DE GRUYTER Open Math. 2019; 17:689–699

Open Mathematics

Research Article

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Razumikhin-type theorem on time-changed stochastic functional differential equations with Markovian switching

https://doi.org/10.1515/math-2019-0055
Received July 21, 2018; accepted May 16, 2019

Abstract: This work is mainly concerned with the exponential stability of time-changed stochastic functional differential equations with Markovian switching. By expanding the time-changed Itô formula and the Razumikhin theorem, we obtain the exponential stability results for the time-changed stochastic functional differential equations with Markovian switching. What's more, we get many useful stability results by applying our new results to several important types of functional differential equations. Finally, an example is given to demonstrate the effectiveness of the main results.

Keywords: Exponential stability; time-changed stochastic differential equations; Razumikhin theorem; Markovian switching

MSC: 34D20, 34K50

1 Introduction

The research for stochastic differential equations (SDEs) is a mature field, which plays an important role in modeling dynamic system considering uncertainty noise in many applied areas such as economics and finance, physics, engineering and so on. Many qualitative properties of the solution of stochastic functional differential equations (SFDEs) have been received much attention. In particular, the stability or asymptotic stability of SFDEs has been studied widely by more and more researchers ([1–5]).

Recently, Chlebak et al.[6] discussed sub-diffusion process and its associated fractional Fokker-Planck-Kolmogorov equations. The fractional partial differential equations are well known to be connected with limit process arising from continuous-time random walks. The limit process is time-changed Lev́y process, which is the first hitting time process of a stable subordinator (see [7–9] for details). The existence and stability of SDE with respect to time-changed Brownian motion recently have received much attention([10, 11]). Wu [12, 13] established the time-changed Itô formula of time-changed SDE, and then obtained the stability results. Subsequently, Nane and Ni [14] established the Itô formula for time-changed Lévy noise, then discussed the stability of the solution.

However, to the best of our knowledge, there are no results for the time-changed stochastic functional differential equations with Markovian switching published till now. Motivated strongly by the above, in this paper, we will study the stability of time-changed SFDEs with Markovian switching. By applying the time-changed Itô formula and Lyapunov function, we present the Razumikhin-type theorem([15, 16]) of the time-

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changed SFDEs with Markovian switching. More precisely, we consider the following SFDEs with Markovian switching driven by time-changed Brownian motions:

$$dx(t) = h(x_t, t, E_t, r(t))dt + f(x_t, t, E_t, r(t))dE_t + g(x_t, t, E_t, r(t))dB_{E_t} \tag{1.1}$$

on $t \geq 0$ with $\{x(\theta) : -\tau \leq \theta \leq 0\} = \xi \in C^b_{\mathscr{F}_0}([-\tau, 0]; \mathbb{R}^n)$, where h, f, g are appropriately specified later.

In the remaining parts of this paper, further needed concepts and related background will be presented in Section 2. In Section 3, the exponential stability results of the time-changed SFDEs with Markovian switching will be given. Many useful types of results of stochastic delay differential equations and stochastic differential equations are presented in Section 4 and Section 5 respectively. Finally, an example is given to show the availability of the main results.

2 Preliminary

Throughout this paper, let $(\Omega, \mathscr{F}, \{\mathscr{F}\}_{t\geq 0}, P)$ be a complete probability space with the filtration $\{\mathscr{F}\}_{t\geq 0}$ which satisfies the usual condition(i.e. $\{\mathscr{F}\}_{t\geq 0}$ is right continuous and $\mathscr{F}$ contains all the P-null sets in $\mathscr{F}$). Let $\{U(t), t \geq 0\}$ be a right continuous with left limit (RCLL) increasing Lévy process that is called subordinator starting from 0. For a subordinator $U(t)$, in particular, is a β-stable subordinator if it is a strictly increasing process denoted by $U_\beta(t)$ and characterized by Laplace transform

$$E[\exp(-sU_\beta(t))] = \exp(-ts^\beta), \; s > 0, \beta \in (0, 1).$$

For an adapted β-stable subordinator $U_\beta(t)$, define its generalized inverse as

$$E_t := E^\beta_t = \inf\{s > 0 : U_\beta(s) > t\},$$

which means the first hitting time process. And E_t is continuous since $U_\beta(t)$ is strictly increasing.

Let B_t be a standard Brownian motion independent on E_t, define the following filtration as

$$\mathscr{F}_t = \bigcap_{s>t} \left\{\sigma[B_r : 0 \leq r \leq s] \vee \sigma[E_r : r \geq 0]\right\},$$

where $\sigma_1 \vee \sigma_2$ denotes the σ-algebra generated by the union of σ-algebras σ_1 and σ_2. It concludes that the time-changed Brownian motion B_{E_t} is a square integrable martingale with respect to the filtration $\{\mathscr{F}_{E_t}\}_{t\geq 0}$. And its quadratic variation satisfies $< B_{E_t}, B_{E_t} >= E_t$.([17])

Let $r(t), t \geq 0$ be a right continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, \ldots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ by

$$P\{r(t + \Delta) = j|r(t) = i\} = \begin{cases} r_{ij}\Delta + o(\Delta) & if\ i \neq j, \\ 1 + r_{ij}\Delta + o(\Delta) & if\ i = j, \end{cases}$$

where $\Delta > 0$, γ_{ij} is the transition rate from i to j if $i \neq j$ and $\gamma_{ii} = -\sum_{i\neq j} \gamma_{ij}$. We assume that the Markov chain $r(t)$ is independent on Brownian motion, it is well known that almost each sample path of $r(t)$ is a right-continuous step function.

