)dz\right)^{1/p}\\
&\quad\times\left(\int_{Q(y,2^{j+1}\sqrt{n}r)}v(z)^{-p'/p}dz\right)^{1/p'}\\
&\le C\sum_{j=1}^{\infty}\left(\frac{v(Q(y,r))}{v(Q(y,2^{j+1}\sqrt{n}r))}\right)^{1/\beta-1/s}v(Q(y,2^{j+1}\sqrt{n}r))^{1/\beta-1/q-1/s}\|f\chi_{Q(y,2^{j+1}\sqrt{n}r)}\|_{L^p(v)}\\
&\quad\times\frac{v(Q(y,2^{j+1}\sqrt{n}r))^{1/q}}{|Q(y,2^{j+1}\sqrt{n}r)|^{1-\gamma/n}}\left(\int_{Q(y,2^{j+1}\sqrt{n}r)}v(z)^{-p'/p}dz\right)^{1/p'}\\
&\le C\sum_{j=1}^{\infty}\left(\frac{v(Q(y,r))}{v(Q(y,2^{j+1}\sqrt{n}r))}\right)^{1/\beta-1/s}v(Q(y,2^{j+1}\sqrt{n}r))^{1/\beta-1/q-1/s}\|f\chi_{Q(y,2^{j+1}\sqrt{n}r)}\|_{L^p(v)}\\
&\quad\times\frac{|Q(y,2^{j+1}\sqrt{n}r)|^{1/r'q}}{|Q(y,2^{j+1}\sqrt{n}r)|^{1-\gamma/n}}\left(\int_{Q(y,2^{j+1}\sqrt{n}r)}v(z)^r dz\right)^{1/(rq)}\left(\int_{Q(y,2^{j+1}\sqrt{n}r)}v(z)^{-p'/p}dz\right)^{1/p'}\\
&\le C\sum_{j=1}^{\infty}\left(\frac{v(Q(y,r))}{v(Q(y,2^{j+1}\sqrt{n}r))}\right)^{1/\beta-1/s}v(Q(y,2^{j+1}\sqrt{n}r))^{1/\alpha-1/p-1/s}\|f\chi_{Q(y,2^{j+1}\sqrt{n}r)}\|_{L^p(v)}\\
&\le C\sum_{j=1}^{\infty}v(Q(y,2^{j+1}\sqrt{n}r))^{1/\alpha-1/p-1/s}\|f\chi_{Q(y,2^{j+1}\sqrt{n}r)}\|_{L^p(v)}.
\end{aligned}
$$

For the term II_2, it is similar to Theorem 4.4. Together with Chebyshev's inequality, we can obtain

$$
\begin{aligned}
II_2 &\le 4v(Q(y,r))^{1/\beta-1/q-1/s}\left(\int_{Q(y,r)}|\eta(x)|^q v(x)dx\right)^{1/q}\\
&\le Cv(Q(y,r))^{1/\beta-1/q-1/s}\left(\int_Q\left|\int_{(2Q)^c}\frac{|b(y)-b_Q||f(y)|}{|x-y|^{n-\gamma}}dy\right|^q v(x)dx\right)^{1/q}\\
&\le Cv(Q(y,r))^{1/\beta-1/q-1/s}\left(\int_Q\left|\int_{(2Q)^c}\frac{|b(y)-b_Q||f(y)|}{|x_0-y|^{n-\gamma}}dy\right|^q v(x)dx\right)^{1/q}\\
&\le Cv(Q(y,r))^{1/\beta-1/s}\sum_{j=1}^{\infty}\frac{1}{|2^{j+1}Q|^{1-\gamma/n}}\int_{2^{j+1}Q}|b(y)-b_Q||f(y)|dy\\
&\le Cv(Q(y,r))^{1/\beta-1/s}\sum_{j=1}^{\infty}\frac{1}{|2^{j+1}Q|^{1-\gamma/n}}\int_{2^{j+1}Q}|b(y)-b_{2^{j+1}Q}||f(y)|dy\\
&\quad+Cv(Q(y,r))^{1/\beta-1/s}\sum_{j=1}^{\infty}\frac{1}{|2^{j+1}Q|^{1-\gamma/n}}\int_{2^{j+1}Q}|b_{2^{j+1}Q}-b_Q||f(y)|dy\\
&=: II_{21}+II_{22}.
\end{aligned}
$$

