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Random Polygons and Estimations of π

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Abstract: In this paper, we study the approximation of π through the semiperimeter or area of a *random* n-sided polygon inscribed in a unit circle in $\mathbb{R}^2$. We show that, with probability 1, the approximation error goes to 0 as $n \to \infty$, and is roughly sextupled when compared with the classical Archimedean approach of using a regular n-sided polygon. By combining both the semiperimeter and area of these random inscribed polygons, we also construct extrapolation improvements that can significantly speed up the convergence of these approximations.

Keywords: Archimedean polygon; random polygon; random division; extrapolation; Borel-Cantelli lemma

MSC: Primary 00A05; Secondary 60D05, 65C50

1 Introduction

The classical approach to estimate π, the ratio of the circumference of a circle to its diameter, based on the semiperimeter (or area) of *regular* polygons inscribed in or circumscribed about a unit circle in $\mathbb{R}^2$ can be traced to Archimedes more than 2000 years ago [1]. Although the lower bound $\pi \approx 3$ and better estimates such as $\pi \approx 3.125$ were known to the Babylonians and the Egyptians as early as 4000 years ago, it was Archimedes who first used the polygonal method to calculate π to any desired degree of accuracy. On the one hand, Archimedes correctly recognized that π lies between the semiperimeter $\mathcal{S}_n$ of a regular n-sided polygon inscribed in the unit circle and the semiperimeter $\mathcal{S}'_n$ of a similar regular n-gon circumscribed about the circle; On the other hand, being a master of the method of exhaustion, he certainly knew that as n gets larger and larger, both $\mathcal{S}_n$ and $\mathcal{S}'_n$ get closer and closer to π. Furthermore, with the doubling of the sides of the polygons, Archimedes also discovered the following harmonic-geometric-mean relations

$$1/\mathcal{S}_n + 1/\mathcal{S}'_n = 2/\mathcal{S}'_{2n}, \quad \mathcal{S}_n\mathcal{S}'_{2n} = \mathcal{S}^2_{2n}$$

satisfied by the semiperimeters $\mathcal{S}_n = n\sin(\pi/n)$ and $\mathcal{S}'_n = n\tan(\pi/n)$ of the respective regular n-sided polygons inscribed in and circumscribed about the unit circle. These recurrence relations allowed him to actually compute $\mathcal{S}_n$ and $\mathcal{S}'_n$ for $n = 6, 12, 24, 48, 96$ and obtain the famous bounds $223/71 < \pi < 22/7$ (and provided essentially the only tool to obtain more accurate estimates of π for later mathematicians until about the seventeenth century).

To introduce some modern flavor to the ancient Archimedean approach, we consider in this paper the problem of approximating π using the semiperimeter $\mathcal{S}_n$ or area $\mathcal{A}_n$ of an n-sided *random* polygon inscribed in the unit circle. For simplicity, we assume that all vertices are independently and uniformly distributed on the circle. By connecting these vertices consecutively, we then obtain a random polygon inscribed in the unit

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circle. Note that although such random polygons will rarely be regular (when the vertices happen to be all equally spaced on the circle), it is intuitively clear that, as n becomes large, these random vertices tend to spread out and become "evenly" distributed on the circle so that the semiperimeter or area of the circle may still be well approximated by the corresponding semiperimeter or area of the inscribed random polygon. This is confirmed by the strong convergence results stated in the theorem below.

Theorem 1.1. *Given $n \geq 3$, let $\mathcal{S}_n$ and $\mathcal{A}_n$ be the semiperimeter and area of a random inscribed polygon generated by n independent points uniformly distributed on the unit circle. Then, with probability 1, both $\mathcal{S}_n$ and $\mathcal{A}_n$ converge to π as $n \to \infty$.*

Note that Theorem 1.1 improves on the weak convergence results previously obtained by Bélisle [2]. In fact, for n large, we can also obtain the error estimates

$$\mathbb{E}(\mathcal{S}_n) = \pi - \pi^3/n^2 + O(n^{-3}), \quad \mathbb{E}(\mathcal{A}_n) = \pi - 4\pi^3/n^2 + O(n^{-3}).$$

Thus, compared with a regular n-gon which happens to minimize the approximation error, on average, the approximation error is roughly sextupled when a random n-gon is used. Additionally, we will also show that, for both Archimedean and our random approximations of π, by applying extrapolation type techniques [3], it is possible to construct some simple linear combinations of $\mathcal{S}_n$ and $\mathcal{A}_n$ that can greatly improve the accuracy of these approximations.