For the future use, we formulate the following generalized time-changed Itô formula.

Lemma 2.1. *(The generalized time-changed Itô formula) Suppose $U_\beta(t)$ is a β-stable subordinator and E_t is its associated inverse stable subordinator. Let $x(t)$ be a $\mathscr{F}_{E_t}$ adapted process defined in (1.1). If $V : \mathbb{R}^n\times\mathbb{R}_+\times\mathbb{R}_+\times S \to \mathbb{R}$ is a $C^{2,1,1}(\mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}_+ \times S; \mathbb{R})$ function, let*

$$L_1 V(x_t, t, E_t, i) = V_t(x, t, E_t, i) + V_x(x, t, E_t, i)h(x_t, t, E_t, i) + \sum_{j=1}^{N} \gamma_{ij} V(x, t, E_t, j)$$

and

$$L_2V(x_t,t,E_t,i) = V_{E_t}(x,t,E_t,i) + V_x(x,t,E_t,i)f(x_t,t,E_t,i) + \frac{1}{2}trace[g^TV_{xx}g(x_t,t,E_t,i)],$$

then with probability one

$$\begin{aligned}V(x(t),t,E_t,r(t)) = {} & V(x_0,0,0,r(0)) + \int_0^t L_1V(x_s,s,E_s,r(s))ds \\ & + \int_0^t L_2V(x_s,s,E_s,r(s))dE_s \\ & + \int_0^t V_x(x(s),s,E_s,r(s))g(x_s,s,E_s,r(s))dB_{E_s} \\ & + \int_0^t\int_R [V(x(s),s,E_s,i_0+h(r(s),l)) - V(x(s),s,E_s,r(s))]\mu(ds,dl),\end{aligned}$$

where $\mu(ds,dl) = \nu(ds,dl) - m(dl)ds$ is a martingale measure, $\nu(ds,dl)$ is a Poisson random measure with density $dt\times m(dl)$, in which m is the Lebesgue measure on $\mathbb{R}$.

Proof Let $y = [x,t_1,t_2]^T = [x,t,E_t]^T$, and $G(y(t),r(t)) = V(x_t,t,E_t,r(t))$. Based on the computation rules ([8]), we have

$$dt\cdot dt = dE_t\cdot dE_t = dt\cdot dE_t = dt\cdot dB_{E_t} = dE_t\cdot dB_{E_t} = 0,\ dB_{E_t}\cdot dB_{E_t} = dE_t.$$

Applying the multi-dimensional Itô formula([18]) to $G(y(t),r(t))$ yields that

$$\begin{aligned}G(y(t),r(t)) = {} & G(y(0),r(0)) + \int_0^t G_y(y(s),r(s))dy(s) + \int_0^t \frac{1}{2}dy^TG_{yy}dy + \int_0^t\sum_{j=1}^N\gamma_{ij} \\ & G(y(s),j)ds + \int_0^t\int_R [G(y(s),i_0+h(r(s),l),x(s)) - G(y(s),r(s))]\mu(ds,dl) \\ = {} & G(y(0),r(0)) + \int_0^T [V_x\ V_{t_1}\ V_{t_2}]\begin{bmatrix} hdt+fdE_t+gdB_{E_t} \\ dt_1 \\ dt_2\end{bmatrix} + \int_0^t \frac{1}{2}trace[g^TV_{xx}g]dE_t \\ & + \int_0^t\sum_{j=1}^N\gamma_{ij}V(x(s),s,E_s,j)ds \\ & + \int_0^t\int_R [V(x(s),s,E_s,i_0+h(r(s),l)) - V(x(s),s,E_s,r(s))]\mu(ds,dl) \\ = {} & V(x_0,0,0,r(0)) + \int_0^t V_x(x(s),s,E_s,r(s))g(x_s,s,E_s,r(s))dB_{E_s} \\ & + \int_0^t \left[V_{E_s}(x(s),s,E_s,r(s)) + V_xf(x_s,s,E_s,r(s)) + \frac{1}{2}trace(g^TV_{xx}g)\right]dE_s \\ & + \int_0^t \left[V_t(x(s),s,E_s,r(s)) + V_xh(x_s,s,E_s,r(s)) + \sum_{j=1}^N\gamma_{ij}V(x(s),s,E_s,j)\right]ds\end{aligned}$$

$$+\int_0^t\int_R [V(x(s),s,E_s,i_0+h(r(s),l))-V(x(s),s,E_s,r(s))]\mu(ds,dl).$$

This completes the proof. □

Corollary 2.1. *Suppose $U_\beta(t)$ is a β-stable subordinator and E_t is its associated inverse. Let $x(t)$ be an $\mathscr{F}_{E_t}$ adapted process defined in (1.1). If $V:\mathbb{R}^n\times\mathbb{R}_+\times\mathbb{R}_+\times S\to\mathbb{R}$ is a $C^{2,1,1}(\mathbb{R}^n\times\mathbb{R}_+\times\mathbb{R}_+\times S;\mathbb{R})$ function, then for any stopping time $0\le t_1\le t_2<\infty$*

$$\mathbb{E}V(x(t_2),t_2,E_{t_2},r(t_2))=\mathbb{E}V(x(t_1),t_1,E_{t_1},r(t_1))+\mathbb{E}\int_{t_1}^{t_2}L_1V(x_s,s,E_s,r(s))ds$$
$$+\mathbb{E}\int_{t_1}^{t_2}L_2V(x_s,s,E_s,r(s))dE_s$$

where L_1 and L_2 are defined in the lemma above.

