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## Research Article

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# Algebraic loop structures on algebra comultiplications

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**Abstract:** In this paper, we study the algebraic loop structures on the set of Lie algebra comultiplications. More specifically, we investigate the fundamental concepts of algebraic loop structures and the set of Lie algebra comultiplications which have inversive, power-associative and Moufang properties depending on the Lie algebra comultiplications up to all the possible quadratic and cubic Lie algebra comultiplications. We also apply those notions to the rational cohomology of Hopf spaces.

**Keywords:** Lie algebra comultiplication, perturbation, algebraic loop, inversive property, power-associativity, Moufang property, Eilenberg-MacLane space, cohomology algebra

**MSC:** Primary 17B70; Secondary 16T05, 55P45, 20N05

## 1 Introduction

The theory of Lie algebras is an outgrowth of the Lie theory of continuous groups and is playing an important role for many reasons. Moreover, it intervenes in other areas of science such as different branches of physics and chemistry. It is an active domain of current research and employs at the same time algebra, algebraic topology, geometry and so on. Lie algebras arise as vector spaces of linear transformations equipped with a new operation which is in general neither commutative nor associative. Indeed, Lie algebras are vector spaces endowed with a particular non-associative product which is called a Lie bracket. Recently, the necessary and sufficient conditions for a family of functors from the category of partial group entwining modules to the category of modules over a suitable algebra to be separable were given in [1], and the homotopy relationship between the space of proper maps and the space of local maps was constructed in [2].

In the topological point of view, co-H-spaces, which are also called spaces with a comultiplication, are important objects of study for at least two reasons (see [3-8]). First of all, they are the duals of H-spaces in the sense of Eckmann and Hilton (see also [9-11]). They have played a significant and central role in algebraic topology for many years. Secondly, there is a large class of examples, namely the suspensions and the spheres, which are co-H-spaces. They have the co-H-group structures which enable one to add homotopy classes of maps using the suspension structure and which give rise to the homotopy group of a space in algebraic topology.

In this paper, as an algebraic version of comultiplications of a space, we think about the algebraic comultiplications of algebraic objects [12] and we are particularly interested in the algebraic loop structures on the set of some Lie algebra comultiplications. The main purpose of this paper is to construct the algebraic loops derived from a Lie algebra comultiplication, and investigate the basic properties of those algebraic

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loops and the set of Lie algebra comultiplications which have the inversive, power-associative and Moufang properties depending on the Lie algebra comultiplications. We also apply these notions to those of the rational cohomology of Hopf spaces.

The paper is organized as follows: In Section 2, we define certain Lie algebra comultiplications on the Lie algebras and construct the abelian group structure on the set of Lie algebra comultiplications. In Section 3, we determine concretely when the algebraic loop structures on the set of Lie algebra comultiplications have the inversive, power-associative and Moufang properties for up to all the possible quadratic and cubic Lie algebra comultiplications. Finally, in Section 4, we apply these results to those of some algebra comultiplications on the rational cohomology algebras derived from the Hopf structures.

## 2 Preliminaries

We begin this section with the basic notions of comultiplications of topological spaces. A pair  $(X, \eta)$  consisting of a pointed space  $X$  and a pointed map  $\eta : X \rightarrow X \vee X$  is called a *co-H-space* if  $p_1\eta \simeq 1$  and  $p_2\eta \simeq 1$ , where  $p_1$  and  $p_2$  are the projections  $X \vee X \rightarrow X$  onto the first and second summands of the wedge, respectively, and  $1$  is the identity map of  $X$ , and  $\simeq$  is the homotopy relation. In this case, the map  $\eta : X \rightarrow X \vee X$  is said to be a *comultiplication*.

Let  $A$  be a set with binary operation ‘+’ and zero element  $0 \in A$ ; that is,

$$a + 0 = a = 0 + a$$

for every  $a \in A$ . Then  $A$  is called an *algebraic loop* if for every  $a, b \in A$ , the equations  $a + x = b$  and  $y + a = b$  have unique solutions  $x, y$  in  $A$ .

The following result (see [3, Theorem 2.3] and [13, Proposition 1.13]) is the dual of a well known theorem of James [9] for H-spaces and provides a connection between co-H-spaces and algebraic loops. For the moment, we adopt the following notation: If  $(X, \eta)$  is a co-H-space with a comultiplication  $\eta : X \rightarrow X \vee X$  and  $Y$  is any space, then the set  $[X, Y]$  of homotopy classes of maps from  $X$  to  $Y$  with the binary operation ‘ $+_\eta$ ’ induced by  $\eta$  will be denoted by  $[X, Y]_\eta$ ; that is,

$$f +_\eta g = \nabla(f \vee g)\eta$$

for  $f, g \in [X, Y]_\eta$ , where  $\nabla : Y \vee Y \rightarrow Y$  is the folding map.

**Proposition 2.1.** *If  $(X, \eta)$  is a 1-connected co-H-space and  $Y$  is any space, then the set  $[X, Y]_\eta$  becomes an algebraic loop.*

Moreover, it is well known in [3] that  $[X, Y]_\eta$  is also an algebraic loop if  $(X, \eta)$  is a co-H-space and  $Y$  is a nilpotent space.

We now move on to the algebraic version of the topological comultiplication above. In this section, we consider Lie algebras over a field  $\mathbb{F}$  of characteristic 0. Let  $V = \langle v_1, v_2, \dots, v_n \rangle$  and  $W = \langle w_1, w_2, \dots, w_m \rangle$  be graded vector spaces over  $\mathbb{F}$ , and write  $|v_s| = d_s$  for the degree of  $v_s$ ,  $s = 1, 2, \dots, n$  with  $d_1 \leq d_2 \leq \dots \leq d_n$ , and similarly for  $w_t$ ,  $t = 1, 2, \dots, m$ . Let  $L(V)$  and  $L(W)$  be the free graded Lie algebras generated by  $V$  and  $W$ , respectively, with the Lie brackets  $[\ , \ ]$ . We then define the coproduct  $L(V) \sqcup L(W)$  of Lie algebras  $L(V)$  and  $L(W)$  by

$$L(V) \sqcup L(W) = L(V \oplus W).$$

In particular, if  $V = W$ , then the elements of  $L(V) \sqcup L(V)$  in either summand are distinguished from each other by using a prime on elements from the second factor. Therefore, if  $L(V) = L(v_1, v_2, \dots, v_n)$ , then

$$L(V) \sqcup L(V) = L(v_1, v_2, \dots, v_n, v'_1, v'_2, \dots, v'_n).$$

**Example 2.2.** Let  $X^{(n)}$  be the  $n$ th Postnikov approximation of  $X$  [14, 15], and let  $\mathbb{Q}$  be the field of rational numbers. Then the graded rational homotopy groups of the Postnikov approximations of the function spaces

based on the infinite complex and quaternionic projective spaces with the Samelson products as the Lie brackets become the free graded Lie algebras over  $\mathbb{Q}$ ; that is,

$$\pi_{\sharp}((\Sigma \mathbb{C}P^{\infty}, \star)^{(S^1, s_0)}, \omega_0)^{(2n)} \otimes \mathbb{Q} = L(v_1, v_2, \dots, v_n)$$

and

$$\pi_{\sharp}((\Sigma \mathbb{H}P^{\infty}, \star)^{(S^1, s_0)}, \omega_0)^{(4m)} \otimes \mathbb{Q} = L(w_1, w_2, \dots, w_m),$$

where  $\Sigma$  is the suspension functor, and  $\omega_0$  is the constant map as the base point of the function spaces, and  $\dim(v_i) = 2i$  for  $i = 1, 2, \dots, n$ , and  $\dim(w_j) = 4j$  for  $j = 1, 2, \dots, m$ .

**Definition 2.3.** A homomorphism of Lie algebras

$$\varphi : L(V) \longrightarrow L(V) \sqcup L(V)$$

is called a *Lie algebra comultiplication* of  $L(V)$  if

$$\pi_1 \circ \varphi = 1_{L(V)} = \pi_2 \circ \varphi,$$

where  $\pi_1$  and  $\pi_2$  are the projections  $L(V) \sqcup L(V) \rightarrow L(V)$  onto the first and second factors, respectively.

Throughout this paper, we will make use of polynomials whose multiplications are the Lie brackets  $[\cdot, \cdot]$  in the Lie algebras. We now consider the following.

**Definition 2.4.** ([16]) We define a *quadratic Lie algebra comultiplication*  $\varphi : L(V) \rightarrow L(V) \sqcup L(V)$  of the free graded Lie algebra  $L(V)$  by

$$\varphi(v_s) = v_s + v'_s + P_s,$$

where  $P_s = 0$  for  $s = 1, 2$  and  $P_s = \sum_{i,j} (v_i v'_j + v'_i v_j)$  for  $s = 3, 4, \dots, n$  and  $i, j \in \{1, 2, \dots, n-1\}$  with  $d_i + d_j = d_s$ .

Indeed, it can be seen that  $\varphi : L(V) \rightarrow L(V) \sqcup L(V)$  is a Lie algebra comultiplication, since

$$\pi_1 \circ \varphi(v_s) = \pi_1(v_s + v'_s + P_s) = v_s = 1_{L(V)}(v_s)$$

for each  $s = 1, 2, \dots, n$ , and similarly  $\pi_2 \circ \varphi = 1_{L(V)}$ .

As a special case, we define the following:

**Definition 2.5.** The Lie algebra comultiplication

$$\varphi_0 : L(V) \longrightarrow L(V) \sqcup L(V)$$

given by

$$\varphi_0(v_s) = v_s + v'_s$$

for each  $v_s \in V$ ,  $s = 1, 2, \dots, n$  is said to be the *standard comultiplication* (see [17, page 67] and [16, page 84]).

