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A. Ghareeb, H. S. Al-Saadi, and O. H. Khalil*

A new representation of α-openness, α-continuity, α-irresoluteness, and α-compactness in L-fuzzy pretopological spaces

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Abstract: This paper presents a new representation of α-openness, α-continuity, α-irresoluteness, and α-compactness based on L-fuzzy α-open operators introduced by Nannan and Ruiying [1] and implication operation. The proposed representation extends the properties of α-openness, α-continuity, α-irresoluteness, and α-compactness to the setting of L-fuzzy pretopological spaces based on graded concepts. Moreover, we introduce and establish the relationships among the new concepts.

Keywords: L-fuzzy pretopology, L-fuzzy α-open operator, L-fuzzy α-openness degree, L-fuzzy α-continuty degree, L-fuzzy α-irresolutness degree, L-fuzzy α-compactness degree

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1 Introduction

Continuity is an important concept in topology, which has developed extensively with the emergence of fuzzy mathematics. In [2, 3], Šostak considered the degrees to which a mapping is continuous, open, and closed between two (L, M)-fuzzy topological spaces (including the fuzzifying case) for the first time. Subsequently, the degrees of continuity, openness, and closeness of mappings between L-fuzzifying topological spaces were discussed in detail by Pang [4]. Later on, Liang and Shi [5] clarified the relationship among these degrees and the degree of compactness and connectedness in the case of (L, M)-fuzzy setting.

Recently, Shi [6] measured preopenness and semiopenness of L-subset by introducing the concepts of L-fuzzy preopen operators and L-fuzzy semiopen operators, respectively. In [7], Shi and Li used L-fuzzy semiopen operators to introduce and characterize the semicompactness. Later on in [8] the degree of preconnectedness was introduced with the help of L-fuzzy preopen operators. In addition, he used Shi's operators to define new operators such as L-fuzzy semipreopen operators [9] and L-fuzzy **F**-open operators [10]. These

A. Ghareeb: Department of Mathematics, College of Science, Al-Baha University, Al-Baha, Saudi Arabia
Permanent address: Department of Mathematics, Faculty of Science, South Valley University, Qena, Egypt; E-mail: a.ghareeb@sci.svu.edu.eg, nasserfuzt@hotmail.com
H. S. Al-Saadi: Mathematics Department, Faculty of Applied Sciences, Umm Al-Qura University, Makkah 21955, P. O Box 11155, Saudi Arabia; Email : hasa112@hotmail.com
***Corresponding Author: O. H. Khalil:** Department of Mathematics, College of Science, Majmaah University, Al-Majmaah, 11952, Saudi Arabia
Permanent address: Department of Mathematics, Faculty of Science, Beni-Suef University, Beni-Suef, Egypt; E-mail: o.khalil@mu.edu.sa

operators have proved to be of great importance in studying the characteristics of many concepts of L-fuzzy topology (see [11–14]).

In [1], Nannan and Ruiying introduced L-fuzzy α-open operators in L-fuzzy topological spaces and used it to study L-fuzzy α-compactness. Moreover, the concept of open cover and α-fuzzy α-compact are given and its related properties are discussed. Also, the relationship between L-fuzzy α-compactness and fuzzy α-compactness are discussed.

This paper first discusses some important properties of L-fuzzy α-open operators. It then introduces α-openness, α-continuity, α-irresoluteness, and α-compactness degree based on the implication operation and L-fuzzy α-open operators. Further, some important properties of α-openness, α-continuity, α-irresoluteness, and α-compactness degree were extended to the setting of L-fuzzy pretopology based on graded concepts. Moreover, it presents a systematic discussion on the relationship among the new concepts.