In this paper, the following hypothesis is imposed on the coefficients h,f and g.

(H_1) Both $h,f:\mathbb{R}^n\times\mathbb{R}_+\times\mathbb{R}_+\times S\to\mathbb{R}^n$ and $g:\mathbb{R}^n\times\mathbb{R}_+\times\mathbb{R}_+\times S\to\mathbb{R}^{n\times m}$ are Borel-measurable functions. They satisfy the Lipschitz condition. That is, there is $L>0$ such that

$$|h(\phi_1,t_1,t_2,i)-h(\phi_2,t_1,t_2,i)|\vee|f(\phi_1,t_1,t_2,i)-f(\phi_2,t_1,t_2,i)|$$
$$\vee|g(\phi_1,t_1,t_2,i)-g(\phi_2,t_1,t_2,i)|\le L||\phi_1-\phi_2||$$

for all $t\ge0,\ i\in S$ and $\phi_1,\phi_2\in C([-\tau,0];\mathbb{R}^n)$.

(H_2) If $x(t)$ is an RCLL and $\mathscr{F}_{E_t}$-adapted process, then $h(x_t,t,E_t,r(t)), f(x_t,t,E_t,r(t))$, $g(x_t,t,E_t,r(t))\in\mathcal{L}(\mathscr{F}_{E_t})$, where $\mathcal{L}(\mathscr{F}_{E_t})$ denotes the class of RCLL and $\mathscr{F}_{E_t}$-adapted process.

3 Main results

In this section, we aim to establish the stability results of the system equation (1.1). Firstly, we have to guarantee the existence of the solution of the equation (1.1).

Lemma 3.1. *Under the conditions of* (H_1) *and* (H_2), *for any initial data $\{x(\theta):-\tau\le\theta\le0\}=\xi\in C^b_{\mathscr{F}_0}([-\tau,0];\mathbb{R}^n)$, the equation (1.1) has a unique global solution.*

Proof Let $T>0$ be arbitrary. It is known that ([18]) there is a sequence $\{\tau_k\}_{k\ge0}$ of stopping times such that $0<\tau_0<\tau_1<\cdots<\tau_k\to\infty$ and $r(t)$ is constant on each interval $[\tau_k,\tau_{k+1})$, that is, for each $k\ge0$,

$$r(t)=r(\tau_k),\qquad\qquad \tau_k\le t<\tau_{k+1}.$$

We first consider the equation on $t\in[0,\tau_1\wedge T]$, it becomes

$$dx(t)=h(x_t,t,E_t,r(0))dt+f(x_t,t,E_t,r(0))dE_t+g(x_t,t,E_t,r(0))dB_{E_t}$$

with initial data $x_0=\xi\in C^b_{\mathscr{F}_0}([-\tau,0])$ has a unique solution on $[-\tau,\tau_1\wedge T]$([4, 8]). Next, for $t\in[\tau_1\wedge T,\tau_2\wedge T]$, the equation becomes

$$dx(t)=h(x_t,t,E_t,r(\tau_1\wedge T))dt+f(x_t,t,E_t,r(\tau_1\wedge T))dE_t+g(x_t,t,E_t,r(\tau_1\wedge T))dB_{E_t}$$

with initial data $x_{\tau_1\wedge T}$ given above. Again we know the equation has a unique continuous solution on $[\tau_1\wedge T-\tau,\tau_2\wedge T]$. Repeating the progress, we can see the equation has a unique solution $x(t)$ on $[-\tau,T]$. Since T is arbitrary, the existence and uniqueness have been proved. □

Now, let us consider the exponential stability of equation (1.1). We fix the Markov chain $r(t)$ and let the initial data ξ vary in $C^b_{\mathscr{F}_0}([-\tau,0];\mathbb{R}^n)$. The solution of equation (1.1) is denoted as $x(t;\xi)$ throughout this paper. Assume that $h(0,t,E_t,i)=0, f(0,t,E_t,i)=0, g(0,t,E_t,i)=0$, so the equation (1.1) have a trivial solution $x(t;0)=0$. Next, we establish a new Razumikhin theorem on p-th moment exponential stability for the time-changed SFDEs with Markovian switching.

Theorem 3.1. *Let* (H_1) *and* (H_2) *hold. Let* $\lambda_1,\lambda_2,p,c_1,c_2,\alpha$ *be all positive numbers and* $q>1$. *Assume that there exists a function* $V(x,t,E_t,i)\in C^{2,1,1}(R^n\times[-\tau,\infty)\times[0,\infty)\times S;R_+)$ *such that*

$$c_1|x|^p \le V(x,t,E_t,i)\le c_2|x|^p,\quad (x,t,E_t,i)\in R^n\times[-\tau,\infty)\times[0,\infty)\times S \tag{3.1}$$

and for all $t>0$,

$$\mathbb{E}\left[\max_{1\le i\le N} e^{\alpha E_t}L_jV(\phi,t,E_t,i)\right]\le -\lambda_j\mathbb{E}\left[\max_{1\le i\le N} e^{\alpha E_t}V(\phi(0),t,E_t,i)\right]\ (j=1,2) \tag{3.2}$$