**Definition 2.6.** The element  $P_s \in L(V) \sqcup L(V)$  is called the *sth perturbation* of  $\varphi : L(V) \rightarrow L(V) \sqcup L(V)$ . We call  $P = (P_1, P_2, \dots, P_n)$  the *perturbation* of  $\varphi$  and write  $\varphi = \varphi_P$ . We call  $P$  or  $\varphi$  *one-stage* if there is an integer  $s$  with  $1 \leq s \leq n$  such that  $P_i = 0$  for all  $i \neq s$ .

We now describe the group structure on the set of Lie algebra comultiplications of the free graded Lie algebras as follows.

**Proposition 2.7.** Let  $C(L(V))$  be the set of all Lie algebra comultiplications of  $L(V)$ . Then it has the structure of an abelian group.

*Proof.* We first define the binary operation and the corresponding inverse at the perturbation level. Given two perturbations  $P = (P_1, P_2, \dots, P_n)$  and  $Q = (Q_1, Q_2, \dots, Q_n)$  of some Lie algebra comultiplications of  $L(V)$ , we can add them in the obvious way

$$P + Q = (P_1 + Q_1, P_2 + Q_2, \dots, P_n + Q_n)$$

and take the negative of each perturbation  $P = (P_1, P_2, \dots, P_n)$  as

$$-P = (-P_1, -P_2, \dots, -P_n).$$

This allows us to add and subtract Lie algebra comultiplications of  $L(V)$  by adding and subtracting the corresponding perturbations as follows: If  $\varphi_P$  and  $\varphi_Q$  are the two Lie algebra comultiplications of  $L(V)$  with perturbations  $P = (P_1, P_2, \dots, P_n)$  and  $Q = (Q_1, Q_2, \dots, Q_n)$ , respectively, then we define the operations by

$$\varphi_P + \varphi_Q = \varphi_{P+Q}$$

and

$$-\varphi = \varphi_{-P},$$

where  $\varphi_{P+Q} : L(V) \rightarrow L(V) \sqcup L(V)$  is the Lie algebra comultiplication with perturbation  $P + Q$ . Indeed, we can show that  $\varphi_{P+Q}$  and  $-\varphi$  are Lie algebra comultiplications of  $L(V)$ . This gives the set of Lie algebra comultiplications  $\mathcal{C}(L(V))$  the abelian group structure with the standard comultiplication  $\varphi_0$  as the unit, where  $0 = (0, 0, \dots, 0)$  is the perturbation.  $\square$

More generally, we let  $\mathcal{L}$  and  $\mathcal{M}$  be Lie algebras and let  $\varphi : \mathcal{L} \rightarrow \mathcal{L} \sqcup \mathcal{L}$  be a Lie algebra comultiplication. Then we can define an addition  $+\varphi$  on the set of Lie algebra homomorphisms as follows:

**Definition 2.8.** For the Lie algebra homomorphisms  $f, g : \mathcal{L} \rightarrow \mathcal{M}$ , we define the addition  $f + \varphi g$  by the composition of algebra homomorphisms

$$\mathcal{L} \xrightarrow{\varphi} \mathcal{L} \sqcup \mathcal{L} \xrightarrow{f \sqcup g} \mathcal{M} \sqcup \mathcal{M} \xrightarrow{\nabla} \mathcal{M},$$

where  $\nabla$  is the folding homomorphism; that is,  $\nabla \circ (f \sqcup g)(v) = f(v)$  and  $\nabla \circ (f \sqcup g)(v') = g(v')$  for each  $v$  and  $v'$  in  $\mathcal{L}$ .

From the definition above, we note that if  $i_1, i_2 : \mathcal{L} \rightarrow \mathcal{L} \sqcup \mathcal{L}$  are the first and second inclusions, respectively, then  $\varphi = i_1 + \varphi i_2 : \mathcal{L} \rightarrow \mathcal{L} \sqcup \mathcal{L}$ .

### 3 Algebraic loop structures

Let  $A$  be an algebraic loop, and let  $l(a)$  and  $r(a)$  be the left and right inverses of  $a \in A$ , respectively, under the operation  $+$ ; that is,

$$l(a) + a = 0 = a + r(a).$$

Then an algebraic loop  $A$  is said to be *inversive* if  $l(a) = r(a)$  for all  $a \in A$ . An algebraic loop  $A$  is called *power-associative* if  $(a + a) + a = a + (a + a)$ , and is called *Moufang* if  $(a + b) + (c + a) = (a + (b + c)) + a$  for all  $a, b, c \in A$  (see [4, 18]). It can be shown that an associative binary operation is Moufang.

**Convention.** From now on, we substitute the free graded Lie algebra  $L(V)$  by  $\mathcal{L}$  for a notational convenience, where  $V = \langle v_1, v_2, \dots, v_n \rangle$  is the graded vector space over a field  $\mathbb{F}$  of characteristic zero. If  $\varphi : \mathcal{L} \rightarrow \mathcal{L} \sqcup \mathcal{L}$  is a Lie algebra comultiplication and  $\mathcal{M}$  is any Lie algebra, then the set of Lie algebra homomorphisms  $h : \mathcal{L} \rightarrow \mathcal{M}$  with the binary operation above induced by the Lie algebra comultiplication  $\varphi$  will be denoted by  $\text{Hom}(\mathcal{L}, \mathcal{M})_\varphi$ .

**Lemma 3.1.** Under the Lie algebra comultiplication  $\varphi : \mathcal{L} \rightarrow \mathcal{L} \sqcup \mathcal{L}$ ,  $\text{Hom}(\mathcal{L}, \mathcal{M})_\varphi$  becomes an algebraic loop.

*Proof.* For the Lie algebra homomorphisms  $f, g \in \text{Hom}(\mathcal{L}, \mathcal{M})_\varphi$ , we need to find unique solutions  $x, y \in \text{Hom}(\mathcal{L}, \mathcal{M})_\varphi$  for the equations  $f +_\varphi x = g$  and  $y +_\varphi f = g$ . The Lie algebra homomorphism  $x : \mathcal{L} \rightarrow \mathcal{M}$  is constructed on generators of  $\mathcal{L}$  by induction on the degree  $|v_s| = d_s$  of  $v_s$ . The equation shows that

$$\begin{aligned} g(v_s) &= (f +_\varphi x)(v_s) \\ &= \nabla \circ (f \sqcup x) \circ \varphi(v_s) \\ &= \nabla \circ (f \sqcup x)(v_s + v'_s + P_s) \\ &= f(v_s) + x(v_s) + \nabla \circ (f \sqcup x)(P_s), \end{aligned}$$

where  $P_s$  is the  $s$ th perturbation of  $\varphi$ . In other words,

$$x(v_s) = g(v_s) - f(v_s) - \nabla \circ (f \sqcup x)(P_s).$$

The induction gives the existence and uniqueness of  $x$ . Similarly, the existence and uniqueness of  $y$  follows.  $\square$

We recall that there is a bijection as sets

$$\text{Hom}(\mathcal{L}, \mathcal{M})_\varphi \xrightarrow{\approx} \bigoplus_{s=1}^n \text{Hom}(L(v_s), \mathcal{M})_{\varphi|_{L(v_s)}}$$

under the correspondence  $f \mapsto (f_1, f_2, \dots, f_n)$ . Here,

1. the left-hand side is an algebraic loop, while the right-hand side is a direct sum of vector spaces over a field  $\mathbb{K}$  of characteristic zero;
2.  $\varphi|_{L(v_s)} : L(v_s) \rightarrow \mathcal{L} \sqcup \mathcal{L}$  is the standard comultiplication defined by  $\varphi|_{L(v_s)}(v_s) = v_s + v'_s$ ; and
3.  $f_s \in \text{Hom}(L(v_s), \mathcal{M})_{\varphi|_{L(v_s)}}$  for each  $s = 1, 2, \dots, n$ .

**Definition 3.2.** Let  $\varphi = \varphi_P : \mathcal{L} \rightarrow \mathcal{L} \sqcup \mathcal{L}$  be a Lie algebra comultiplication of  $\mathcal{L}$  with perturbation  $P = (P_1, P_2, \dots, P_n)$ . Then  $\varphi_P$  or  $P$  is said to be *purely cubic* if each  $P_i, i = 1, 2, \dots, n$  has only polynomials of length 3, and is said to be *cubic* if each  $P_i, i = 1, 2, \dots, n$  has only polynomials of length  $\leq 3$ .

**Definition 3.3.** We define *decomposable maps*

$$D_{ij} : \mathcal{L} \longrightarrow \mathcal{L} \sqcup \mathcal{L}$$

between vector spaces by

$$D_{ij}(v_1) = 0 = D_{ij}(v_2)$$

and

$$D_{ij}(v_s) = v_i v'_j + v'_i v_j$$

for each  $s = 3, 4, \dots, n$ , and  $i, j = 1, 2, \dots, n-1$  with  $d_i + d_j = d_s$ .

It is natural for us to consider the following fundamental question: What kind of operations are there in the algebraic loop  $\text{Hom}(\mathcal{L}, \mathcal{M})_\varphi$ ? The following lemma gives the answer to this query. We denote by  $\sum_{i,j}$  the summation in which  $i$  and  $j$  run from 1 through  $n-1$  with  $d_i + d_j = d_s$  for  $s = 3, 4, \dots, n$ .

**Lemma 3.4.** Let  $\varphi : \mathcal{L} \rightarrow \mathcal{L} \sqcup \mathcal{L}$  be a Lie algebra comultiplication as in Definition 2.4 and let  $f = (f_1, f_2, \dots, f_n)$  and  $g = (g_1, g_2, \dots, g_n)$  be as in the algebraic loop  $\text{Hom}(\mathcal{L}, \mathcal{M})_\varphi$ . Then we have

$$(f_1, f_2, \dots, f_n) +_\varphi (g_1, g_2, \dots, g_n) = (h_1, h_2, \dots, h_n),$$

where

$$h_s = f_s + g_s$$

for  $s = 1, 2$ , and

$$h_s = f_s + g_s + \sum_{i,j} \nabla(f \sqcup g) D_{ij}$$

on  $L(v_s)$  for  $s = 3, 4, \dots, n$  with  $d_i + d_j = d_s$ , and  $+_\varphi$  means the binary operation on  $\text{Hom}(\mathcal{L}, \mathcal{M})_\varphi$  induced by the Lie algebra comultiplication  $\varphi$  with perturbation  $P = (P_1, P_2, \dots, P_n)$ .