2 Preliminaries

In the sequel, $X \neq \emptyset$, and L refers to a completely distributive De Morgan algebra (briefly, CDDA). Let $\mathbf{1}_L$ and $\mathbf{0}_L$ denote the greatest and smallest elements of L, respectively. For each $u, v \in L$, the element u is wedge below v [15], written $u \lhd v$, if for each $\mathcal{D} \subseteq L$, $\bigvee \mathcal{D} \geq v$ yields to $w \geq u$ for some $w \in \mathcal{D}$. We say the complete lattice L is completely distributive (briefly, CD) if and only if $v = \bigvee\{u \in L | u \lhd v\}$ for any $v \in L$. A member $u \in L$ is said to be co-prime if $u \leq v \vee w$ yields to $u \leq v$ or $u \leq w$. $P(L)$ and $J(L)$ refer to the family of non-unit prime members and non-zero co-prime members of L respectively. The greatest minimal family and the greatest maximal family of $v \in L$ are denoted by $\alpha(v)$ and $\beta(v)$ respectively. Moreover, $\alpha^\star(v) = \alpha(v) \cap J(L)$ and $\beta^\star(v) = \beta(v) \cap P(L)$. By L^X we refer to the set of all L-subsets on X. $2^{\mathcal{U}}$ denotes the collection of all finite sub-collections of $\mathcal{U} \subseteq L^X$. Evidently, L^X is a CDDA when it inherits the structure of the lattice L in a natural way, by defining $\bigvee$, $\bigwedge$, $\leq$ and $'$ pointwisely. Further, $\{x_u | u \in J(L)\}$ denotes the collection of non-zero co-primes of L^X.

For each CDDA L, there exists an implication operation $\mapsto: L \times L \longrightarrow L$ as the right adjoint for the meet operation $\wedge$ is defined by

$$u \mapsto v = \bigvee\{w \in L | u \wedge w \leq v\}.$$

Further, the operation $\leftrightarrow$ is given by

$$u \leftrightarrow v = (u \mapsto v) \wedge (v \mapsto u).$$

The following lemma lists some important properties of implication operation.

Lemma 2.1. *[16] Let $(L, \bigvee, \bigwedge)$ be a CD lattice and $\mapsto$ be the implication operation corresponding to $\wedge$. Then for all $u, v, w \in L$, $\{u_i\}_{i\in\Gamma}$, and $\{v_i\}_{i\in\Gamma} \subseteq L$, we have the following statements:*

- ***(I1)*** $(u \mapsto v) \geq w \Longleftrightarrow u \wedge w \leq v$.
- ***(I2)*** $u \leq v \Leftrightarrow u \mapsto v = \mathbf{1}_L$.
- ***(I3)*** $u \mapsto (v \mapsto w) = (u \wedge v) \mapsto w$.
- ***(I4)*** $(w \mapsto u) \wedge (u \mapsto v) \leq w \mapsto v$.
- ***(I5)*** $w \mapsto u \leq (u \mapsto v) \mapsto (w \mapsto v)$.
- ***(I6)*** $u \mapsto \bigwedge_{i\in\Gamma} u_i = \bigwedge_{i\in\Gamma} (u \mapsto u_i)$, *hence* $u \mapsto v \leq u \mapsto w$ *whenever* $v \leq w$.
- ***(I7)*** $\bigvee_{i\in\Gamma} u_i \mapsto v = \bigwedge_{i\in\Gamma} (u_i \mapsto v)$, *hence* $u \mapsto w \geq v \mapsto w$ *whenever* $u \leq v$.

An L-fuzzy inclusion [17, 18] on X is defined by the function $\tilde{\subset} : L^X \times L^X \longrightarrow L$, where $\tilde{\subset}(A_1, A_2) = \bigwedge_{x\in X}(A_1'(x) \vee A_2(x))$. We shall denote an L-fuzzy inclusion by $[A_1 \tilde{\subset} A_2]$. For each function $f : X \longrightarrow Y$ and $\mathcal{C} \subseteq L^Y$, the next equality is defined in [19]:

$$\bigwedge_{y\in Y}\left\{f^{\rightarrow}(A)'(y)\vee\bigvee_{B\in\mathcal{C}}B(y)\right\}=\bigwedge_{x\in X}\left\{B'(x)\vee\bigvee_{B\in\mathcal{C}}f^{\leftarrow}(B)(x)\right\}.$$

An L-topological space (briefly, L-ts) is a pair (X,τ), where the subfamily $\tau\subseteq L^X$ contains $\underline{\mathbf{0}}_{L^X}$, $\underline{\mathbf{1}}_{L^X}$, and closed for any suprema and finite infima. Elements of τ are called open L-subsets and their complements are called closed L-subsets. For an L-subset A of an L-topological space (X,τ) we denote by $\bar{A}$ and A° the closure and the interior of A, respectively.