For II_{21}, putting $\mathcal{C}=t^{p'}$ is a Young function. It is similar to Theorem 4.4 and together with generalized Hölder's inequality and Hölder's inequality to get,

$$
\begin{aligned}
II_{21} &\le Cv(Q(y,r))^{1/\beta-1/s}\sum_{j=1}^{\infty}\frac{1}{|Q(y,2^{j+1}\sqrt{n}r)|^{1-\gamma/n}}\left(\int_{Q(y,2^{j+1}\sqrt{n}r)}|f(z)|^p u(z)dz\right)^{1/p}\\
&\quad\times\left(\int_{Q(y,2^{j+1}\sqrt{n}r)}|b(y)-b_{Q(y,2^{j+1}\sqrt{n}r)}|^{p'}u(z)^{-p'/p}dz\right)^{1/p'}
\end{aligned}
$$

$$
\begin{aligned}
&\le Cv(Q(y,r))^{1/\beta-1/s}\sum_{j=1}^{\infty}\frac{\|f\chi_{Q(y,2^{j+1}\sqrt{n}r)}\|_{L^p(u)}}{|Q(y,2^{j+1}\sqrt{n}r)|^{1-\gamma/n}}\\
&\quad\times|Q(y,2^{j+1}\sqrt{n}r)|^{1/p'}\|(b-b_{Q(y,2^{j+1}\sqrt{n}r)})u^{-1/p}\|_{\mathcal{C},Q(y,2^{j+1}\sqrt{n}r)}\\
&\le Cv(Q(y,r))^{1/\beta-1/s}\sum_{j=1}^{\infty}\frac{\|f\chi_{Q(y,2^{j+1}\sqrt{n}r)}\|_{L^p(u)}}{|Q(y,2^{j+1}\sqrt{n}r)|^{1/p-\gamma/n}}\|b\|_{BMO}\|u^{-1/p}\|_{\mathcal{A},Q(y,2^{j+1}\sqrt{n}r)}\\
&= C\sum_{j=1}^{\infty}\left(\frac{Q(y,r)}{Q(y,2^{j+1}\sqrt{n}r)}\right)^{1/\beta-1/s}v(Q(y,2^{j+1}\sqrt{n}r))^{1/\beta-1/q-1/s}\|f\chi_{Q(y,2^{j+1}\sqrt{n}r)}\|_{L^p(u)}\\
&\quad\times\frac{v(Q(y,2^{j+1}\sqrt{n}r))^{1/q}}{|Q(y,2^{j+1}\sqrt{n}r)|^{1/p-\gamma/n}}\|u^{-1/p}\|_{\mathcal{A},Q(y,2^{j+1}\sqrt{n}r)}\\
&\le C\sum_{j=1}^{\infty}\left(\frac{Q(y,r)}{Q(y,2^{j+1}\sqrt{n}r)}\right)^{1/\beta-1/s}v(Q(y,2^{j+1}\sqrt{n}r))^{1/\beta-1/q-1/s}\|f\chi_{Q(y,2^{j+1}\sqrt{n}r)}\|_{L^p(u)}\\
&\quad\times|Q(y,2^{j+1}\sqrt{n}r)|^{\gamma/n+1/q-1/p}\left(\frac{1}{|2^{j+1}|}\int_{Q(y,2^{j+1}\sqrt{n}r)}v(y)^r\,dy\right)^{1/(rq)}\|u^{-1/p}\|_{\mathcal{A},Q(y,2^{j+1}\sqrt{n}r)}\\
&\le C\sum_{j=1}^{\infty}v(Q(y,2^{j+1}\sqrt{n}r))^{1/\beta-1/q-1/s}\|f\chi_{Q(y,2^{j+1}\sqrt{n}r)}\|_{L^p(u)}.
\end{aligned}
$$

where we have used $v\in A_\infty$. For the term II_{22}, we make use of Lemma 4.1 and Hölder's inequality, then