2 Basic convergence estimates for the Archimedean approximations of π

By using the following well-known elementary estimates (which can be derived, for example, by comparing the areas of ΔOAB, sector OAB and ΔOAD, or somewhat differently, by comparing the lengths of BC, arc $\overset{\frown}{AB}$, and AD, in a unit circle as shown in Fig. 1 below)

$$\sin\theta < \theta < \tan\theta, \qquad 0 < \theta < \pi/2, \tag{1}$$

it is easy to see that $\mathcal{S}_n < \pi < \mathcal{S}'_n$ for all $n \geq 3$. By further applying the related limit

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1,$$

it follows that

$$\lim_{n\to\infty} \mathcal{S}_n = \lim_{n\to\infty} \mathcal{S}'_n = \pi.$$

Moreover, since the function $(\sin x)/x$ is monotone decreasing on the interval $(0, \pi/2)$, the sequence $\{\mathcal{S}_n\}$ increases with n. On the other hand, since the function $(\tan x)/x$ is monotone increasing for $0 < x < \pi/2$, the sequence $\{\mathcal{S}'_n\}$ decreases with n. Thus, as n becomes larger, the estimates provided by $\mathcal{S}_n < \pi < \mathcal{S}'_n$ indeed become more and more accurate. Additionally, we note that while the corresponding areas $\mathcal{A}_n$ and $\mathcal{A}'_n$ of these Archimedean polygons also provide useful approximations of π, with $\mathcal{A}_n = \frac{1}{2}n\sin\frac{2\pi}{n} < \mathcal{S}_n$ and $\mathcal{A}'_n = n\tan\frac{\pi}{n} = \mathcal{S}'_n$, there seems to be no clear advantage in doing so—something Archimedes might have reasonably concluded.

The following lemma provides some improved higher-order estimates for the sine function and will be useful for deriving error estimates for various approximations of π.

Lemma 2.1. *Let $\theta > 0$. Then $\sin\theta < \theta$, $\sin\theta > \theta - \frac{1}{3!}\theta^3$, $\sin\theta < \theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5$, $\sin\theta > \theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \frac{1}{7!}\theta^7$, $\sin\theta < \theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \frac{1}{7!}\theta^7 + \frac{1}{9!}\theta^9$, etc.*

Note that these inequalities correspond precisely to estimates given by the partial sums of the alternating Taylor series of the sine function. By using $\sin\theta > \theta - \theta^3/6$ and $\sin\theta < \theta - \theta^3/6 + \theta^5/120$ for $\theta > 0$, we can

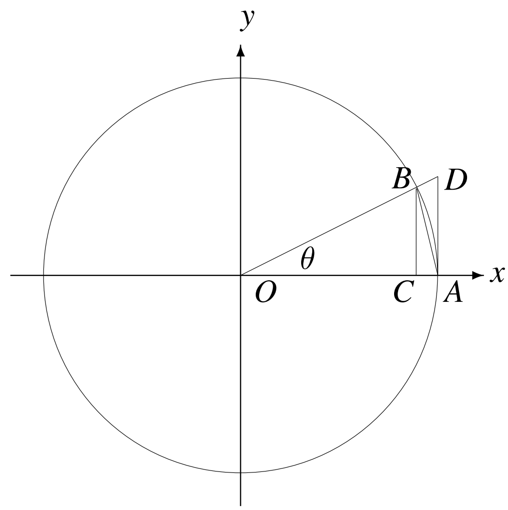

Figure 1: Comparison of areas and lengths in a unit circle: The areas of $\triangle OAB$, sector OAB, and $\triangle OAD$ equal $\frac{1}{2}\sin\theta$, $\frac{1}{2}\theta$ and $\frac{1}{2}\tan\theta$ respectively, hence $\sin\theta = |BC| < \theta = |\overset{\frown}{AB}| < \tan\theta = |AD|$ for all $0 < \theta < \pi/2$. Note that in the case of a unit circle, θ measures exactly the length of the subtending arc $\overset{\frown}{AB}$. In general, the angle θ, measured in radians, is defined as the ratio of the length of arc $\overset{\frown}{AB}$ to the radius of the arc, a quantity that is dimensionless and independent of the radius of the arc.