Definition 2.2. [16, 20–22] An L-fuzzy pretopology is given by the function $\sigma : L^X \longrightarrow L$ satisfies the following statements:

- **($\mathcal{O}$1)** $\sigma(\underline{\mathbf{1}}_{L^X})=\sigma(\underline{\mathbf{0}}_{L^X})=\mathbf{1}_L$.
- **($\mathcal{O}$2)** $\sigma\left(\bigvee_{i\in I}A_i\right)\geq\bigwedge_{i\in I}\sigma(A_i),\ \forall\{A_i\}_{i\in I}\subseteq L^X$.

For any L-subset $A\in L^X$, $\sigma(A)$ refers to the degree of openness of A. $\sigma^{\star}(A)=\sigma(A')$ is the closeness degree of A. The pair (X,σ) is said to be an L-fuzzy pretopological space (briefly, L-pfts). A function $f:(X,\sigma_1)\longrightarrow(Y,\sigma_2)$ is said to be L-fuzzy continuous with respect to L-fpts's (X,σ_1) and (Y,σ_2) if and only if $\sigma_1(f^{\leftarrow}(B))\geq\sigma_2(B)$ for each $B\in L^Y$, where $f^{\leftarrow}(B)(x)=B(f(x))$.

Definition 2.3. [1] Let σ be an L-fpt on X and let the mapping $\mathscr{A}:L^X\longrightarrow L$ defined as follows:

$$\mathscr{A}(A)=\bigvee_{B\leq A}\left\{\sigma(B)\wedge\bigwedge_{x_u\lhd A}\bigvee_{x_u\lhd C}\left\{\sigma(C)\wedge\bigwedge_{y_v\lhd C}\bigwedge_{y_v\not\leq D\geq B}(\sigma(D'))'\right\}\right\}.$$

In this case, $\mathscr{A}$ is the induced L-fuzzy α-open operator by σ. $\mathscr{A}(A)$ is called the degree of α-openness of A and $\mathscr{A}^{\star}(A)=\mathscr{A}(A')$ can be regarded as the α-closeness degree of A.

Corollary 2.4. *If σ is an L-fpt on X and $A\in L^X$, then:*

$$\mathscr{A}(A)=\bigvee_{B\leq A}\left\{\sigma(B)\wedge\bigwedge_{x_u\lhd A}\bigvee_{x_u\lhd C}\left\{\sigma(C)\wedge\bigwedge_{y_v\lhd C}Cl^{\sigma}(B)(y_v)\right\}\right\},$$

where Cl^{σ} refers to the L-fuzzy closure operator induced by σ (see [23]).

Theorem 2.5. *[1] Let σ be an L-fpt on X, $A\in L^X$, and $u\in J(L)$, then $A\in\mathscr{A}_{[u]}$ if and only if A is an α-open set in $\mathscr{A}_{[u]}$, where $\mathscr{A}_{[u]}=\{A\in L^X\mid\mathscr{A}(A)\geq u\}$.*

Theorem 2.6. *Let $\sigma:L^X\longrightarrow\{\mathbf{0}_L,\mathbf{1}_L\}$ be an L-pts and let $\mathscr{A}:L^X\longrightarrow\{\mathbf{0}_L,\mathbf{1}_L\}$ be the corresponding L-α-open operator. Then $\mathscr{A}(A)=\mathbf{1}_L$ if and only if A is α-open L-subset.*

Proof. We can prove the theorem by using the following fact:

$$\mathscr{A}(A)=\mathbf{1}_L\Leftrightarrow\bigvee_{B\leq A}\left\{\sigma(B)\wedge\bigwedge_{x_u\lhd A}\bigvee_{x_u\lhd C}\left\{\sigma(C)\wedge\bigwedge_{y_v\lhd C}Cl^{\sigma}(B)(y_\mu)\right\}\right\}=\mathbf{1}_L$$

$$\Leftrightarrow\exists\,B\leq A\text{ such that }\sigma(B)=\mathbf{1}_L\text{ and }\bigwedge_{x_u\lhd A}\bigvee_{x_u\lhd C}\left\{\sigma(C)\wedge\bigwedge_{y_v\lhd C}Cl^{\sigma}(B)(y_\mu)\right\}=\mathbf{1}_L$$