provided $\phi=\{\phi(\theta;-\tau\le\theta\le 0)\}$ *satisfying*

$$\mathbb{E}\left[\min_{1\le i\le N} e^{\alpha E_{t+\theta}}V(\phi(\theta),t+\theta,E_{t+\theta},i)\right]\le q\mathbb{E}\left[\max_{1\le i\le N} e^{\alpha E_t}V(\phi(0),t,E_t,i)\right] \tag{3.3}$$

for all $-\tau\le\theta\le 0$. *Then for all* $\xi\in C^b_{\mathscr{F}_0}([-\tau,0],R^n)$

$$\mathbb{E}|x(t;\xi)|^p\le\frac{c_2}{c_1}\mathbb{E}||\xi||^p e^{-\gamma t},\quad t\ge 0, \tag{3.4}$$

where $\gamma=\min\{\lambda_1,\lambda_2,\log(q)/\tau\}$. *In other words, the trivial solution of equation (1.1) is pth moment exponentially stable and the pth moment Lyapunov exponent is not greater than* $-\gamma$.

Proof For the initial data $\xi\in C^b_{\mathscr{F}_0}([-\tau,0],R^n)$ arbitrarily and we write $x(t;\xi)=x(t)$ simply. Extend $r(t)$ to $[-\tau,0)$ by setting $r(t)=r(0)$, and extend E_t to $[-\tau,0)$ by setting $E_t=E_0$. Let $\varepsilon\in(0,\gamma)$ be arbitrary then set $\overline{\gamma}=\gamma-\varepsilon$. Define

$$U(t)=\sup_{-\tau\le\theta\le 0}\mathbb{E}\left[e^{\overline{\gamma}(t+\theta+E_{t+\theta})}V(x(t+\theta),t+\theta,E_{t+\theta},r(t+\theta))\right]\quad for\ \ t\ge 0.$$

Since $r(t)$ is right continuous, the fact that both E_t and $x(t)$ is continuous and $\mathbb{E}(\sup_{-\tau\le s\le t}|x(s)|^p)<\infty$ for $t\ge 0$, we can see $\mathbb{E}V(x(t),t,E_t,r(t))$ is right continuous on $t\ge-\tau$. Hence $U(t)$ is well defined and right continuous. We claim that

$$D_+U(t):=\limsup_{l\to 0^+}\frac{U(t+l)-U(t)}{t}\le 0\quad for\ all\ t\ge 0. \tag{3.5}$$

To show this, we know that for each $t\ge 0$, either $U(t)>\mathbb{E}[e^{\overline{\gamma}(t+E_t)}V(x(t),t,E_t,r(t))]$ or $U(t)=\mathbb{E}[e^{\overline{\gamma}(t+E_t)}V(x(t),t,E_t,r(t))]$.

Case 1: If $U(t)>\mathbb{E}[e^{\overline{\gamma}(t+E_t)}V(x(t),t,E_t,r(t))]$, it follows from the right continuity of $\mathbb{E}[e^{\overline{\gamma}(t+E_t)}V(x(t),t,E_t,r(t))]$ that for each $l>0$ sufficiently small

$$U(t)>\mathbb{E}[e^{\overline{\gamma}(t+l+E_{t+l})}V(x(t+l),t+l,E_{t+l},r(t+l))]. \tag{3.6}$$

Noting that

$$U(t+l)=\sup_{-\tau\le\theta\le 0}\mathbb{E}\left[e^{\overline{\gamma}(t+l+\theta+E_{t+l+\theta})}V(x(t+l+\theta),t+l+\theta,E_{t+l+\theta},r(t+l+\theta))\right]\quad for\ \ t\ge 0,$$

if $l+\theta>0$, by (3.6),we have

$$\mathbb{E}\left[e^{\overline{\gamma}(t+l+\theta+E_{t+l+\theta})}V(x(t+l+\theta),t+l+\theta,E_{t+l+\theta},r(t+l+\theta))\right]\le U(t).$$

Therefore, $U(t+l)\le U(t)$. On the other hand, if $l+\theta\le 0$, we set $\theta'=l+\theta$, then

$$U(t+l)=\sup_{l-\tau\le\theta'\le 0}\mathbb{E}\left[e^{\overline{\gamma}(t+\theta'+E_{t+\theta'})}V(x(t+\theta'),t+\theta',E_{t+\theta'},r(t+\theta'))\right]$$

$$\leq \sup_{-\tau\leq\theta'\leq 0} \mathbb{E}\left[e^{\overline{\gamma}(t+\theta'+E_{t+\theta'})} V(x(t+\theta'), t+\theta', E_{t+\theta'}, r(t+\theta'))\right] = U(t).$$

Therefore, for each $t > 0$, $U(t+l) \leq U(t)$ and $D_+U(t) \leq 0$.

Case 2: If $U(t) = \mathbb{E}[e^{\overline{\gamma}(t+E_t)} V(x(t), t, E_t, r(t))]$, by the definition of $U(t)$, one obtains that for $-\tau \leq \theta \leq 0$,

$$\mathbb{E}\left[e^{\overline{\gamma}(t+\theta+E_{t+\theta})} V(x(t+\theta), t+\theta, E_{t+\theta}, r(t+\theta))\right] \leq \mathbb{E}\left[e^{\overline{\gamma}(t+E_t)} V(x(t), t, E_t, r(t))\right],$$

it follows that

$$\begin{aligned}\mathbb{E}\left[e^{\overline{\gamma}E_{t+\theta}} V(x(t+\theta), t+\theta, E_{t+\theta}, r(t+\theta))\right] &\leq e^{-\overline{\gamma}\theta}\mathbb{E}\left[e^{\overline{\gamma}E_t} V(x(t), t, E_t, r(t))\right] \\ &\leq e^{\overline{\gamma}\tau}\mathbb{E}\left[e^{\overline{\gamma}E_t} V(x(t), t, E_t, r(t))\right].\end{aligned}$$