establish the following error estimates for $\mathcal{S}_n = n\sin(\pi/n)$

$$\pi - \frac{\pi^3}{6n^2} < \mathcal{S}_n < \pi - \frac{\pi^3}{6n^2} + \frac{\pi^5}{120n^4} < \pi \quad \text{for all } n \geq 3.$$

Thus, the approximation error associated with $\mathcal{S}_n$, an under-estimate of π, is slightly less than, but almost precisely $\pi^3/(6n^2)$. On the other hand, for the over-estimate of π given by $\mathcal{S}'_n = n\tan\pi/n$, by using the monotone Taylor series expansion $\tan\theta = \theta + \frac{1}{3}\theta^3 + \frac{2}{15}\theta^5 + \frac{17}{315}\theta^7 + \frac{62}{2835}\theta^9 + \cdots$ for the tangent function, we can obtain

$$\mathcal{S}'_n = \pi + \frac{\pi^3}{3n^2} + \frac{2\pi^5}{15n^4} + \frac{17\pi^7}{315n^6} + \frac{62\pi^9}{2835n^8} + \cdots \approx \pi + \frac{\pi^3}{3n^2} + \frac{2\pi^5}{15n^4},$$

with the approximation error slightly more than $\pi^3/(3n^2)$. In particular, for $n = 96$, we find $\mathcal{S}_{96} - \pi \approx -\pi^3/55296 \approx -5.6 \times 10^{-4}$ and $\mathcal{S}'_{96} - \pi \approx \pi^3/27648 \approx 1.1 \times 10^{-3}$.

It is interesting to note that, as one of the greatest mathematicians of all time, Archimedes was wise enough to have stopped at $n = 96$, but instead suggested taking the average of $\mathcal{S}_{96}$ and $\mathcal{S}'_{96}$ for a better approximation of π. However, had it ever occurred to him that for $\theta = \pi/n$ small, while $\sin\theta < \theta < \tan\theta$, that is, the area of sector OAB is "sandwiched" between those of $\triangle OAB$ and $\triangle OAD$, the difference between the areas of $\triangle OAD$ and sector OAB, is *not the same, but about twice as large* as the difference between the areas of sector OAB and $\triangle OAB$ (see Fig. 2 below for a more complete comparison), he would have arrived at the more useful estimate $\tan\theta - \theta \approx 2(\theta - \sin\theta)$ for θ small; consequently, instead of the simple average $\frac{1}{2}\mathcal{S}_n + \frac{1}{2}\mathcal{S}'_n$, he would have used the weighted average $\mathcal{X}_n = \frac{2}{3}\mathcal{S}_n + \frac{1}{3}\mathcal{S}'_n$ to produce a significantly more accurate estimate of π. (Not until the seventeenth century was such an improvement first pointed out and then rigorously proved by Dutch mathematicians Snellius and Huygens respectively [1].) From the Taylor expansions for $\sin\theta$ and $\tan\theta$, we see that

$$\mathcal{X}_n = \frac{2}{3}\mathcal{S}_n + \frac{1}{3}\mathcal{S}'_n = \pi + \frac{\pi^5}{20n^4} + \frac{\pi^7}{56n^6} + \frac{7\pi^9}{960n^8} + \cdots$$

Thus, even with the modest value of $n = 96$, this would yield $\pi \approx \mathcal{X}_{96} - \pi^5/1698693120$ with an approximation error of about 1.8×10^{-7}, a historic feat that was first achieved by Chinese mathematician Zu Chongzhi more than 7 centuries later by calculating $\mathcal{S}_n$ with $n = 2^{12} \times 3 = 12,288$!