$$\Leftrightarrow\exists\,B\leq A\text{ such that }\sigma(B)=\mathbf{1}_L\text{ and }\forall x_u\lhd A,\exists\,C\text{ with }x_u\lhd C\text{ such that }\sigma(C)=\mathbf{1}_L$$
$$\text{and }\bigwedge_{y_v\lhd C}Cl^{\sigma}(B)(y_\mu)$$

$$\Leftrightarrow\exists\,B\leq A\text{ such that }\sigma(B)=\mathbf{1}_L\text{ and }\forall x_u\lhd A,\exists\,C\text{ with }x_u\lhd C\text{ such that }\sigma(C)=\mathbf{1}_L$$

$$\begin{aligned}
&\text{and } \forall y_v \lhd C,\ Cl^{\sigma}(B)(y_v) = \mathbf{1}_L \\
\Leftrightarrow\ &\exists\, B \le A \text{ such that } \sigma(B) = \mathbf{1}_L \text{ and } \forall x_u \lhd A, \exists\, C \text{ with } x_u \lhd C \text{ such that } \sigma(C) = \mathbf{1}_L \\
&\text{and } C \le Cl^{\sigma}(B) \\
\Leftrightarrow\ &\exists\, B \in \sigma,\ B \le A \le (\bar{B})^{\circ} \\
\Leftrightarrow\ &A \text{ is } \alpha\text{-open } L\text{-subset.}
\end{aligned}$$

Where $^{-}$and $^{\circ}$ refer to the closure and the interior operator, respectively. □

Theorem 2.7. *Let σ be an L-fpt on X and let $\mathscr{A}$ be its corresponding L-fuzzy α-open operator. Then $\sigma(A) \le \mathscr{A}(A)$ for all $A \in L^X$.*

Proof. The proof can be obtained from the following inequality:

$$\begin{aligned}
\mathscr{A}(A) &= \bigvee_{B\le A}\left\{\sigma(B) \wedge \bigwedge_{x_u \lhd A}\bigvee_{x_u \lhd C}\left\{\sigma(C) \wedge \bigwedge_{y_v \lhd C} Cl^{\sigma}(B)(y_\mu)\right\}\right\} \\
&\ge \sigma(A) \wedge \bigwedge_{x_u \lhd A}\bigvee_{x_u \lhd C}\left\{\sigma(C) \wedge \bigwedge_{y_v \lhd C} Cl^{\sigma}(A)(y_\mu)\right\} \\
&\ge \sigma(A) \wedge \bigwedge_{x_u \lhd A}\left\{\sigma(A) \wedge \bigwedge_{y_v \lhd A} Cl^{\sigma}(A)(y_\mu)\right\} \\
&= \sigma(A) \wedge \sigma(A) \wedge \mathbf{1}_L \\
&= \sigma(A).
\end{aligned}$$

□

Corollary 2.8. *Let σ be an L-fpt on X and let $\mathscr{A}$ be its corresponding L-fuzzy α-open operator. Then $\sigma^{\star}(A) \le \mathscr{A}^{\star}(A)$ for all $A \in L^X$.*

Theorem 2.9. *Let $\mathscr{A} : L^X \longrightarrow L$ be an L-fuzzy α-open operator induced by L-fpt σ on X. Then $\mathscr{A}$ satisfies the following conditions:*

- ***(1)*** $\mathscr{A}(\underline{\mathbf{0}}_{L^X}) = \mathscr{A}(\underline{\mathbf{1}}_{L^X}) = \mathbf{1}_L$.
- ***(2)*** $\mathscr{A}(\bigvee_{i\in I} A_i) \ge \bigwedge_{i\in I} \mathscr{A}(A_i)$ *for any* $\{A_i\}_{i\in I} \subseteq L^X$.