We note that the image of generators of the graded vector space  $V$  under the composition  $(f \sqcup g) D_{ij}$  of  $D_{ij} : \mathcal{L} \rightarrow \mathcal{L} \sqcup \mathcal{L}$  with  $f \sqcup g : \mathcal{L} \sqcup \mathcal{L} \rightarrow \mathcal{M} \sqcup \mathcal{M}$  is as follows:

$$(f \sqcup g) D_{ij}(v_s) = (f \sqcup g)(v_i v'_j + v'_i v_j) = f_i(v_i) g_j(v'_j) + g_i(v'_i) f_j(v_j)$$

for  $s = 3, 4, \dots, n$ .

*Proof.* We first recall that the sum ' $f +_\varphi g$ ' is the composite of the homomorphisms

$$\mathcal{L} \xrightarrow{\varphi} \mathcal{L} \sqcup \mathcal{L} \xrightarrow{f \sqcup g} \mathcal{M} \sqcup \mathcal{M} \xrightarrow{\nabla} \mathcal{M},$$

where  $\nabla$  is the folding homomorphism. It can easily be seen that the following diagram is commutative:

$$\begin{array}{ccc} L(v_s) \sqcup L(v_s) & \xrightarrow{r_s \sqcup r_s} & \mathcal{L} \sqcup \mathcal{L} \\ & \searrow f_s \sqcup g_s & \downarrow f \sqcup g \\ & & \mathcal{M} \sqcup \mathcal{M}, \end{array}$$

where  $r_s : L(v_s) \hookrightarrow \mathcal{L}$  is the canonical inclusion for each  $s = 1, 2, \dots, n$ . For  $s = 1, 2$ , we have

$$\begin{aligned} (f +_\varphi g)(v_s) &= \nabla \circ (f \sqcup g) \circ \varphi(v_s) \\ &= \nabla \circ (f \sqcup g)(v_s + v'_s) \\ &= \nabla \circ ((f_1, f_2, \dots, f_n) \sqcup (g_1, g_2, \dots, g_n))(v_s + v'_s) \\ &= \nabla \circ (f_s(v_s) + g_s(v'_s)) \\ &= f_s(v_s) + g_s(v_s). \end{aligned}$$

We now get

$$\begin{aligned} (f +_\varphi g)(v_s) &= \nabla \circ (f \sqcup g) \circ \varphi(v_s) \\ &= \nabla \circ (f \sqcup g)(v_s + v'_s + \sum_{i,j} (v_i v'_j + v'_i v_j)) \\ &= \nabla \circ ((f_1, f_2, \dots, f_n) \sqcup (g_1, g_2, \dots, g_n))(v_s + v'_s + \sum_{i,j} (v_i v'_j + v'_i v_j)) \\ &= \nabla \circ (f_s(v_s) + g_s(v'_s) + \sum_{i,j} (f_i(v_i) g_j(v'_j) + g_i(v'_i) f_j(v_j))) \\ &= f_s(v_s) + g_s(v_s) + \sum_{i,j} (f_i(v_i) g_j(v_j) + g_i(v_i) f_j(v_j)) \\ &= (f_s + g_s)(v_s) + \sum_{i,j} \nabla(f \sqcup g) D_{ij}(v_s) \end{aligned}$$

for each  $v_s \in L(v_s)$ ,  $s = 3, 4, \dots, n$  with  $d_i + d_j = d_s$ . This completes the proof.  $\square$

**Definition 3.5.** ([16]) We define a *one-stage quadratic Lie algebra comultiplication*

$$\varphi : \mathcal{L} \longrightarrow \mathcal{L} \sqcup \mathcal{L}$$

of  $\mathcal{L}$  by

$$\varphi(v_s) = v_s + v'_s + P_s,$$

where  $P_s = 0$  for  $s = 1, 2, \dots, n-1$  and  $P_n = \sum_{i,j} (v_i v'_j + v'_i v_j)$  with  $d_i + d_j = d_n$ ; that is, the perturbation of the Lie algebra comultiplication  $\varphi : \mathcal{L} \rightarrow \mathcal{L} \sqcup \mathcal{L}$  is  $P = (0, 0, \dots, 0, P_n)$ .

**Theorem 3.6.** Let  $\varphi : \mathcal{L} \rightarrow \mathcal{L} \sqcup \mathcal{L}$  be the one-stage quadratic Lie algebra comultiplication given by Definition 3.5. Then the algebraic loop  $\text{Hom}(\mathcal{L}, \mathcal{M})_\varphi$  has the inversive, power-associative and Moufang properties.

*Proof.* We first prove the inverse property. Let  $l(f_1, f_2, \dots, f_n)$  denote the left inverse of a Lie algebra homomorphism  $f = (f_1, f_2, \dots, f_n)$ , namely

$$l(f_1, f_2, \dots, f_n) = l(f) = \bar{f} = (\bar{f}_1, \bar{f}_2, \dots, \bar{f}_n),$$

in the algebraic loop  $\text{Hom}(\mathcal{L}, \mathcal{M})_\varphi$ . Then, by Lemma 3.4, we have

$$\begin{aligned} (0, \dots, 0) &= l(f_1, f_2, \dots, f_n) +_\varphi (f_1, f_2, \dots, f_n) \\ &= (\bar{f}_1, \bar{f}_2, \dots, \bar{f}_n) +_\varphi (f_1, f_2, \dots, f_n) \\ &= (\bar{f}_1 + f_1, \bar{f}_2 + f_2, \dots, \bar{f}_{n-1} + f_{n-1}, \bar{f}_n + f_n + \sum_{i,j} \nabla(\bar{f} \sqcup f) D_{ij}). \end{aligned}$$

Thus,  $\bar{f}_1 = -f_1, \bar{f}_2 = -f_2, \dots, \bar{f}_{n-1} = -f_{n-1}$  and

$$\bar{f}_n = -f_n - \sum_{i,j} \nabla(\bar{f} \sqcup f) D_{ij} = -f_n + \sum_{i,j} \nabla(f \sqcup \bar{f}) D_{ij},$$

where we have used the following:

$$\begin{aligned} (\bar{f} \sqcup f) D_{ij}(v_n) &= (\bar{f} \sqcup f)(v_i v'_j + v'_i v_j) \\ &= \bar{f}_i(v_i) f_j(v'_j) + f_i(v'_i) \bar{f}_j(v_j) \\ &= -f_i(v_i) f_j(v'_j) - f_i(v'_i) f_j(v_j) \\ &= -(f \sqcup f)(v_i v'_j + v'_i v_j) \\ &= -(f \sqcup f) D_{ij}(v_n). \end{aligned}$$

Now, if  $r(f_1, f_2, \dots, f_n)$  is the right inverse of  $(f_1, f_2, \dots, f_n)$  in the algebraic loop  $\text{Hom}(\mathcal{L}, \mathcal{M})_\varphi$ , say  $r(f_1, f_2, \dots, f_n) = (\hat{f}_1, \hat{f}_2, \dots, \hat{f}_n)$ . Then

$$\begin{aligned} (0, \dots, 0) &= (f_1, f_2, \dots, f_n) +_\varphi r(f_1, f_2, \dots, f_n) \\ &= (f_1, f_2, \dots, f_n) +_\varphi (\hat{f}_1, \hat{f}_2, \dots, \hat{f}_n) \\ &= (f_1 + \hat{f}_1, f_2 + \hat{f}_2, \dots, f_{n-1} + \hat{f}_{n-1}, f_n + \hat{f}_n + \sum_{i,j} \nabla(f \sqcup \hat{f}) D_{ij}). \end{aligned}$$

Thus,  $\hat{f}_1 = -f_1, \hat{f}_2 = -f_2, \dots, \hat{f}_{n-1} = -f_{n-1}$  and

$$\hat{f}_n = -f_n - \sum_{i,j} \nabla(f \sqcup \hat{f}) D_{ij} = -f_n + \sum_{i,j} \nabla(\hat{f} \sqcup f) D_{ij}.$$

Therefore,  $\bar{f} = \hat{f}$  in the algebraic loop  $\text{Hom}(\mathcal{L}, \mathcal{M})_\varphi$  so that it has the inversivity property.

Secondly, we compute the following:

$$\begin{aligned} &((f_1, f_2, \dots, f_n) +_\varphi (f_1, f_2, \dots, f_n)) +_\varphi (f_1, f_2, \dots, f_n) \\ &= (2f_1, 2f_2, \dots, 2f_{n-1}, 2f_n + \sum_{i,j} \nabla(f \sqcup f) D_{ij}) +_\varphi (f_1, f_2, \dots, f_n) \\ &= (F_1, F_2, \dots, F_n), \end{aligned}$$

where  $F_i = 3f_i$  for  $i = 1, 2, \dots, n-1$  and

$$F_n = 3f_n + \sum_{i,j} \nabla(f \sqcup f) D_{ij} + \sum_{i,j} \nabla(2f \sqcup f) D_{ij} = 3f_n + 3 \sum_{i,j} \nabla(f \sqcup f) D_{ij}.$$

Similarly, we have

$$\begin{aligned} &(f_1, f_2, \dots, f_n) +_\varphi ((f_1, f_2, \dots, f_n) +_\varphi (f_1, f_2, \dots, f_n)) \\ &= (f_1, f_2, \dots, f_n) +_\varphi (2f_1, 2f_2, \dots, 2f_{n-1}, 2f_n + \sum_{i,j} \nabla(f \sqcup f) D_{ij}) \\ &= (G_1, G_2, \dots, G_n), \end{aligned}$$

where  $G_i = 3f_i$  for  $i = 1, 2, \dots, n-1$  and

$$G_n = 3f_n + \sum_{i,j} \nabla(f \sqcup f) D_{ij} + \sum_{i,j} \nabla(f \sqcup 2f) D_{ij} = 3f_n + 3 \sum_{i,j} \nabla(f \sqcup f) D_{ij}.$$

Thus

$$(f +_{\varphi} f) +_{\varphi} f = (F_1, F_2, \dots, F_n) = (G_1, G_2, \dots, G_n) = f +_{\varphi} (f +_{\varphi} f) : \mathcal{L} \longrightarrow \mathcal{M}$$

so that the algebraic loop  $\text{Hom}(\mathcal{L}, \mathcal{M})_{\varphi}$  is power-associative.