We conclude this discussion by noting that, based on a similar approximate $1:3$ ratio between the area bounded by AB and $\overset{\frown}{AB}$ and the area of $\triangle ACB$, a slightly more accurate estimate for π can be achieved by

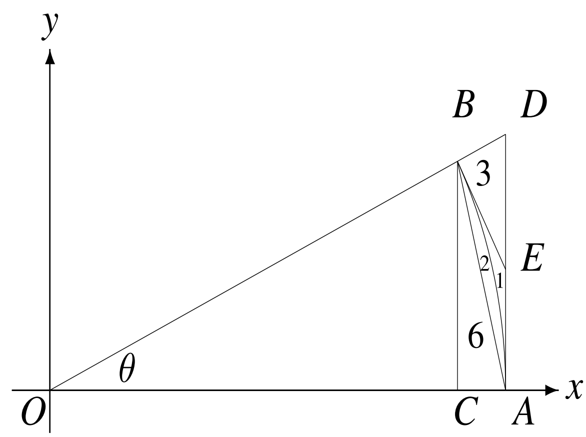

Figure 2: The approximate $1:2:3:6$ ratio for the areas of the four small regions in the trapezoid $ACBD$ separated by AB, arc $\widehat{AB}$, and tangent line BE. The region bounded by $\widehat{AB}$, BE and EA has the smallest area, followed by the region bounded by AB and $\widehat{AB}$, and then $\triangle BED$, and then $\triangle ACB$.

using the following combination of $\mathcal{S}_n = \mathcal{A}_{2n}$ and $\mathcal{A}_n$ (which may also be viewed as an application of modern extrapolation techniques in numerical analysis [3])

$$\mathcal{Y}_n = \frac{4}{3}\mathcal{S}_n - \frac{1}{3}\mathcal{A}_n = \frac{4}{3}\mathcal{A}_{2n} - \frac{1}{3}\mathcal{A}_n = \pi - \frac{\pi^5}{30n^4} + \frac{\pi^7}{252n^6} - \frac{\pi^9}{4320n^8} + \cdots \tag{2}$$

and further improvements can be obtained by combining $\mathcal{S}_n$, $\mathcal{S}'_n$ and $\mathcal{A}_n$ in the form

$$\mathcal{Z}_n = \frac{2}{5}\mathcal{X}_n + \frac{3}{5}\mathcal{Y}_n = \frac{16}{15}\mathcal{S}_n + \frac{2}{15}\mathcal{S}'_n - \frac{1}{5}\mathcal{A}_n = \pi + \frac{\pi^7}{105n^6} + \frac{\pi^9}{360n^8} + \cdots$$

and in numerous more ways by also utilizing earlier values such as $\mathcal{S}_{n/2}$, $\mathcal{S}'_{n/2}$, $\mathcal{A}_{n/2}$, etc.

3 Approximation of π through the semiperimeter or area of a random cyclic n-gon

We now turn to the related but more interesting problem of approximating π through the semiperimeter or area of a randomly selected n-gon inscribed in a unit circle, adding another modern twist to Archimedes' ancient approach. For definiteness, we assume that the vertices of the n-gon are independently and uniformly distributed on the circle. Our main goal is to show that, as $n \to \infty$, the semiperimeter $\mathcal{S}_n$ and area $\mathcal{A}_n$ of such a random n-gon each converges to π with probability 1, that is, $\mathbb{P}(\mathcal{S}_n \to \pi) = \mathbb{P}(\mathcal{A}_n \to \pi) = 1$. This in turn implies convergence of $\mathcal{S}_n \to \pi$ and $\mathcal{A}_n \to \pi$ in probability and in mean square as well.

Suppose the vertices of such an n-gon are labeled $P_0, P_1, \ldots, P_{n-1}, P_n$ in counterclockwise direction with $\theta_0 < \theta_1 < \cdots < \theta_{n-1} < \theta_n = \theta_0 + 2\pi$ and P_n representing the same point as P_0 on the circle. Here θ_i equals the length of the arc from the fixed reference point $(1, 0)$ to P_i, while $\theta_{i+1} - \theta_i$ gives the length of the arc $\widehat{P_iP_{i+1}}$ on the unit circle. The semiperimeter $\mathcal{S}_n$ and area $\mathcal{A}_n$ of the n-gon are then given by

$$\mathcal{S}_n = \sum_{i=1}^{n} \sin\frac{\theta_i - \theta_{i-1}}{2}, \quad \mathcal{A}_n = \frac{1}{2}\sum_{i=1}^{n} \sin(\theta_i - \theta_{i-1}).$$