Proof. The proof of **(1)** is clear. To prove **(2)**, suppose that $w \in L$ and $w \lhd \bigwedge_{i\in I} \mathscr{A}(A_i)$. Then for any $i \in I$, there is $B_i \le A_i$ such that

$$w \lhd \sigma(B_i) \quad \text{and} \quad w \lhd \bigwedge_{x_u \lhd A_i}\bigvee_{x_u \lhd C_i}\left\{\sigma(C_i) \wedge \bigwedge_{y_v \lhd C_i}\bigwedge_{y_{\not v} \le D \ge B_i} (\sigma(D'))'\right\},$$

i.e., $w \lhd \sigma(B_i)$ and for any $i \in I$ and $x_u \lhd A_i$, there is $C_i \in L^X$ such that $x_u \lhd C_i$, $w \lhd \sigma(C_i)$ and $w \lhd \bigwedge_{y_v \lhd C_i} \bigwedge_{y_{\not v} \le D \ge \bigvee_{i\in I} B_i} (\sigma(D'))'$. Hence

$$w \le \bigwedge_{i\in I} \sigma(B_i) \le \sigma\left(\bigvee_{i\in I} B_i\right), \quad w \le \bigwedge_{i\in I} \sigma(C_i) \le \sigma\left(\bigvee_{i\in I} C_i\right),$$

and

$$w \le \bigwedge_{i\in I}\bigwedge_{y_v \lhd C_i}\bigwedge_{y_{\not v} \le D \ge \bigvee_{i\in I} B_i} (\sigma(D'))'.$$

Since $\left\{x_u | x_u \lhd \bigvee_{i\in I} A_i\right\} = \bigcup_{i\in I}\left\{x_u | x_u \lhd A_i\right\}$ and $\left\{y_v | y_v \lhd \bigvee_{i\in I} C_i\right\} = \bigcup_{i\in I}\left\{y_v | y_v \lhd C_i\right\}$, we have

$$\begin{aligned}
\mathscr{A}\left(\bigvee_{i\in I} A_i\right) &= \bigvee_{B\le \bigvee_{i\in I} A_i}\left\{\sigma(B)\wedge \bigwedge_{x_u\lhd \bigvee_{i\in I} A_i}\ \bigvee_{x_u\lhd C}\left\{\sigma(C)\wedge \bigwedge_{y_v\lhd C}\ \bigwedge_{y_v\not\le D\ge B}(\sigma(D'))'\right\}\right\}\\
&\ge \sigma\left(\bigvee_{i\in I} B_i\right)\wedge \bigwedge_{i\in I}\ \bigwedge_{x_u\lhd \bigvee_{i\in I} A_i}\left\{\sigma\left(\bigvee_{i\in I} C_i\right)\wedge \bigwedge_{y_v\lhd \bigvee_{i\in I} C_i}\ \bigwedge_{y_v\not\le D\ge \bigvee_{i\in I} B_i}(\sigma(D'))'\right\}\\
&= \sigma\left(\bigvee_{i\in I} B_i\right)\wedge \bigwedge_{i\in I}\ \bigwedge_{x_u\lhd \bigvee_{i\in I} A_i}\left\{\sigma\left(\bigvee_{i\in I} C_i\right)\wedge \bigwedge_{i\in I}\ \bigwedge_{y_v\lhd C_i}\ \bigwedge_{y_v\not\le D\ge \bigvee_{i\in I} B_i}(\sigma(D'))'\right\}\\
&\ge \sigma\left(\bigvee_{i\in I} B_i\right)\wedge \bigwedge_{i\in I}\ \bigwedge_{x_u\lhd A_i}\left\{\sigma\left(\bigvee_{i\in I} C_i\right)\wedge \bigwedge_{i\in I}\ \bigwedge_{y_v\lhd C_i}\ \bigwedge_{y_v\not\le D\ge \bigvee_{i\in I} B_i}(\sigma(D'))'\right\}\\
&\ge w.
\end{aligned}$$

This shows $\mathscr{A}\left(\bigvee_{i\in I} A_i\right) \ge \bigwedge_{i\in I}\mathscr{A}(A_i)$. □

In the following definition, we use L-fuzzy α-open operators to introduce generalized definitions for L-fuzzy α-open, L-fuzzy α-continuous and L-fuzzy α-irresolute functions.