If $\mathbb{E}\left[e^{\overline{\gamma}E_t} V(x(t), t, E_t, r(t))\right] = 0$, from (3.1) we can see that

$$\mathbb{E}[e^{\overline{\gamma}E_{t+\theta}} c_1|x(t+\theta)|^p] \leq 0,$$

which yields that $x(t+\theta) = 0,\ -\tau \leq \theta \leq 0$. Since $h(0, t, E_t, i) = 0,\ f(0, t, E_t, i) = 0$ and $g(0, t, E_t, i) = 0$ a.s. for all $-\tau \leq \theta \leq 0$, one obtains that $x(t+l) = 0$ a.s. for all $l > 0$, hence $U(t+l) = 0$ and $D_+U(t) = 0$.

On the other hand, if $\mathbb{E}\left[e^{\overline{\gamma}E_t} V(x(t), t, E_t, r(t))\right] > 0$, one can see that

$$\mathbb{E}\left[e^{\overline{\gamma}E_{t+\theta}} V(x(t+\theta), t+\theta, E_{t+\theta}, r(t+\theta))\right] < q\mathbb{E}\left[e^{\overline{\gamma}E_t} V(x(t), t, E_t, r(t))\right]$$

for all $-\tau \leq \theta \leq 0$ since $e^{\overline{\gamma}\tau} < q$. It follows from the condition (3.2) that

$$\mathbb{E}\left[\max_{1\leq i\leq N} e^{\overline{\gamma}E_t} L_j V(\phi, t, E_t, i)\right] < -\lambda_j \mathbb{E}\left[\max_{1\leq i\leq N} e^{\overline{\gamma}E_t} V(\phi(0), t, E_t, i)\right],\ j = 1, 2.$$

It means that

$$\mathbb{E}\left[e^{\overline{\gamma}E_t} L_j V(x_t, t, E_t, r(t))\right] < -\lambda_j \mathbb{E}\left[e^{\overline{\gamma}E_t} V(x(t), t, E_t, r(t))\right],\ j = 1, 2,$$

then

$$\mathbb{E}\left[e^{\overline{\gamma}E_t}(\overline{\gamma}V(x(t), t, E_t, r(t)) + L_j V(x_t, t, E_t, r(t)))\right] \leq -(\lambda_j - \overline{\gamma})\mathbb{E}[e^{\overline{\gamma}E_t} V(x(t), t, E_t, r(t))] < 0.$$

By the right continuity of the process involved one can see that for all $l > 0$ sufficiently small,

$$\mathbb{E}\left[e^{\overline{\gamma}E_s}(\overline{\gamma}V(x(s), s, E_s, r(s)) + L_j V(x_s, s, E_s, r(s)))\right] \leq 0,\ t \leq s \leq t+l.$$

By the generalized time-changed Itô formula, we get that

$$\begin{aligned}&\mathbb{E}\left[e^{\overline{\gamma}(t+l+E_{t+l})} V(x(t+l), t+l, E_{t+l}, r(t+l))\right] - \mathbb{E}\left[e^{\overline{\gamma}(t+E_t)}(V(x(t), t, E_t, r(t)))\right] \\ &= \mathbb{E}\int_t^{t+l} e^{\overline{\gamma}(s+E_s)}[\overline{\gamma}V(x(s), s, E_s, r(s)) + L_1 V(x_s, s, E_s, r(s))]ds \\ &\quad + \mathbb{E}\int_t^{t+l} e^{\overline{\gamma}(s+E_s)}[\overline{\gamma}V(x(s), s, E_s, r(s)) + L_2 V(x_s, s, E_s, r(s))]dE_s \\ &= \int_t^{t+l} e^{\overline{\gamma}s}\mathbb{E}e^{\overline{\gamma}E_s}[\overline{\gamma}V(x(s), s, E_s, r(s)) + L_1 V(x_s, s, E_s, r(s))]ds \\ &\quad + \int_t^{t+l} e^{\overline{\gamma}s}\mathbb{E}e^{\overline{\gamma}E_s}[\overline{\gamma}V(x(s), s, E_s, r(s)) + L_2 V(x_s, s, E_s, r(s))]dE_s \\ &\leq 0. \end{aligned} \tag{3.7}$$

Then $U(t+l) \le U(t)$ for $l > 0$ sufficiently small.

Since

$$U(t+l) = \sup_{-\tau\le\theta\le 0} \mathbb{E}\left[e^{\overline{\gamma}(t+\theta+l+E_{t+\theta+l})} V(x(t+l+\theta), t+l+\theta, E_{t+l+\theta}, r(t+l+\theta))\right],$$

here we set $\theta' = \theta + l$, if $l+\theta > 0$, then $\mathbb{E}\left[e^{\overline{\gamma}(t+\theta'+E_{t+\theta'})} V(x(t+\theta'), t+\theta', E_{t+\theta'}, r(t+\theta'))\right] \le U(t)$ from (3.7), otherwise, since $U(t) = \mathbb{E}[e^{\overline{\gamma}(t+E_t)} V(x(t), t, E_t, r(t))]$, then