Note that, since $\sin\theta < \theta$ for all $\theta > 0$, again we have $\mathcal{A}_n < \mathcal{S}_n < \pi$. In fact, we also have $\mathcal{S}_n \le n\sin\frac{\pi}{n}$, $\mathcal{A}_n \le \frac{1}{2}n\sin\frac{2\pi}{n}$. For $\mathcal{S}_n$, this follows easily from the inequality $\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \le 2\sin\frac{\alpha+\beta}{2}$ for all $0 \le \alpha, \beta \le \pi$. For $\mathcal{A}_n$, the same argument applies if $\theta_i - \theta_{i-1} \le \pi$ holds for all i; and while this is not true, the exception $\theta_i - \theta_{i-1} > \pi$ occurs with only one index, say $i = i_\star$, then $\mathcal{A}_n \le \frac{1}{2}\sum_{i \ne i_\star} \sin(\theta_i - \theta_{i-1}) \le$

$\frac{1}{2}(n-1)\sin\frac{2\pi-(\theta_{i_*}-\theta_{i_*-1})}{n-1} \le \frac{1}{2}(n-1)\sin\frac{\pi}{n-1} \le \frac{1}{2}n\sin\frac{2\pi}{n}$ for all $n \ge 3$. Thus, in terms of approximating π through either $\mathcal{S}_n$ or $\mathcal{A}_n$, the regular n-gon always outperforms a random n-gon. (In an extreme case when all n vertices are highly clustered, both $\mathcal{S}_n$ and $\mathcal{A}_n$ can be nearly 0.) Nevertheless, it turns out, convergence remains true for both $\mathcal{S}_n$ and $\mathcal{A}_n$; and in fact, it is not at all a bad idea to use the semiperimeter or area of a random n-gon to approximate π since a typical approximation error is only about 6 times that of the regular n-gon.

Before we proceed, we mention that the main difficulty in establishing the convergence of $\mathcal{S}_n \to \pi$ or $\mathcal{A}_n \to \pi$ as $n \to \infty$ arises from the lack of independence among $\theta_i - \theta_{i-1}$ for $1 \le i \le n$ (with their sum being 2π). The key to our proof is to use Lemma 2.1 to establish a tight lower bound for $\mathbb{E}(\mathcal{S}_n)$ and $\mathbb{E}(\mathcal{A}_n)$ with $\mathbb{E}(|\mathcal{S}_n - \pi|) \to 0$ and $\mathbb{E}(|\mathcal{A}_n - \pi|) \to 0$ sufficiently fast as $n \to \infty$. In particular, we will exploit the symmetry (all vertices are independent and identically distributed) which implies that all $\theta_i - \theta_{i-1}$ are also identically distributed.

Without loss of generality, we assume $\theta_0 = 0$. To further simplify the calculations below, we also write $\theta_i = 2\pi X_i$, $0 \le i \le n$ so that $0 = X_0 < X_1 < X_2 < \cdots < X_{n-1} < X_n = 1$ corresponds to a random division [4–6] of the unit interval by $n-1$ uniformly distributed random points, with the lengths of the resulting n segments $X_i - X_{i-1} = (2\pi)^{-1}(\theta_i - \theta_{i-1})$ all identically distributed. Since $X_1 = \min\{X_1, X_2, \ldots, X_{n-1}\}$, it follows that, for any $0 < x < 1$, $\mathbb{P}(X_1 > x) = \mathbb{P}(X_i > x \text{ for all } 1 \le i \le n-1) = (1-x)^{n-1}$, and thus the probability density function of X_1, and hence of each $X_i - X_{i-1}$, is given by $f(x) = (n-1)(1-x)^{n-2}$. Consequently,

$$\mathbb{E}(|X_i - X_{i-1}|^k) = (n-1)\int_0^1 x^k(1-x)^{n-2}\,dx = (n-1)\frac{k!(n-2)!}{(k+n-1)!} = \frac{k!(n-1)!}{(k+n-1)!}. \tag{3}$$

In particular, for $k = 1, 2, 3$, we have

$$\mathbb{E}(|X_i - X_{i-1}|) = \frac{1}{n}, \quad \mathbb{E}(|X_i - X_{i-1}|^2) = \frac{2}{n(n+1)}, \quad \mathbb{E}(|X_i - X_{i-1}|^3) = \frac{6}{n(n+1)(n+2)}. \tag{4}$$