Definition 2.10. If (X, σ_1) and (Y, σ_2) are L-fpts's and $f : (X, \sigma_1) \longrightarrow (Y, \sigma_2)$ is a function, then:

(1) f is an L-fuzzy α-open function iff $\sigma_1(A) \le \mathscr{A}_2(f^{\rightarrow}(A))$ for any $A \in L^X$.
(2) f is an L-fuzzy α-continuous function iff $\sigma_2(B) \le \mathscr{A}_1(f^{\leftarrow}(B))$ holds for any $B \in L^Y$.
(3) f is an L-fuzzy α-irresolute iff $\mathscr{A}_2(B) \le \mathscr{A}_1(f^{\leftarrow}(B))$ holds for any $B \in L^Y$.

Corollary 2.11. *If (X, σ_1) and (Y, σ_2) are L-fpts's and $f : (X, \sigma_1) \longrightarrow (Y, \sigma_2)$ is a function, then:*

(1) *f is an L-fuzzy α-continuous iff $\sigma_2^\star(B) \le \mathscr{A}_1^\star(f^{\leftarrow}(B))$ for any $B \in L^Y$.*
(2) *f is an L-fuzzy α-irresolute iff $\mathscr{A}_2^\star(B) \le \mathscr{A}_1^\star(f^{\leftarrow}(B))$ for any $B \in L^Y$.*

Definition 2.12. [24] For an L-fpt σ on X and an L-subset $A \in L^X$, the degree of fuzzy compactness $\mathbf{com}(A)$ of A is given by:

$$\begin{aligned}
\mathbf{com}(A) &= \bigwedge_{\mathcal{U}\subseteq L^X}\left\{\left\{\bigwedge_{B\in\mathcal{U}}\sigma(B)\wedge\bigwedge_{x\in X}\left(A'\vee\bigvee_{B\in\mathcal{U}}B\right)(x)\right\}\mapsto \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}}\ \bigwedge_{x\in X}\left(A'\vee\bigvee_{B\in\mathcal{V}}B\right)(x)\right\}\\
&= \bigwedge_{\mathcal{U}\subseteq L^X}\left\{\left\{\bigwedge_{B\in\mathcal{U}}\sigma(B)\wedge\left[A\tilde{\subseteq}\bigvee\mathcal{U}\right]\right\}\mapsto\bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}}\left[A\tilde{\subseteq}\bigvee\mathcal{V}\right]\right\}.
\end{aligned}$$

In this case, an L-subset A is said to be fuzzy compact if and only if $\mathbf{com}(A) = \mathbf{1}_L$.

Definition 2.13. [1] Let σ be an L-fpt on X. An L-subset $A \in L^X$ is called α-compact if

$$\bigwedge_{B\in\mathcal{U}}\mathscr{A}(B)\wedge\bigwedge_{x\in X}\left(A'\vee\bigvee_{B\in\mathcal{U}}B\right)(x)\le\bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}}\ \bigwedge_{x\in X}\left(A'\vee\bigvee_{B\in\mathcal{V}}B\right)(x)$$

for every $\mathcal{U} \subset L^X$.

Definition 2.14. [25, 26] For an L-pt τ on X, $u \in L\backslash\{\mathbf{1}_L\}$ and $A \in L^X$, a family $\mathcal{U} \subseteq L^X$ is said to be a α_u-cover of A if for each $x \in X$, we have $u \in \alpha(A'(x) \vee \bigvee_{B\in\mathcal{U}} B(x))$. The family $\mathcal{U}$ is said to be a strong α_u-cover of A if $u \in \alpha(\bigwedge_{x\in X}(A'(x) \vee \bigvee_{B\in\mathcal{U}} B(x)))$.

Definition 2.15. [25, 26] For an L-pt τ on X, $u \in L\backslash\{\mathbf{1}_L\}$ and $A \in L^X$, a family $\mathcal{U} \subseteq L^X$ is said to be a Q_u-cover of A if for each $x \in X$, we have $A'(x) \vee \bigvee_{B\in\mathcal{U}} B(x) \geq u$.