Finally here, we show that the algebraic loop  $\text{Hom}(\mathcal{L}, \mathcal{M})_{\varphi}$  has the Moufang property. Let  $f = (f_1, f_2, \dots, f_n)$ ,  $g = (g_1, g_2, \dots, g_n)$  and  $h = (h_1, h_2, \dots, h_n)$  be elements of  $\text{Hom}(\mathcal{L}, \mathcal{M})_{\varphi}$ . Then we have

$$\begin{aligned} & \left( (f_1, f_2, \dots, f_n) +_{\varphi} ((g_1, g_2, \dots, g_n) +_{\varphi} (h_1, h_2, \dots, h_n)) \right) +_{\varphi} (f_1, f_2, \dots, f_n) \\ &= ((f_1, f_2, \dots, f_n) +_{\varphi} (g_1 + h_1, g_2 + h_2, \dots, g_{n-1} + h_{n-1}, \\ & \quad g_n + h_n + \sum_{i,j} \nabla(g \sqcup h) D_{ij})) +_{\varphi} (f_1, f_2, \dots, f_n) \\ &= (f_1 + g_1 + h_1, f_2 + g_2 + h_2, \dots, f_{n-1} + g_{n-1} + h_{n-1}, f_n + g_n + h_n \\ & \quad + \sum_{i,j} \nabla(g \sqcup h) D_{ij} + \sum_{i,j} \nabla(f \sqcup (g + h)) D_{ij}) +_{\varphi} (f_1, f_2, \dots, f_n) \\ &= (2f_1 + g_1 + h_1, 2f_2 + g_2 + h_2, \dots, 2f_{n-1} + g_{n-1} + h_{n-1}, 2f_n + g_n + h_n \\ & \quad + \sum_{i,j} \nabla(g \sqcup h) D_{ij} + \sum_{i,j} \nabla(f \sqcup g) D_{ij} + \sum_{i,j} \nabla(f \sqcup h) D_{ij} \\ & \quad + \sum_{i,j} \nabla(f \sqcup f) D_{ij} + \sum_{i,j} \nabla(g \sqcup f) D_{ij} + \sum_{i,j} \nabla(h \sqcup f) D_{ij}). \end{aligned}$$

We now compute

$$\begin{aligned} & ((f_1, f_2, \dots, f_n) +_{\varphi} (g_1, g_2, \dots, g_n)) +_{\varphi} ((h_1, h_2, \dots, h_n) +_{\varphi} (f_1, f_2, \dots, f_n)) \\ &= ((f_1 + g_1, f_2 + g_2, \dots, f_{n-1} + g_{n-1}, f_n + g_n + \sum_{i,j} \nabla(f \sqcup g) D_{ij}) \\ & \quad +_{\varphi} (h_1 + f_1, h_2 + f_2, \dots, h_{n-1} + f_{n-1}, h_n + f_n + \sum_{i,j} \nabla(h \sqcup f) D_{ij})) \\ &= (2f_1 + g_1 + h_1, 2f_2 + g_2 + h_2, \dots, 2f_{n-1} + g_{n-1} + h_{n-1}, 2f_n + g_n + h_n \\ & \quad + \sum_{i,j} \nabla(f \sqcup g) D_{ij} + \sum_{i,j} \nabla(h \sqcup f) D_{ij} + \sum_{i,j} \nabla(f \sqcup h) D_{ij} \\ & \quad + \sum_{i,j} \nabla(f \sqcup f) D_{ij} + \sum_{i,j} \nabla(g \sqcup h) D_{ij} + \sum_{i,j} \nabla(g \sqcup f) D_{ij}). \end{aligned}$$

Therefore, we get

$$(f +_{\varphi} (g +_{\varphi} h)) +_{\varphi} f = (f +_{\varphi} g) +_{\varphi} (h +_{\varphi} f)$$

as required.  $\square$

In general, a Lie algebra comultiplication

$$\varphi = \varphi_{(P_1, P_2, \dots, P_n)} : \mathcal{L} \longrightarrow \mathcal{L} \sqcup \mathcal{L}$$

containing at least two non-zero perturbations, say  $P_s \neq P_t$ , cannot be guaranteed to have the inversive, power-associative and Moufang properties. This is why we restrict our range in Theorem 3.6 to the one-stage quadratic perturbation.

From now on,  $\sum_{i,j,k}$  denotes the summation in which  $i, j$  and  $k$  are the elements of the set  $\{1, 2, \dots, n-1\}$  with  $d_i + d_j + d_k = d_s$  for  $s = 3, 4, \dots, n$ .

**Definition 3.7.** We define a *purely cubic Lie algebra comultiplication*  $\psi : \mathcal{L} \rightarrow \mathcal{L} \sqcup \mathcal{L}$  by

$$\psi(v_s) = v_s + v'_s + Q_s,$$

where  $Q_s = 0$  for  $s = 1, 2$  and

$$Q_s = \sum_{i,j,k} (v_i v_j v'_k + v_i v'_j v'_k + v'_i v_j v_k + v'_i v'_j v_k + v_i v'_j v_k + v'_i v_j v'_k)$$

for each  $s = 3, 4, \dots, n$ , and  $i, j, k \in \{1, 2, \dots, n-1\}$  with  $d_i + d_j + d_k = d_s$ .

**Definition 3.8.** We define the decomposable maps

$$D_{ijk}^l : \mathcal{L} \longrightarrow \mathcal{L} \sqcup \mathcal{L}$$

between vector spaces by

$$D_{ijk}^l(v_1) = 0 = D_{ijk}^l(v_2)$$



for  $l = 1, 2, \dots, 6$ , and

$$\begin{cases} D_{ijk}^1(v_s) = v_i v_j v'_k \\ D_{ijk}^2(v_s) = v_i v'_j v'_k \\ D_{ijk}^3(v_s) = v'_i v_j v_k \\ D_{ijk}^4(v_s) = v'_i v'_j v_k \\ D_{ijk}^5(v_s) = v_i v'_j v_k \\ D_{ijk}^6(v_s) = v'_i v_j v'_k \end{cases}$$

for each  $s = 3, 4, \dots, n$ , and  $i, j, k = 1, 2, \dots, n-1$  with  $d_i + d_j + d_k = d_s$ .

**Lemma 3.9.** Let  $\psi : \mathcal{L} \rightarrow \mathcal{L} \sqcup \mathcal{L}$  be the purely cubic Lie algebra comultiplication as in Definition 3.7 and let  $f = (f_1, f_2, \dots, f_n)$  and  $g = (g_1, g_2, \dots, g_n)$  be in the algebraic loop  $\text{Hom}(\mathcal{L}, \mathcal{M})_\psi$ . Then we have

$$(f_1, f_2, \dots, f_n) +_\psi (g_1, g_2, \dots, g_n) = (h_1, h_2, \dots, h_n)$$

where  $h_s = f_s + g_s$  for  $s = 1, 2$ , and

$$\begin{aligned} h_s = f_s + g_s + \sum_{i,j,k} \nabla((f_i, f_j) \sqcup g_k) D_{ijk}^1 + (f_i \sqcup (g_j, g_k)) D_{ijk}^2 + ((f_j, f_k) \sqcup g_i) D_{ijk}^3 \\ + (f_k \sqcup (g_i, g_j)) D_{ijk}^4 + ((f_i, f_k) \sqcup g_j) D_{ijk}^5 + (f_j \sqcup (g_i, g_k)) D_{ijk}^6 \end{aligned}$$

on  $L(v_s)$  for  $s = 3, 4, \dots, n$ , and  $+_\psi$  means the binary operation on  $\text{Hom}(\mathcal{L}, \mathcal{M})_\psi$  induced by the Lie algebra comultiplication  $\psi$ .

We note that the notation  $(f_i, f_j)$  etc. stands for the direct sums of  $f_i$  and  $f_j$  in a similar fashion to the discussion right after the proof of Lemma 3.1.

*Proof.* A straightforward argument just like Lemma 3.4 completes the proof.  $\square$

**Definition 3.10.** We define a one-stage purely cubic Lie algebra comultiplication  $\psi : \mathcal{L} \rightarrow \mathcal{L} \sqcup \mathcal{L}$  by

$$\psi(v_s) = v_s + v'_s + Q_s,$$

where  $Q_s = 0$  for  $s = 1, 2, \dots, n-1$ , and

$$Q_n = \sum_{i,j,k} (v_i v_j v'_k + v_i v'_j v'_k + v'_i v_j v_k + v'_i v'_j v_k + v_i v'_j v_k + v'_i v_j v'_k)$$

for  $i, j, k = 1, 2, \dots, n-1$  with  $d_i + d_j + d_k = d_n$ .