We now turn to estimate $\mathbb{E}(|\mathcal{S}_n - \pi|)$. First, by using the inequality $\sin\theta > \theta - \frac{1}{3!}\theta^3$ for all $\theta > 0$, we can easily obtain

$$|\mathcal{S}_n - \pi| = \pi - \mathcal{S}_n \le \frac{\pi^3}{6}\sum_{i=1}^{n}(X_i - X_{i-1})^3.$$

With (4), this yields

$$\mathbb{E}(|\mathcal{S}_n - \pi|) \le \frac{\pi^3}{6}\sum_{i=1}^{n}\mathbb{E}(|X_i - X_{i-1}|^3) = \frac{\pi^3}{(n+1)(n+2)} \to 0 \quad \text{as } n \to \infty.$$

Thus, by Markov inequality [7, 8], we have, for any $\varepsilon > 0$,

$$\mathbb{P}(|\mathcal{S}_n - \pi| > \varepsilon) \le \frac{1}{\varepsilon}\mathbb{E}(|\mathcal{S}_n - \pi|) \le \frac{\pi^3}{(n+1)(n+2)\varepsilon} \to 0 \quad \text{as } n \to \infty.$$

This proves $\mathcal{S}_n \to \pi$ in probability as $n \to \infty$. Furthermore, we have

$$\sum_{n=3}^{\infty}\mathbb{P}(|\mathcal{S}_n - \pi| > \varepsilon) \le \sum_{n=3}^{\infty}\frac{\pi^3}{(n+1)(n+2)\varepsilon} = \frac{\pi^3}{4\varepsilon} < \infty.$$

By applying Borel-Cantelli lemma [7, 8], we see that $|\mathcal{S}_n - \pi| > \varepsilon$ occurs finitely often. This implies $\mathcal{S}_n \to \pi$ with probability 1, that is, $\mathbb{P}(\mathcal{S}_n \to \pi) = 1$. Additionally, since $|\mathcal{S}_n - \pi| \le \pi$, we also have the following mean square convergence of $\mathcal{S}_n \to \pi$ as $n \to \infty$:

$$\mathbb{E}(|\mathcal{S}_n - \pi|^2) \le \pi\mathbb{E}(|\mathcal{S}_n - \pi|) = \frac{\pi^4}{(n+1)(n+2)} \to 0 \quad \text{as } n \to \infty.$$

With slight modifications in the calculations above, we can obtain similar convergence results for $\mathcal{A}_n$:

$$\mathbb{E}(|\mathcal{A}_n - \pi|) \le \frac{4\pi^3}{(n+1)(n+2)}, \quad \mathbb{E}(|\mathcal{A}_n - \pi|^2) \le \frac{4\pi^4}{(n+1)(n+2)},$$

and for all $\varepsilon > 0$,

$$\mathbb{P}(|\mathcal{A}_n - \pi| > \varepsilon) \le \frac{1}{\varepsilon}\mathbb{E}(|\mathcal{A}_n - \pi|) \le \frac{4\pi^3}{(n+1)(n+2)\varepsilon} \to 0 \quad \text{as } n \to \infty,$$

$$\sum_{n=3}^{\infty} \mathbb{P}(|\mathcal{A}_n - \pi| > \varepsilon) \le \sum_{n=3}^{\infty} \frac{4\pi^3}{(n+1)(n+2)\varepsilon} = \frac{\pi^3}{\varepsilon} < \infty.$$

Similar to (2), we can further show that, the combination $\mathcal{Y}_n = \frac{4}{3}\mathcal{S}_n - \frac{1}{3}\mathcal{A}_n$ satisfies

$$\mathcal{Y}_n = \pi - \frac{\pi^5}{30}\sum_{i=1}^{n}(X_i - X_{i-1})^5 + \frac{\pi^7}{252}\sum_{i=1}^{n}(X_i - X_{i-1})^7 - \frac{\pi^9}{4320}\sum_{i=1}^{n}(X_i - X_{i-1})^9 + \cdots,$$

$$\mathbb{E}(|\mathcal{Y}_n - \pi|) \le \frac{\pi^5}{30}\sum_{i=1}^{n}\mathbb{E}(|X_i - X_{i-1}|^5) = \frac{4\pi^5}{(n+1)(n+2)(n+3)(n+4)} \to 0 \quad \text{as } n \to \infty,$$