$$
\begin{aligned}
II_{22}&\le Cv(Q(y,r))^{1/\beta-1/s}\sum_{j=1}^{\infty}(j+1)\frac{\|b\|_{BMO}}{|Q(y,2^{j+1}\sqrt{n}r)|^{1-\gamma/n}}\int_{Q(y,2^{j+1}\sqrt{n}r)}|f(y)|\,dy\\
&\le Cv(Q(y,r))^{1/\beta-1/s}\sum_{j=1}^{\infty}(j+1)\frac{1}{|Q(y,2^{j+1}\sqrt{n}r)|^{1-\gamma/n}}\left(\int_{Q(y,2^{j+1}\sqrt{n}r)}|f(y)|^p u(y)\,dy\right)^{1/p}\\
&\quad\times\left(\int_{Q(y,2^{j+1}\sqrt{n}r)}u(y)^{-p'/p}\,dy\right)^{1/p'}\\
&= C\sum_{j=1}^{\infty}v(Q(y,2^{j+1}\sqrt{n}r))^{1/\beta-1/q-1/s}\|f\chi_{Q(y,2^{j+1}\sqrt{n}r)}\|_{L^p(u)}\\
&\quad\times(j+1)\left(\frac{v(Q(y,r))}{v(Q(y,2^{j+1}\sqrt{n}r))}\right)^{1/\beta-1/s}\frac{v(Q(y,2^{j+1}\sqrt{n}r))^{1/q}}{|Q(y,2^{j+1}\sqrt{n}r)|^{1-\gamma/n}}\left(\int_{Q(y,2^{j+1}\sqrt{n}r)}u(z)^{-p'/p}\,dz\right)^{1/p'}.
\end{aligned}
$$

It is similar to Theorem 4.4 and notice that the condition (4.3) is stronger than the condition (3.3). From these and Hölder's inequality yields

$$
\begin{aligned}
&\le C\sum_{j=1}^{\infty}v(Q(y,2^{j+1}\sqrt{n}r))^{1/\beta-1/q-1/s}\|f\chi_{Q(y,2^{j+1}\sqrt{n}r)}\|_{L^p(u)}(j+1)\left(\frac{v(Q(y,r))}{v(Q(y,2^{j+1}\sqrt{n}r))}\right)^{1/\beta-1/s}\\
&\quad\times\frac{|Q(y,2^{j+1}\sqrt{n}r)|^{1/r'q}}{|Q(y,2^{j+1}\sqrt{n}r)|^{1-\gamma/n}}\left(\int_{Q(y,2^{j+1}\sqrt{n}r)}v(y)^r\,dy\right)^{1/rq}\left(\int_{Q(y,2^{j+1}\sqrt{n}r)}u(y)^{-p'/p}\,dy\right)^{1/p'}\\
&\le C\sum_{j=1}^{\infty}v(Q(y,2^{j+1}\sqrt{n}r))^{1/\beta-1/q-1/s}\|f\chi_{Q(y,2^{j+1}\sqrt{n}r)}\|_{L^p(u)}(j+1)\left(\frac{v(Q(y,r))}{v(Q(y,2^{j+1}\sqrt{n}r))}\right)^{1/\beta-1/s}.
\end{aligned}
$$

Summing up all the above estimates and from the fact that $1/\beta-1/q-1/s=1/\alpha-1/p-1/s$ and $v\in A_\infty$, we obtain

$$
\begin{aligned}
II&\le C\sum_{j=1}^{\infty}v(Q(y,2^{j+1}\sqrt{n}r))^{1/\beta-1/q-1/s}\|f\chi_{Q(y,2^{j+1}\sqrt{n}r)}\|_{L^p(u)}(j+1)\left(\frac{v(Q(y,r))}{v(Q(y,2^{j+1}\sqrt{n}r))}\right)^{1/\beta-1/s}\\
&= C\sum_{j=1}^{\infty}v(Q(y,2^{j+1}\sqrt{n}r))^{1/\alpha-1/p-1/s}\|f\chi_{Q(y,2^{j+1}\sqrt{n}r)}\|_{L^p(u)}(j+1)\left(\frac{v(Q(y,r))}{v(Q(y,2^{j+1}\sqrt{n}r))}\right)^{1/\beta-1/s}
\end{aligned}
$$

$$\leq C \sum_{j=1}^{\infty} v(Q(y, 2^{j+1}\sqrt{n}r))^{1/\alpha-1/p-1/s} \|f\chi_{Q(y,2^{j+1}\sqrt{n}r)}\|_{L^p(u)}.$$

which is desired inequality. Combining the above estimates for I and II, and taking the $L^s(\omega)$-norm of both sides of with respect to the variable y, we finish the proof of Theorem. □

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