**Corollary 3.11.** Let  $\psi : \mathcal{L} \rightarrow \mathcal{L} \sqcup \mathcal{L}$  be the Lie algebra comultiplication as in Definition 3.10 and let  $f = (f_1, f_2, \dots, f_n)$ ,  $g = (g_1, g_2, \dots, g_n)$  and  $h = (h_1, h_2, \dots, h_n)$  be the elements of the algebraic loop  $\text{Hom}(\mathcal{L}, \mathcal{M})_\psi$ . Then we have

$$\begin{aligned} (f +_\psi (g +_\psi h)) \\ = (f_1 + g_1 + h_1, \dots, f_{n-1} + g_{n-1} + h_{n-1}, f_n + g_n + h_n \\ + \sum \nabla((g_i, g_j) \sqcup h_k) D^1 + \sum \nabla(g_i \sqcup (h_j, h_k)) D^2 + \sum \nabla((g_j, g_k) \sqcup h_i) D^3 \\ + \sum \nabla(g_k \sqcup (h_i, h_j)) D^4 + \sum \nabla((g_i, g_k) \sqcup h_j) D^5 + \sum \nabla(g_j \sqcup (h_i, h_k)) D^6 \\ + \sum \nabla(((f_i, f_j) \sqcup (g_k + h_k)) D^1 + (f_i \sqcup (g_j + h_j, g_k + h_k)) D^2 + ((f_j, f_k) \sqcup (g_i + h_i)) D^3 \\ + (f_k \sqcup (g_i + h_i, g_j + h_j)) D^4 + ((f_i, f_k) \sqcup (g_j + h_j)) D^5 + (f_j \sqcup (g_i + h_i, g_k + h_k)) D^6); \end{aligned}$$

and

$$\begin{aligned} (f +_\psi g) +_\psi (h +_\psi f) \\ = (f_1 + g_1, \dots, f_{n-1} + g_{n-1}, f_n + g_n + \sum \nabla(((f_i, f_j) \sqcup g_k) D^1 + (f_i \sqcup (g_j, g_k)) D^2 \\ + ((f_j, f_k) \sqcup g_i) D^3 + (f_k \sqcup (g_i, g_j)) D^4 + ((f_i, f_k) \sqcup g_j) D^5 + (f_j \sqcup (g_i, g_k)) D^6) \\ +_\psi (h_1 + f_1, \dots, h_{n-1} + f_{n-1}, h_n + f_n + \sum \nabla(((h_i, h_j) \sqcup f_k) D^1 + (h_i \sqcup (f_j, f_k)) D^2 \\ + ((h_j, h_k) \sqcup f_i) D^3 + (h_k \sqcup (f_i, f_j)) D^4 + ((h_i, h_k) \sqcup f_j) D^5 + (h_j \sqcup (f_i, f_k)) D^6)). \end{aligned}$$

*Proof.* The proof follows from Lemma 3.9.  $\square$

**Theorem 3.12.** Let  $\psi : \mathcal{L} \rightarrow \mathcal{L} \sqcup \mathcal{L}$  be the Lie algebra comultiplication as in Definition 3.10 and let  $f = (f_1, f_2, \dots, f_n)$ ,  $g = (g_1, g_2, \dots, g_n)$  and  $h = (h_1, h_2, \dots, h_n)$  be the elements of  $\text{Hom}(\mathcal{L}, \mathcal{M})_\psi$ . Then

1.  $\text{Hom}(\mathcal{L}, \mathcal{M})_\psi$  is inversive or power-associative if and only if

$$\sum_{i,j,k} \nabla((f_i, f_j) \sqcup f_k) D_{ijk}^1 + ((f_j, f_k) \sqcup f_i) D_{ijk}^3 + ((f_i, f_k) \sqcup f_j) D_{ijk}^5 \\ = \sum_{i,j,k} \nabla((f_i \sqcup (f_j, f_k)) D_{ijk}^2 + (f_k \sqcup (f_i, f_j)) D_{ijk}^4 + (f_j \sqcup (f_i, f_k)) D_{ijk}^6);$$

and

2. it has the Moufang property if and only if

$$0 = \sum_{i,j,k} \nabla((f_i, h_j) \sqcup f_k) D_{ijk}^1 + ((g_i, h_j) \sqcup f_k) D_{ijk}^1 + ((h_i, f_j) \sqcup f_k) D_{ijk}^1 \\ + ((h_i, g_j) \sqcup f_k) D_{ijk}^1 - ((f_i, g_j) \sqcup h_k) D_{ijk}^1 - ((g_i, f_j) \sqcup h_k) D_{ijk}^1 \\ + (f_i \sqcup (g_j, h_k)) D_{ijk}^2 + (f_i \sqcup (h_j, g_k)) D_{ijk}^2 - (f_i \sqcup (h_j, f_k)) D_{ijk}^2 \\ - (f_i \sqcup (f_j, h_k)) D_{ijk}^2 - (g_i \sqcup (h_j, f_k)) D_{ijk}^2 - (g_i \sqcup (f_j, h_k)) D_{ijk}^2 \\ + ((f_j, h_k) \sqcup f_i) D_{ijk}^3 + ((g_j, h_k) \sqcup f_i) D_{ijk}^3 + ((h_j, f_k) \sqcup f_i) D_{ijk}^3 \\ + ((h_j, g_k) \sqcup f_i) D_{ijk}^3 - ((f_j, g_k) \sqcup h_i) D_{ijk}^3 - ((g_j, f_k) \sqcup h_i) D_{ijk}^3 \\ + (f_k \sqcup (g_i, h_j)) D_{ijk}^4 + (f_k \sqcup (h_i, g_j)) D_{ijk}^4 - (f_k \sqcup (h_i, f_j)) D_{ijk}^4 \\ - (f_k \sqcup (f_i, h_j)) D_{ijk}^4 - (g_k \sqcup (h_i, f_j)) D_{ijk}^4 - (g_k \sqcup (f_i, h_j)) D_{ijk}^4 \\ + ((f_i, h_k) \sqcup f_j) D_{ijk}^5 + ((g_i, h_k) \sqcup f_j) D_{ijk}^5 + ((h_i, f_k) \sqcup f_j) D_{ijk}^5 \\ + ((h_i, g_k) \sqcup f_j) D_{ijk}^5 - ((f_i, g_k) \sqcup h_j) D_{ijk}^5 - ((g_i, f_k) \sqcup h_j) D_{ijk}^5 \\ + (f_j \sqcup (g_i, h_k)) D_{ijk}^6 + (f_j \sqcup (h_i, g_k)) D_{ijk}^6 - (f_j \sqcup (h_i, f_k)) D_{ijk}^6 \\ - (f_j \sqcup (f_i, h_k)) D_{ijk}^6 - (g_j \sqcup (h_i, f_k)) D_{ijk}^6 - (g_j \sqcup (f_i, h_k)) D_{ijk}^6),$$

where  $d_i + d_j + d_k = d_n$ .

*Proof.* For a notational convenience, we will also make use of the simple notation  $\sum$  for  $\sum_{i,j,k}$ , and  $D^l$  for  $D_{ijk}^l$  in the proof, where  $l = 1, 2, \dots, 6$ .

1. For the power-associative property, we calculate the following:

$$((f_1, f_2, \dots, f_n) +_\psi (f_1, f_2, \dots, f_n)) +_\psi (f_1, f_2, \dots, f_n) \\ = \left( 2f_1, \dots, 2f_{n-1}, 2f_n + \sum_{i,j,k} \nabla((f_i, f_j) \sqcup f_k) D^1 + (f_i \sqcup (f_j, f_k)) D^2 \right. \\ \left. + ((f_j, f_k) \sqcup f_i) D^3 + (f_k \sqcup (f_i, f_j)) D^4 + ((f_i, f_k) \sqcup f_j) D^5 + (f_j \sqcup (f_i, f_k)) D^6 \right) \\ +_\psi (f_1, f_2, \dots, f_n) \\ = \left( 3f_1, \dots, 3f_{n-1}, 3f_n + \sum \nabla((f_i, f_j) \sqcup f_k) D^1 + (f_i \sqcup (f_j, f_k)) D^2 + ((f_j, f_k) \sqcup f_i) D^3 \right. \\ \left. + (f_k \sqcup (f_i, f_j)) D^4 + ((f_i, f_k) \sqcup f_j) D^5 + (f_j \sqcup (f_i, f_k)) D^6 \right. \\ \left. + ((2f_i, 2f_j) \sqcup f_k) D^1 + (2f_i \sqcup (f_j, f_k)) D^2 + ((2f_j, 2f_k) \sqcup f_i) D^3 \right. \\ \left. + (2f_k \sqcup (f_i, f_j)) D^4 + ((2f_i, 2f_k) \sqcup 2f_j) D^5 + (2f_j \sqcup (f_i, f_k)) D^6 \right) \\ = (3f_1, \dots, 3f_{n-1}, 3f_n + 5 \sum \nabla((f_i, f_j) \sqcup f_k) D^1 + 3 \sum \nabla(f_i \sqcup (f_j, f_k)) D^2 \\ + 5 \sum \nabla((f_j, f_k) \sqcup f_i) D^3 + 3 \sum \nabla(f_k \sqcup (f_i, f_j)) D^4 \\ + 5 \sum \nabla((f_i, f_k) \sqcup f_j) D^5 + 3 \sum \nabla(f_j \sqcup (f_i, f_k)) D^6);$$