and for any $\varepsilon > 0$,

$$\mathbb{P}(|\mathcal{Y}_n - \pi| > \varepsilon) \le \frac{1}{\varepsilon}\mathbb{E}(|\mathcal{Y}_n - \pi|) \le \frac{4\pi^5}{(n+1)(n+2)(n+3)(n+4)\varepsilon} \to 0 \quad \text{as } n \to \infty,$$

$$\sum_{n=3}^{\infty} \mathbb{P}(|\mathcal{Y}_n - \pi| > \varepsilon) \le \sum_{n=3}^{\infty} \frac{4\pi^5}{(n+1)(n+2)(n+3)(n+4)\varepsilon} = \frac{\pi^5}{90\varepsilon} < \infty.$$

Note that while the average approximation error for $\mathcal{Y}_n$ is now about 120 times that associated with a regular n-gon, it converges to π much faster than $\mathcal{S}_n$ and $\mathcal{A}_n$ for large n. It should be clear that, with the doubling of the sides of such a random n-gon, further extrapolation improvements may be obtained [9] by combining $\mathcal{S}_n$ and $\mathcal{A}_n$ with the corresponding semiperimeter and area of a suitably constructed $2n$-sided random polygon inscribed in the unit circle. In fact, besides the above mentioned strong convergence results, central limit theorem type (weak) convergence estimates also hold for these random approximations of π [2, 9].

On the other hand, by using (3) and the uniform and absolute convergence of the Taylor series of sine function on the interval $[0, 2\pi]$ (or tighter estimates described in Section 2), we can obtain

$$\mathbb{E}(\mathcal{S}_n) = n(n-1)\int_0^1 (\sin \pi x)(1-x)^{n-2}\,dx = \pi + \sum_{k=1}^{\infty}(-1)^k \frac{n!}{(n+2k)!}\pi^{2k+1} = \pi - \frac{\pi^3}{n^2} + O(n^{-3}),$$

$$\mathbb{E}(\mathcal{A}_n) = \frac{1}{2}n(n-1)\int_0^1 (\sin 2\pi x)(1-x)^{n-2}\,dx = \pi + \frac{1}{2}\sum_{k=1}^{\infty}(-1)^k \frac{n!}{(n+2k)!}(2\pi)^{2k+1} = \pi - \frac{4\pi^3}{n^2} + O(n^{-3}),$$

$$\mathbb{E}(\mathcal{Y}_n) = \frac{4}{3}\mathbb{E}(\mathcal{S}_n) - \frac{1}{3}\mathbb{E}(\mathcal{A}_n) = \pi - \sum_{k=2}^{\infty}(-1)^k \frac{4^k - 4}{3}\frac{n!}{(n+2k)!}\pi^{2k+1} = \pi - \frac{4\pi^5}{n^4} + O(n^{-5}),$$

or alternatively, by repeatedly using integration by parts, the following finite sum expression

$$\mathbb{E}(\mathcal{S}_n) = \begin{cases} \displaystyle\sum_{k=1}^{(n-1)/2} (-1)^{k-1}\frac{n!}{(n-2k)!}\frac{1}{\pi^{2k-1}} & \text{for } n \text{ odd},\\ \displaystyle\sum_{k=1}^{n/2}(-1)^{k-1}\frac{n!}{(n-2k)!}\frac{1}{\pi^{2k-1}} + (-1)^{n/2-1}\frac{n!}{\pi^{n-1}} & \text{for } n \text{ even}, \end{cases}$$

$$\mathbb{E}(\mathcal{A}_n) = \frac{1}{2}\sum_{k=1}^{\lfloor (n-1)/2 \rfloor} (-1)^{k-1}\frac{n!}{(n-2k)!}\frac{1}{(2\pi)^{2k-1}} \quad \text{for all } n \ge 3.$$

We mention that, while only random inscribed polygons are considered in this paper, most of our convergence results actually also hold for random *circumscribing* polygons [10] that are tangent to the circle at each of the prescribed random points. However, unlike the classical Archimedean case, such a circumscribing random polygon is not always well-defined (when all random points fall on a semicircle), and even if it exists, its semiperimeter or area can still be unbounded. Finally, similar convergence results also hold for certain random cyclic polygons whose vertices are no longer independently and uniformly distributed on the circle. We refer to [10, 11] for details.

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