$$\mathbb{E}\left[e^{\overline{\gamma}(t+\theta'+E_{t+\theta'})} V(x(t+\theta'), t+\theta', E_{t+\theta'}, r(t+\theta'))\right] \le U(t),$$

so, by the definition of supremum, $U(t+l) = U(t)$ for $l > 0$ sufficiently small and $D_+U(t) = 0$. Therefore, the inequality (3.5) has been proved. It follows that

$$U(t) \le U(0), \ for\ t \ge 0.$$

$$\mathbb{E}e^{\overline{\gamma}t} c_1|x|^p \le \mathbb{E}e^{\overline{\gamma}(t+E_t)} V(x(t), t, E_t, r(t)) \le U(t) \le U(0) \le c_2\mathbb{E}||\xi||^p$$

this means

$$\mathbb{E}|x|^p \le \frac{c_2}{c_1} e^{-\overline{\gamma}t}\mathbb{E}||\xi||^p = \frac{c_2}{c_1}\mathbb{E}||\xi||^p e^{-(\gamma-\varepsilon)t}.$$

Since ε is arbitrary, the required inequality (3.4) must hold. The proof is completed. □

4 Stochastic delay differential equations with Markovian switching

In this section, as a special case of equation (1.1), we consider the time-changed stochastic delay differential equation with Marking switching as follows,

$$\begin{aligned} dx(t) = {} & H(x(t), x(t-\delta(t)), t, E_t, r(t))dt + F(x(t), x(t-\delta(t)), t, E_t, r(t))dE_t \\ & + G(x(t), x(t-\delta(t)), t, E_t, r(t))dB_{E_t} \end{aligned} \tag{4.1}$$

on $t \ge 0$ with $x_0 = \xi \in C^b_{\mathscr{F}_0}([-\tau, 0]; \mathbb{R}^n)$, where $\delta : \mathbb{R}_+ \to [0, \tau]$ is Borel measure while

$$H, F : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}_+ \times S \to \mathbb{R}^n$$

and

$$G : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}_+ \times S \to \mathbb{R}^{n\times m}.$$

We impose the following hypotheses:

(H_3) Both $H, F : \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}_+ \times S \to \mathbb{R}^n$ and $G : \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}_+ \times S \to \mathbb{R}^{n\times m}$ are Borel-measurable functions. They satisfy the Lipschitz condition. That is, there is $L > 0$ such that

$$\begin{aligned} & |H(x, y, t_1, t_2, i) - H(\overline{x}, \overline{y}, t_1, t_2, i)| \vee |F(x, y, t_1, t_2, i) - F(\overline{x}, \overline{y}, t_1, t_2, i)| \\ & \vee |G(x, y, t_1, t_2, i) - G(\overline{x}, \overline{y}, t_1, t_2, i)| \le L(|x - \overline{x}| + |y - \overline{y}|) \end{aligned}$$

for all $t \ge 0,\ i \in S$ and $x, y, \overline{x}, \overline{y} \in \mathbb{R}^n$.

(H_4) If $x(t)$ is an RCLL and $\mathcal{F}_{E_t}$-adapted process, then $H(x(t), x(t-\delta(t)), t, E_t, r(t)), F(x(t), x(t-\delta(t)), t, E_t, r(t)),\ G(x(t), x(t-\delta(t)), t, E_t, r(t)) \in \mathcal{L}(\mathscr{F}_{E_t})$, where $\mathcal{L}(\mathscr{F}_{E_t})$ denotes the class of RCLL and $\mathscr{F}_{E_t}$-adapted process.

If we define, for $(\phi, t, E_i, i) \in C([-\tau, 0]; \mathbb{R}^n) \times \mathbb{R}^+ \times \mathbb{R}^+ \times S$,

$$\begin{aligned} h(\phi, t, E_t, i) &= H(\phi(0), \phi(-\delta(t)), t, E_t, i), \\ g(\phi, t, E_t, i) &= G(\phi(0), \phi(-\delta(t)), t, E_t, i), \\ f(\phi, t, E_t, i) &= F(\phi(0), \phi(-\delta(t)), t, E_t, i), \end{aligned}$$

then the equation (4.1) becomes the equation (1.1) and (H$_3$) (H$_4$) imply (H$_1$) (H$_2$). So, by Lemma 3.1 ,the equation (4.1) has a unique global solution which is again denoted by $x(t;\xi)$. Furthermore, assume that $H(0,0,t,E_t,i)=0$, $F(0,0,t,E_t,i)=0$, $G(0,0,t,E_t,i)=0$.

If $V\in C^{2,1,1}(\mathbb{R}^n\times[-\tau,\infty)\times[0,\infty)\times S;\mathbb{R}^+)$, define L_1V and L_2V from $\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^+\times\mathbb{R}^+\times S$ to $\mathbb{R}$ respectively by

$$L_1V(x,y,t,E_t,i)=V_t(x,t,E_t,i)+V_x(x,t,E_t,i)H(x,y,t,E_t,i)+\sum_{j=1}^{N}\gamma_{ij}V(x,t,E_t,j),$$

$$L_2V(x,y,t,E_t,i)=V_{E_t}(x,t,E_t,i)+V_x(x,t,E_t,i)F(x,y,t,E_t,i)+\frac{1}{2}trG^TV_{xx}G(x,y,t,E_t,i).$$