and

$$\begin{aligned}
 & (f_1, f_2, \dots, f_n) +_{\psi} ((f_1, f_2, \dots, f_n) +_{\psi} (f_1, f_2, \dots, f_n)) \\
 &= (f_1, f_2, \dots, f_n) +_{\psi} \left( 2f_1, \dots, 2f_{n-1}, 2f_n + \sum \nabla((f_i, f_j) \sqcup f_k) D^1 + (f_i \sqcup (f_j, f_k)) D^2 \right. \\
 &\quad \left. + ((f_j, f_k) \sqcup f_i) D^3 + (f_k \sqcup (f_i, f_j)) D^4 + ((f_i, f_k) \sqcup f_j) D^5 + (f_j \sqcup (f_i, f_k)) D^6 \right) \\
 &= \left( 3f_1, \dots, 3f_{n-1}, 3f_n + \sum \nabla((f_i, f_j) \sqcup f_k) D^1 + (f_i \sqcup (f_j, f_k)) D^2 + ((f_j, f_k) \sqcup f_i) D^3 \right. \\
 &\quad \left. + (f_k \sqcup (f_i, f_j)) D^4 + ((f_i, f_k) \sqcup f_j) D^5 + (f_j \sqcup (f_i, f_k)) D^6 \right. \\
 &\quad \left. + ((f_i, f_j) \sqcup 2f_k) D^1 + (f_i \sqcup (2f_j, 2f_k)) D^2 + ((f_j, f_k) \sqcup 2f_i) D^3 \right. \\
 &\quad \left. + (f_k \sqcup (2f_i, 2f_j)) D^4 + ((f_i, f_k) \sqcup 2f_j) D^5 + (f_j \sqcup (2f_i, 2f_k)) D^6 \right) \\
 &= (3f_1, \dots, 3f_{n-1}, 3f_n + 3 \sum \nabla((f_i, f_j) \sqcup f_k) D^1 + 5 \sum \nabla(f_i \sqcup (f_j, f_k)) D^2 \\
 &\quad + 3 \sum \nabla((f_j, f_k) \sqcup f_i) D^3 + 5 \sum \nabla(f_k \sqcup (f_i, f_j)) D^4 \\
 &\quad + 3 \sum \nabla((f_i, f_k) \sqcup f_j) D^5 + 5 \sum \nabla(f_j \sqcup (f_i, f_k)) D^6).
 \end{aligned}$$

Thus, the algebraic loop  $\text{Hom}(\mathcal{L}, \mathcal{M})_{\psi}$  is power-associative if and only if

$$\begin{aligned}
 & \sum \nabla((f_i, f_j) \sqcup f_k) D^1 + ((f_j, f_k) \sqcup f_i) D^3 + ((f_i, f_k) \sqcup f_j) D^5 \\
 &= \sum \nabla(f_i \sqcup (f_j, f_k)) D^2 + (f_k \sqcup (f_i, f_j)) D^4 + (f_j \sqcup (f_i, f_k)) D^6.
 \end{aligned}$$

The inversive property is obtained as in the proof of the power-associative property by applying Lemma 3.9 and Corollary 3.11.

2. We finally find the conditions for the algebraic loop  $\text{Hom}(\mathcal{L}, \mathcal{M})_{\psi}$  to have the Moufang property. Let  $f = (f_1, f_2, \dots, f_n)$ ,  $g = (g_1, g_2, \dots, g_n)$  and  $h = (h_1, h_2, \dots, h_n)$  be the elements of  $\text{Hom}(\mathcal{L}, \mathcal{M})_{\psi}$ . Then, by Lemma 3.9 and Corollary 3.11, we have

$$\begin{aligned}
 & (f +_{\psi} (g +_{\psi} h)) +_{\psi} f \\
 &= (2f_1 + g_1 + h_1, \dots, 2f_{n-1} + g_{n-1} + h_{n-1}, 2f_n + g_n + h_n \\
 &\quad + \sum \nabla((g_i, g_j) \sqcup h_k) D^1 + \sum \nabla(g_i \sqcup (h_j, h_k)) D^2 + \sum \nabla((g_j, g_k) \sqcup h_i) D^3 \\
 &\quad + \sum \nabla(g_k \sqcup (h_i, h_j)) D^4 + \sum \nabla((g_i, g_k) \sqcup h_j) D^5 + \sum \nabla(g_j \sqcup (h_i, h_k)) D^6 \\
 &\quad + \sum \nabla((f_i, f_j) \sqcup g_k) D^1 + \sum \nabla((f_i, f_j) \sqcup h_k) D^1 + \sum \nabla(f_i \sqcup (g_j, g_k)) D^2 \\
 &\quad + \sum \nabla(f_i \sqcup (g_j, h_k)) D^2 + \sum \nabla(f_i \sqcup (h_j, g_k)) D^2 + \sum \nabla(f_i \sqcup (h_j, h_k)) D^2 \\
 &\quad + \sum \nabla((f_j, f_k) \sqcup g_i) D^3 + \sum \nabla((f_j, f_k) \sqcup h_i) D^3 + \sum \nabla(f_k \sqcup (g_i, g_j)) D^4 \\
 &\quad + \sum \nabla(f_k \sqcup (g_i, h_j)) D^4 + \sum \nabla(f_k \sqcup (h_i, g_j)) D^4 + \sum \nabla(f_k \sqcup (h_i, h_j)) D^4 \\
 &\quad + \sum \nabla((f_i, f_k) \sqcup g_j) D^5 + \sum \nabla((f_i, f_k) \sqcup h_j) D^5 + \sum \nabla(f_j \sqcup (g_i, g_k)) D^6 \\
 &\quad + \sum \nabla(f_j \sqcup (g_i, h_k)) D^6 + \sum \nabla(f_j \sqcup (h_i, g_k)) D^6 + \sum \nabla(f_j \sqcup (h_i, h_k)) D^6 \\
 &\quad + \sum \nabla((f_i, f_j) \sqcup f_k) D^1 + \sum \nabla((f_i, g_j) \sqcup f_k) D^1 + \sum \nabla((f_i, h_j) \sqcup f_k) D^1 \\
 &\quad + \sum \nabla((g_i, f_j) \sqcup f_k) D^1 + \sum \nabla((g_i, g_j) \sqcup f_k) D^1 + \sum \nabla((g_i, h_j) \sqcup f_k) D^1 \\
 &\quad + \sum \nabla((h_i, f_j) \sqcup f_k) D^1 + \sum \nabla((h_i, g_j) \sqcup f_k) D^1 + \sum \nabla((h_i, h_j) \sqcup f_k) D^1 \\
 &\quad + \sum \nabla(f_i \sqcup (f_j, f_k)) D^2 + \sum \nabla(g_i \sqcup (f_j, f_k)) D^2 + \sum \nabla(h_i \sqcup (f_j, f_k)) D^2 \\
 &\quad + \sum \nabla((f_j, f_k) \sqcup f_i) D^3 + \sum \nabla((f_j, g_k) \sqcup f_i) D^3 + \sum \nabla((f_j, h_k) \sqcup f_i) D^3 \\
 &\quad + \sum \nabla((g_j, f_k) \sqcup f_i) D^3 + \sum \nabla((g_j, g_k) \sqcup f_i) D^3 + \sum \nabla((g_j, h_k) \sqcup f_i) D^3 \\
 &\quad + \sum \nabla((h_j, f_k) \sqcup f_i) D^3 + \sum \nabla((h_j, g_k) \sqcup f_i) D^3 + \sum \nabla((h_j, h_k) \sqcup f_i) D^3 \\
 &\quad + \sum \nabla(f_k \sqcup (f_i, f_j)) D^4 + \sum \nabla(g_k \sqcup (f_i, f_j)) D^4 + \sum \nabla(h_k \sqcup (f_i, f_j)) D^4 \\
 &\quad + \sum \nabla((f_i, f_k) \sqcup f_j) D^5 + \sum \nabla((f_i, g_k) \sqcup f_j) D^5 + \sum \nabla((f_i, h_k) \sqcup f_j) D^5 \\
 &\quad + \sum \nabla((g_i, f_k) \sqcup f_j) D^5 + \sum \nabla((g_i, g_k) \sqcup f_j) D^5 + \sum \nabla((g_i, h_k) \sqcup f_j) D^5 \\
 &\quad + \sum \nabla((h_i, f_k) \sqcup f_j) D^5 + \sum \nabla((h_i, g_k) \sqcup f_j) D^5 + \sum \nabla((h_i, h_k) \sqcup f_j) D^5 \\
 &\quad + \sum \nabla(f_j \sqcup (f_i, f_k)) D^6 + \sum \nabla(g_j \sqcup (f_i, f_k)) D^6 + \sum \nabla(h_j \sqcup (f_i, f_k)) D^6).
 \end{aligned}$$

Similarly, by Corollary 3.11, we get the following:

$$\begin{aligned}
 & (f +_{\psi} g) +_{\psi} (h +_{\psi} f) \\
 &= (2f_1 + g_1 + h_1, \dots, 2f_{n-1} + g_{n-1} + h_{n-1}, 2f_n + g_n + h_n \\
 &\quad + \sum \nabla((f_i, f_j) \sqcup g_k) D^1 + \sum \nabla(f_i \sqcup (g_j, g_k)) D^2 + \sum \nabla((f_j, f_k) \sqcup g_i) D^3 \\
 &\quad + \sum \nabla(f_k \sqcup (g_i, g_j)) D^4 + \sum \nabla((f_i, f_k) \sqcup g_j) D^5 + \sum \nabla(f_j \sqcup (g_i, g_k)) D^6 \\
 &\quad + \sum \nabla((h_i, h_j) \sqcup f_k) D^1 + \sum \nabla(h_i \sqcup (f_j, f_k)) D^2 + \sum \nabla((h_j, h_k) \sqcup f_i) D^3 \\
 &\quad + \sum \nabla(h_k \sqcup (f_i, f_j)) D^4 + \sum \nabla((h_i, h_k) \sqcup f_j) D^5 + \sum \nabla(h_j \sqcup (f_i, f_k)) D^6 \\
 &\quad + \sum \nabla((f_i + g_i, f_j + g_j) \sqcup (h_k + f_k)) D^1 + ((f_i + g_i) \sqcup (h_j + f_j, h_k + f_k)) D^2 \\
 &\quad + ((f_j + g_j, f_k + g_k) \sqcup (h_i + f_i)) D^3 + ((f_k + g_k) \sqcup (h_i + f_i, h_j + f_j)) D^4 \\
 &\quad + ((f_i + g_i, f_k + g_k) \sqcup (h_j + f_j)) D^5 + ((f_j + g_j) \sqcup (h_i + f_i, h_k + f_k)) D^6) \\
 &= (2f_1 + g_1 + h_1, \dots, 2f_{n-1} + g_{n-1} + h_{n-1}, 2f_n + g_n + h_n \\
 &\quad + \sum \nabla((f_i, f_j) \sqcup g_k) D^1 + \sum \nabla(f_i \sqcup (g_j, g_k)) D^2 + \sum \nabla((f_j, f_k) \sqcup g_i) D^3 \\
 &\quad + \sum \nabla(f_k \sqcup (g_i, g_j)) D^4 + \sum \nabla((f_i, f_k) \sqcup g_j) D^5 + \sum \nabla(f_j \sqcup (g_i, g_k)) D^6 \\
 &\quad + \sum \nabla((h_i, h_j) \sqcup f_k) D^1 + \sum \nabla(h_i \sqcup (f_j, f_k)) D^2 + \sum \nabla((h_j, h_k) \sqcup f_i) D^3 \\
 &\quad + \sum \nabla(h_k \sqcup (f_i, f_j)) D^4 + \sum \nabla((h_i, h_k) \sqcup f_j) D^5 + \sum \nabla(h_j \sqcup (f_i, f_k)) D^6 \\
 &\quad + \sum \nabla((f_i, f_j) \sqcup h_k) D^1 + \sum \nabla((f_i, f_j) \sqcup f_k) D^1 + \sum \nabla((f_i, g_j) \sqcup h_k) D^1 \\
 &\quad + \sum \nabla((f_i, g_j) \sqcup f_k) D^1 + \sum \nabla((g_i, f_j) \sqcup h_k) D^1 + \sum \nabla((g_i, f_j) \sqcup f_k) D^1 \\
 &\quad + \sum \nabla((g_i, g_j) \sqcup h_k) D^1 + \sum \nabla((g_i, g_j) \sqcup f_k) D^1 + \sum \nabla(f_i \sqcup (h_j, h_k)) D^2 \\
 &\quad + \sum \nabla(f_i \sqcup (h_j, f_k)) D^2 + \sum \nabla(f_i \sqcup (f_j, h_k)) D^2 + \sum \nabla(f_i \sqcup (f_j, f_k)) D^2 \\
 &\quad + \sum \nabla(g_i \sqcup (h_j, h_k)) D^2 + \sum \nabla(g_i \sqcup (h_j, f_k)) D^2 + \sum \nabla(g_i \sqcup (f_j, h_k)) D^2 \\
 &\quad + \sum \nabla(g_i \sqcup (f_j, f_k)) D^2 + \sum \nabla((f_j, f_k) \sqcup h_i) D^3 + \sum \nabla((f_j, f_k) \sqcup f_i) D^3 \\
 &\quad + \sum \nabla((f_j, g_k) \sqcup h_i) D^3 + \sum \nabla((f_j, g_k) \sqcup f_i) D^3 + \sum \nabla((g_j, f_k) \sqcup h_i) D^3 \\
 &\quad + \sum \nabla((g_j, f_k) \sqcup f_i) D^3 + \sum \nabla((g_j, g_k) \sqcup h_i) D^3 + \sum \nabla((g_j, g_k) \sqcup f_i) D^3 \\
 &\quad + \sum \nabla(f_k \sqcup (h_i, h_j)) D^4 + \sum \nabla(f_k \sqcup (h_i, f_j)) D^4 + \sum \nabla(f_k \sqcup (f_i, h_j)) D^4 \\
 &\quad + \sum \nabla(f_k \sqcup (f_i, f_j)) D^4 + \sum \nabla(g_k \sqcup (h_i, h_j)) D^4 + \sum \nabla(g_k \sqcup (h_i, f_j)) D^4 \\
 &\quad + \sum \nabla(g_k \sqcup (f_i, h_j)) D^4 + \sum \nabla(g_k \sqcup (f_i, f_j)) D^4 + \sum \nabla((f_i, f_k) \sqcup h_j) D^5 \\
 &\quad + \sum \nabla((f_i, f_k) \sqcup f_j) D^5 + \sum \nabla((f_i, g_k) \sqcup h_j) D^5 + \sum \nabla((f_i, g_k) \sqcup f_j) D^5 \\
 &\quad + \sum \nabla((g_i, f_k) \sqcup h_j) D^5 + \sum \nabla((g_i, f_k) \sqcup f_j) D^5 + \sum \nabla((g_i, g_k) \sqcup h_j) D^5 \\
 &\quad + \sum \nabla((g_i, g_k) \sqcup f_j) D^5 + \sum \nabla(f_j \sqcup (h_i, h_k)) D^6 + \sum \nabla(f_j \sqcup (h_i, f_k)) D^6 \\
 &\quad + \sum \nabla(f_j \sqcup (f_i, h_k)) D^6 + \sum \nabla(f_j \sqcup (f_i, f_k)) D^6 + \sum \nabla(g_j \sqcup (h_i, h_k)) D^6 \\
 &\quad + \sum \nabla(g_j \sqcup (h_i, f_k)) D^6 + \sum \nabla(g_j \sqcup (f_i, h_k)) D^6 + \sum \nabla(g_j \sqcup (f_i, f_k)) D^6).
 \end{aligned}$$

Therefore, the algebraic loop  $\text{Hom}(\mathcal{L}, \mathcal{M})_{\psi}$  has the Moufang property; that is, the equality

$$(f +_{\psi} (g +_{\psi} h)) +_{\psi} f = (f +_{\psi} g) +_{\psi} (h +_{\psi} f)$$

holds if and only if

$$\begin{aligned}
 0 = & \sum \nabla((f_i, h_j) \sqcup f_k) D^1 + \sum \nabla((g_i, h_j) \sqcup f_k) D^1 + \sum \nabla((h_i, f_j) \sqcup f_k) D^1 \\
 & + \sum \nabla((h_i, g_j) \sqcup f_k) D^1 - \sum \nabla((f_i, g_j) \sqcup h_k) D^1 - \sum \nabla((g_i, f_j) \sqcup h_k) D^1 \\
 & + \sum \nabla(f_i \sqcup (g_j, h_k)) D^2 + \sum \nabla(f_i \sqcup (h_j, g_k)) D^2 - \sum \nabla(f_i \sqcup (h_j, f_k)) D^2 \\
 & - \sum \nabla(f_i \sqcup (f_j, h_k)) D^2 - \sum \nabla(g_i \sqcup (h_j, f_k)) D^2 - \sum \nabla(g_i \sqcup (f_j, h_k)) D^2 \\
 & + \sum \nabla((f_j, h_k) \sqcup f_i) D^3 + \sum \nabla((g_j, h_k) \sqcup f_i) D^3 + \sum \nabla((h_j, f_k) \sqcup f_i) D^3 \\
 & + \sum \nabla((h_j, g_k) \sqcup f_i) D^3 - \sum \nabla((f_j, g_k) \sqcup h_i) D^3 - \sum \nabla((g_j, f_k) \sqcup h_i) D^3 \\
 & + \sum \nabla(f_k \sqcup (g_i, h_j)) D^4 + \sum \nabla(f_k \sqcup (h_i, g_j)) D^4 - \sum \nabla(f_k \sqcup (h_i, f_j)) D^4 \\
 & - \sum \nabla(f_k \sqcup (f_i, h_j)) D^4 - \sum \nabla(g_k \sqcup (h_i, f_j)) D^4 - \sum \nabla(g_k \sqcup (f_i, h_j)) D^4 \\
 & + \sum \nabla((f_i, h_k) \sqcup f_j) D^5 + \sum \nabla((g_i, h_k) \sqcup f_j) D^5 + \sum \nabla((h_i, f_k) \sqcup f_j) D^5 \\
 & + \sum \nabla((h_i, g_k) \sqcup f_j) D^5 - \sum \nabla((f_i, g_k) \sqcup h_j) D^5 - \sum \nabla((g_i, f_k) \sqcup h_j) D^5 \\
 & + \sum \nabla(f_j \sqcup (g_i, h_k)) D^6 + \sum \nabla(f_j \sqcup (h_i, g_k)) D^6 - \sum \nabla(f_j \sqcup (h_i, f_k)) D^6 \\
 & - \sum \nabla(f_j \sqcup (f_i, h_k)) D^6 - \sum \nabla(g_j \sqcup (h_i, f_k)) D^6 - \sum \nabla(g_j \sqcup (f_i, h_k)) D^6
 \end{aligned}$$

as required.  $\square$

**Remark 3.13.** A space  $X$  is said to be a *rational space* if  $\pi_*(X)$  is a graded vector space over  $\mathbb{Q}$ . It is well known that  $\pi_*(\Omega X) \otimes \mathbb{Q}$  with the Samelson product  $\langle, \rangle$  is a connected graded Lie algebra  $L_X$  if  $X$  is a 1-connected rational space, where  $\Omega$  is the loop functor from the pointed homotopy category to itself. Moreover, there is a beautiful theorem, originally due to Quillen, which states that any connected graded Lie algebra over  $\mathbb{Q}$  is realized as  $\pi_*(\Omega X) \otimes \mathbb{Q}$  for some simply connected space  $X$ . Therefore, we can obtain the results above with a journey starting from a 1-connected space  $X$ , via the Quillen's theorem (or Quillen minimal model), to the connected graded Lie algebra  $L_X$  with the Samelson product.