Furthermore, we denote $L^p_{\mathscr{F}_t}(\Omega,\mathbb{R}^n)$ as the family of all $\mathscr{F}_t$-measurable $\mathbb{R}^n$-valued random variables X such that $E|X|^p<\infty$. Meanwhile, we set

$$\mathcal{L}_jV(\phi,t,E_t,i)=L_jV(\phi(0),\phi(-\delta(t)),t,E_t,i),\ j=1,2$$

Theorem 4.1. *Let* (H$_3$) *and* (H$_4$) *hold. Let $\lambda_1,\lambda_2,p,c_1,c_2,\alpha$ be all positive numbers and $q>1$. Assume that there exists a function $V(x,t,E_t,i)\in C^{2,1,1}(R^n\times[-\tau,\infty)\times[0,\infty)\times S;R_+)$ such that*

$$c_1|x|^p\le V(x,t,E_t,i)\le c_2|x|^p,\ \ (x,t,E_t,i)\in R^n\times[-\tau,\infty)\times[0,\infty)\times S \tag{4.2}$$

and for all $t>0$,

$$\mathbb{E}\left[\max_{1\le i\le N}e^{\alpha E_t}L_jV(X,Y,t,E_t,i)\right]\le-\lambda_j\mathbb{E}\left[\max_{1\le i\le N}e^{\alpha E_t}V(X,t,E_t,i)\right]\ \ (j=1,2) \tag{4.3}$$

provided $X,Y\in L^p_{\mathscr{F}_t}(\Omega,\mathbb{R}^n)$ satisfying

$$\mathbb{E}\left[\min_{1\le i\le N}e^{\alpha E_{t+\theta}}V(Y,t-\delta(t),E_{t-\delta(t)},i)\right]\le q\mathbb{E}\left[\max_{1\le i\le N}e^{\alpha E_t}V(X,t,E_t,i)\right] \tag{4.4}$$

Then for all $\xi\in C^b_{\mathscr{F}_0}([-\tau,0],R^n)$

$$\mathbb{E}|x(t;\xi)|^p\le\frac{c_2}{c_1}\mathbb{E}||\xi||^pe^{-\gamma t},\ \ t\ge0, \tag{4.5}$$

where $\gamma=\min\{\lambda_1,\lambda_2,\log(q)/\tau\}$. In other words, the trivial solution of equation (4.1) is pth moment exponentially stable and the pth moment Lyapunov exponent is not greater than $-\gamma$.

Proof Let $\phi=\{\phi(\theta):-\tau\le\theta\le0\}\in L^p_{\mathscr{F}_t}([-\tau,0],\mathbb{R}^n)$ satisfy (3.3). For $X=\phi(0)$, $Y=\phi(-\delta(t))\in L^p_{\mathscr{F}_t}(\Omega,\mathbb{R}^n)$ satisfying

$$\mathbb{E}\left[\min_{1\le i\le N}e^{\alpha E_{t+\theta}}V(\phi(-\delta(t)),t-\delta(t),E_{t-\delta(t)},i)\right]\le q\mathbb{E}\left[\max_{1\le i\le N}e^{\alpha E_t}V(\phi(0),t,E_t,i)\right].$$

Then, from (4.3) we have

$$\mathbb{E}\left[\max_{1\le i\le N}e^{\alpha E_t}\mathcal{L}_jV(\phi,t,E_t,i)\right]\le-\lambda_j\mathbb{E}\left[\max_{1\le i\le N}e^{\alpha E_t}V(\phi(0),t,E_t,i)\right]\ \ (j=1,2)$$

which is (3.2). Hence the conditions in Theorem 3.1 are satisfied and the conclusions follow. Applying the Theorem 3.1, the proof is completed. □

Theorem 4.2. *Let* (H$_3$) *and* (H$_4$) *hold. Let p,c_1,c_2,α be all positive numbers and $\lambda_{1j}>\lambda_{2j}\ge0, j=1,2$. Assume that there exists a function $V(x,t,E_t,i)\in C^{2,1,1}(R^n\times[-\tau,\infty)\times[0,\infty)\times S;R_+)$ such that*

$$c_1|x|^p\le V(x,t,E_t,i)\le c_2|x|^p,\ \ (x,t,E_t,i)\in R^n\times[-\tau,\infty)\times[0,\infty)\times S \tag{4.6}$$

and for all $t>0$,

$$\mathbb{E}\left[\max_{1\le i\le N}e^{\alpha E_t}L_jV(X,Y,t,E_t,i)\right]\le-\lambda_{1j}\mathbb{E}\left[\max_{1\le i\le N}e^{\alpha E_t}V(X,t,E_t,i)\right]$$

$$+\lambda_{2j}\mathbb{E}\left[\min_{1\le i\le N} e^{\alpha E_{t+\theta}} V(Y, t-\delta(t), E_{t-\delta(t)}, i)\right] \quad (j=1,2)$$

Then the trivial solution of equation (4.1) is pth moment exponentially stable and the pth moment Lyapunov exponent is not greater than $-\gamma$, where $\gamma = \min\{\lambda_{11} - q\lambda_{21}, \lambda_{12} - q\lambda_{22}, \log(q)/\tau\}$ with $q > 1$.