## 4 Applications to the rational cohomology

If  $(X, m)$  is a rational H-space, then by the Hopf's theorem (see [19, page 269] and [20]) the rational cohomology is the free commutative algebra with the cohomology cup products ' $\cup$ ' on the set of generators  $S = \{x_1, x_2, \dots, x_r\}$ ; that is,

$$(H^*(X; \mathbb{Q}), +, \cup) = \wedge(x_1, x_2, \dots, x_r).$$

We denote this by  $\wedge(S)$  with  $|x_i| = n_i, i = 1, 2, \dots, r$  and  $n_1 \leq n_2 \leq \dots \leq n_r$ . Moreover, the Hopf-Thom theorem says that  $X$  has rational homotopy type of a product of rational Eilenberg-MacLane spaces, i.e.,

$$X \simeq_{\mathbb{Q}} K(\mathbb{Q}, n_1) \times K(\mathbb{Q}, n_2) \times \dots \times K(\mathbb{Q}, n_r).$$

Recall that the multiplication  $m : X_{\mathbb{Q}} \times X_{\mathbb{Q}} \rightarrow X_{\mathbb{Q}}$  and the cohomology cross product pairing induce a homomorphism of rational cohomology algebras

$$m^* : \wedge(S) \longrightarrow \wedge(S) \otimes \wedge(S).$$

In a similar way, we consider the coproduct of rational cohomology algebras as follows:

$$\wedge(x_1, x_2, \dots, x_r) \sqcup \wedge(x_1, x_2, \dots, x_r) = \wedge(x_1, x_2, \dots, x_r, x'_1, x'_2, \dots, x'_r),$$

where  $|x'_i| = n_i$  for  $i = 1, 2, \dots, r$ .

Just like the previous case in Section 3, we define the following.

**Definition 4.1.** A homomorphism of the free commutative algebras

$$\xi : \wedge(S) \longrightarrow \wedge(S) \sqcup \wedge(S)$$

is called a *cohomology algebra comultiplication* of  $\wedge(S)$  if

$$\pi_1 \circ \xi = 1_{\wedge(S)} = \pi_2 \circ \xi,$$

where  $\pi_1$  and  $\pi_2$  are the projections  $\wedge(S) \sqcup \wedge(S) \rightarrow \wedge(S)$  onto the first and second factors, respectively.

We also recall that the following diagram

$$\begin{array}{ccc} H^*(X; \mathbb{Q}) \otimes H^*(X; \mathbb{Q}) & \xrightarrow{\times} & H^*(X \times X; \mathbb{Q}) \\ & \searrow \cup & \downarrow d^* \\ & & H^*(X; \mathbb{Q}) \end{array}$$

is commutative; that is,

$$x_i \cup x_j = d^*(x_i \times x_j)$$

in  $(H^*(X; \mathbb{Q}), +, \cup) = \wedge(S)$ , where  $\times$  is the cohomology cross product and  $d^*$  is an algebra homomorphism induced by the diagonal map  $d : X \rightarrow X \times X$ . Therefore, we will write the cohomology cup product  $x_i \cup x_j$  as ' $x_i x_j$ ' for convenience.

We now construct concrete cohomology algebra comultiplications in more detail below.

**Definition 4.2.** We define cohomology algebra comultiplications

$$\xi, \rho : \wedge(S) \rightarrow \wedge(S) \sqcup \wedge(S)$$

of  $\wedge(S)$  by

$$\xi(x_s) = x_s + x'_s + A_s$$

and

$$\rho(x_t) = x_t + x'_t + B_t,$$

respectively, where

1.  $A_1 = B_1 = B_2 = 0$ ;
2.  $A_s = \sum_{i,j} x_i x'_j$  for  $s = 2, 3, \dots, n$  and  $i, j \in \{1, 2, \dots, r-1\}$  with  $n_i + n_j = n_s$ ; and
3.  $B_t = \sum_{i,j,k} x_i x_j x'_k$  for  $t = 3, 4, \dots, n$  and  $i, j, k \in \{1, 2, \dots, r-1\}$  with  $n_i + n_j + n_k = n_t$ .

We can also construct algebraic loops  $\text{Hom}(\wedge(S), \wedge(T))_\xi$  and  $\text{Hom}(\wedge(S), \wedge(T))_\rho$  induced by the cohomology algebra comultiplications  $\xi, \rho : \wedge(S) \rightarrow \wedge(S) \sqcup \wedge(S)$ , respectively, where  $\wedge(T)$  is the free commutative algebra on the set of generators  $T = \{y_1, y_2, \dots, y_l\}$  with  $|y_j| = m_j, j = 1, 2, \dots, l$  and  $m_1 \leq m_2 \leq \dots \leq m_l$ .

**Remark 4.3.** We note that the first perturbation of  $\xi$ , and the first and second perturbations of  $\rho$  must be defined by  $A_1 = B_1 = B_2 = 0$  in Definition 4.2 because of the anticommutativity of the cohomology cup products and the dimension reason.

Let  $D_{ij'}^1, D_{ijk'}^2 : \wedge(S) \rightarrow \wedge(S) \sqcup \wedge(S)$  be the decomposable maps defined by

$$D_{ij'}^1(x_1) = 0 = D_{ijk'}^2(x_1) = D_{ijk'}^2(x_2);$$

and

$$D_{ij'}^1(x_s) = x_i x'_j$$

for each  $s = 2, 3, \dots, r$  with  $n_i + n_j = n_s$  and

$$D_{ijk'}^2(x_t) = x_i x_j x'_k$$

for each  $t = 3, 4, \dots, r$  with  $n_i + n_j + n_k = n_t$ , where  $i, j, k \in \{1, 2, \dots, r-1\}$ .

Let  $a = (a_1, a_2, \dots, a_r)$  and  $b = (b_1, b_2, \dots, b_r)$  be the elements of the algebraic loops  $\text{Hom}(\wedge(S), \wedge(T))_\xi$  and  $\text{Hom}(\wedge(S), \wedge(T))_\rho$ . Then we have

$$(a_1, a_2, \dots, a_r) +_\xi (b_1, b_2, \dots, b_r) = (c_1, c_2, \dots, c_r)$$

and

$$(a_1, a_2, \dots, a_r) +_\rho (b_1, b_2, \dots, b_r) = (d_1, d_2, \dots, d_r),$$

where

1.  $c_1 = a_1 + b_1 = d_1$  and  $d_2 = a_2 + b_2$ ;
2.  $c_s = a_s + b_s + \sum_{i,j} (a_i \sqcup b_j) D_{ij'}^1$  and  $d_t = a_t + b_t + \sum_{i,j,k} (a_i \sqcup a_j \sqcup b_k) D_{ijk'}^2$  for  $s = 2, 3, \dots, r$  and  $t = 3, 4, \dots, r$ ; and
3. the  $+\xi$  and  $+\rho$  denote the binary operations on the algebraic loops  $\text{Hom}(\wedge(S), \wedge(T))_\xi$  and  $\text{Hom}(\wedge(S), \wedge(T))_\rho$  derived from the cohomology algebra comultiplications  $\xi = \xi_{(A_1, A_2, \dots, A_r)}$  and  $\rho = \rho_{(B_1, B_2, \dots, B_r)}$ , respectively.

**Definition 4.4.** We define the one-stage cohomology algebra comultiplications  $\xi, \rho : \wedge(S) \rightarrow \wedge(S) \sqcup \wedge(S)$  of  $\wedge(S)$  by

$$\xi(x_s) = x_s + x'_s + A_s,$$

and

$$\rho(x_t) = x_t + x'_t + B_t,$$

respectively. Here

1.  $A_s = B_t = 0$  for  $s, t = 1, 2, \dots, r-1$ ;
2.  $A_r = \sum_{i,j} x_i x'_j$  with  $n_i + n_j = n_r$ ; and
3.  $B_r = \sum_{i,j,k} x_i x_j x'_k$  with  $n_i + n_j + n_k = n_r$ .

We end this paper with the application of the Lie algebra comultiplications to the free commutative algebra cases derived from a rational H-space as follows.

**Proposition 4.5.** *Let  $\xi, \rho : \wedge(S) \rightarrow \wedge(S) \sqcup \wedge(S)$  be the cohomology algebra comultiplications given by Definition 4.4, and let  $a = (a_1, a_2, \dots, a_r)$ ,  $b = (b_1, b_2, \dots, b_r)$  and  $c = (c_1, c_2, \dots, c_r)$  be the elements of the algebraic loops  $\text{Hom}(\wedge(S), \wedge(T))_\xi$  and  $\text{Hom}(\wedge(S), \wedge(T))_\rho$ . Then*

1. *the algebraic loop  $\text{Hom}(\wedge(S), \wedge(T))_\xi$  has the inversive, power-associative and Moufang properties;*
2. *the algebraic loop  $\text{Hom}(\wedge(S), \wedge(T))_\rho$  is inversive or power-associative if and only if*

$$\sum_{i,j,k} \nabla((a_i, a_j) \sqcup a_k) D_{ijk'}^2 = 0;$$

and

3.  *$\text{Hom}(\wedge(S), \wedge(T))_\rho$  has the Moufang property if and only if*

$$0 = \sum_{i,j,k} \nabla((a_i, c_j) \sqcup a_k) D_{ijk'}^2 + \sum_{i,j,k} \nabla((b_i, c_j) \sqcup a_k) D_{ijk'}^2 + \sum_{i,j,k} \nabla((c_i, a_j) \sqcup a_k) D_{ijk'}^2 \\ + \sum_{i,j,k} \nabla((c_i, b_j) \sqcup a_k) D_{ijk'}^2 - \sum_{i,j,k} \nabla((a_i, b_j) \sqcup c_k) D_{ijk'}^2 - \sum_{i,j,k} \nabla((b_i, a_j) \sqcup c_k) D_{ijk'}^2,$$

where  $n_i + n_j + n_k = n_r$ , and  $\nabla : \wedge(T) \sqcup \wedge(T) \rightarrow \wedge(T)$  is the folding homomorphism as in Definition 2.8.

*Proof.* The proof is straightforward and is very similar to those of Theorems 3.6 and 3.12. □

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