Proof For $t \ge 0$, $q < \lambda_{1j}/\lambda_{2j}, j = 1, 2$ and $X, Y \in L^p_{\mathscr{F}_t}(\Omega, \mathbb{R}^n)$ satisfying

$$\mathbb{E}\left[\min_{1\le i\le N} e^{\alpha E_{t+\theta}} V(Y, t-\delta(t), E_{t-\delta(t)}, i)\right] \le q\mathbb{E}\left[\max_{1\le i\le N} e^{\alpha E_t} V(X, t, E_t, i)\right],$$

we can arrive that

$$\begin{aligned}
&\mathbb{E}\left[\max_{1\le i\le N} e^{\alpha E_t} L_j V(X, Y, t, E_t, i)\right] \\
&\le -\lambda_{1j}\mathbb{E}\left[\max_{1\le i\le N} e^{\alpha E_t} V(X, t, E_t, i)\right] + \lambda_{2j}\mathbb{E}\left[\min_{1\le i\le N} e^{\alpha E_{t+\theta}} V(Y, t-\delta(t), E_{t-\delta(t)}, i)\right] \\
&\le -(\lambda_{1j} - q\lambda_{2j})\mathbb{E}\left[\max_{1\le i\le N} e^{\alpha E_t} V(X, t, E_t, i)\right],
\end{aligned}$$

that is, (4.3) is satisfied with $\lambda_j = \lambda_{1j} - q\lambda_{2j}, j = 1, 2$. Then the conclusion follows form Theorem 4.1. □

5 Example

Let E_t be generalized inverse of an β-stable subordinator $U_\beta(t)$. Let $B(t)$ be a scalar Brownian motion and $\{r(t)\}$ be a right-continuous Markov chain taking values in $S = \{1, 2\}$ with generator $\Gamma = \{r_{ij}\}_{2\times 2}$, here

$$-\gamma_{11} = \gamma_{12} > 0, \quad \gamma_{21} = -\gamma_{22} > 0.$$

Assume that $B(t)$ and $r(t)$ are independent. Then let us consider the following one-dimensional linear stochastic differential equation with Markovian switching

$$dx(t) = \rho(r(t))x(t)dt + \mu(r(t))x(t-\delta(t))dE_t + \sigma(r(t))x(t-\delta(t))dB_{E_t}, \quad t \ge 0 \tag{5.1}$$

where

$$\rho(1) = -1, \rho(2) = 1; \mu(1) = -\frac{1}{2}, \mu(2) = -\frac{1}{3}; \sigma(1) = 1, \sigma(2) = 1.$$

The equation (5.1) can be regarded as the result of

$$dx(t) = -x(t)dt - \frac{1}{2}x(t-\delta(t))dE_t + x(t-\delta(t))dB_{E_t}, \quad t \ge 0 \tag{5.2}$$

and

$$dx(t) = x(t)dt - \frac{1}{3}x(t-\delta(t))dE_t + x(t-\delta(t))dB_{E_t}, \quad t \ge 0 \tag{5.3}$$

switching to each other according to the movement of the Markovian chain $r(t)$.

We define the function $V : \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+ \times S \to \mathbb{R}_+$ by

$$V(x, t, E_t, i) = c_i|x|^p$$

with $c_i = 1, c_2 = c \in (0, \frac{3}{4})$. The operators have the following forms

$$L_1 V(x, t, E_t, i) = \begin{cases} (c-1-p)|x|^p, & i = 1, \\ (pc + 4 - 4c)|x|^p, & i = 2. \end{cases}$$

$$L_2V(x,t,E_t,i)=\begin{cases}\frac{p(p-1)}{2}|x|^{p-2}|y|^2-\frac{1}{2}p|x|^{p-1}|y|, & i=1,\\ \frac{cp(p-1)}{2}|x|^{p-2}|y|^2-\frac{1}{3}cp|x|^{p-1}|y|, & i=2.\end{cases}$$

Using the following inequality

$$a^\theta b^{1-\theta}\le \theta a+(1-\theta)b,\, a,b>0,\theta\in(0,1),$$

we can see that

$$L_2V(x,t,E_t,i)\le\begin{cases}\frac{(p-1)(p-3)}{2}|x|^p+(p-\frac{3}{2})|y|^p, & i=1,\\ \frac{c(p-1)(p-8)}{2}|x|^p+c(p-4)|y|^p, & i=2.\end{cases}$$

Choose $p=2$, $2<c<3$, then

$$\begin{aligned}L_1V(x,t,E_t,i)&=\begin{cases}(c-3)|x|^p & i=1,\\ (4-2c)|x|^p, & i=2\end{cases}\\ &\le-\min\{3-c,\frac{2c-4}{c}\}\max\{V(x,t,E_t,1),V(x,t,E_t,2)\}.\end{aligned}$$

$$\begin{aligned}L_2V(x,t,E_t,i)&\le\begin{cases}-\frac{1}{2}|x|^p+\frac{1}{2}|y|^p, & i=1,\\ -\frac{c}{3}|x|^p+\frac{2c}{3}|y|^p, & i=2\end{cases}\\ &\le-\frac{1}{2c}\max V(x,t,E_t,1),V(x,t,E_t,2)+\frac{2}{3}\min\{V(x,t,E_t,1),V(x,t,E_t,2)\}.\end{aligned}$$

By the Theorem 4.2 we conclude that the trivial solution of the equation (5.1) is pth moment exponentially stable.

6 Conclusions

The stochastic differential equations(SDEs) driven by time-changed Brownian motions is a new research area for recent years. In this paper, we have studied the exponential stability of the time-changed SDEs with Markovian switching, by expanding the time-changed Itô formula and the time-changed Razumikhin theorem. Our result generalizes that of SDEs in the literature. Due to the more construction of SDEs with time-change than the usual SDEs, our result is not a trivial generalization.

Acknowledgement: The Project is supported by the National Natural Science Foundation of China (11861040,11771198, 11661053).

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