Definition 2.16. [25, 26] For an L-pt τ on X, $u \in L\backslash\{\mathbf{1}_L\}$ and $A \in L^X$, a family $\mathcal{U} \subseteq L^X$ is called:

(1) a u-shading of A if for each $x \in X$, $(A'(x) \vee \bigvee_{B\in\mathcal{U}} B(x))\not\leq u$.
(2) a strong u-shading of A if $\bigwedge_{x\in X}(A'(x) \vee \bigvee_{B\in\mathcal{U}} B(x))\not\leq u$.
(3) a u-remote family of A if for each $x \in X$, $(A(x) \wedge \bigwedge_{B\in\mathcal{U}} B(x))\not\geq u$.
(4) a strong u-remote family of A if $\bigvee_{x\in X}(A(x) \wedge \bigwedge_{B\in\mathcal{U}} B(x))\not\geq u$.

3 Degree of α-openness, α-continuity and α-irresolutness for functions between L-fpts's

In this section, we will introduce the notions of α-openness, α-continuity, and α-irresolutness degree for functions between L-fpts's. Further, we will discuss their properties.

Definition 3.1. If (X, σ_1) and (Y, σ_2) are L-fpts's and $f : (X, \sigma_1) \longrightarrow (Y, \sigma_2)$ is a function, then:

(1) the α-openness degree of f with respect to σ_1 and σ_2 is defined by

$$\alpha\mathbf{o}(f) = \bigwedge_{A\in L^X} \left\{\mathscr{A}_1(A) \mapsto \mathscr{A}_2(f^{\rightarrow}(A))\right\}.$$

(2) the continuity degree of f with respect to σ_1 and σ_2 is defined by

$$\alpha\mathbf{c}(f) = \bigwedge_{B\in L^Y} \left\{\sigma(B) \mapsto \mathscr{A}_1(f^{\leftarrow}(B))\right\}.$$

(3) the irresoluteness degree of f with respect to σ_1 and σ_2 is defined by

$$\alpha\mathbf{i}(f) = \bigwedge_{B\in L^Y} \left\{\mathscr{A}_2(B) \mapsto \mathscr{A}_1(f^{\leftarrow}(B))\right\}.$$

Definition 3.2. For any two L-fpts's (X, σ_1) and (Y, σ_2) and any bijective function $f : (X, \sigma_1) \longrightarrow (Y, \sigma_2)$, the α-homomorphism degree of f with respect to σ_1 and σ_2 is given by

$$\alpha\text{-}\mathbf{Hom}(f) = \alpha\mathbf{i}(f) \wedge \alpha\mathbf{o}(f).$$

Remark 3.3. **(1)** Based on **(2)** of Lemma 2.1, $\alpha\mathbf{c}(f) = \mathbf{1}_L$ implies to $\mathscr{A}_1(f^{\leftarrow}(B)) \geq \sigma_2(B)$ for all $B \in L^Y$. This is exactly the definition of α-continuous function. The cases $\alpha\mathbf{o}(f) = \mathbf{1}_L$ and $\alpha\mathbf{i}(f) = \mathbf{1}_L$ can be shown similarly. Thus **(2)** and **(3)** in Definition 3.1 are precisely the α-open and α-irresolute function's definition as in the sense of Definition 2.10.
(2) For the identity function $i : (X, \sigma_1) \longrightarrow (X, \sigma_1)$, we have $\alpha\mathbf{i}(i) = \alpha\mathbf{o}(i) = \alpha\text{-}\mathbf{Hom}(i) = \mathbf{1}_L$.

By using Definition 3.1 and Corollary 2.11, we can state the following corollary.

Corollary 3.4. *If (X, σ_1) and (Y, σ_2) are L-fpts's and $f : (X, \sigma_1) \longrightarrow (Y, \sigma_2)$ is a function, then:*

(1) *the α-continuity degree of f is characterized by*

$$\alpha\mathbf{c}(f) = \bigwedge_{B \in L^Y} \left\{ \sigma^\star(B) \mapsto \mathscr{A}_1^\star(f^\leftarrow(B)) \right\}.$$

(2) *the α-irresoluteness degree of f is characterized by*

$$\alpha\mathbf{i}(f) = \bigwedge_{B \in L^Y} \left\{ \mathscr{A}_2^\star(B) \mapsto \mathscr{A}_1^\star(f^\leftarrow(B)) \right\}.$$

Definition 3.5. For any function $f : (X, \sigma_1) \longrightarrow (Y, \sigma_2)$ where (X, σ_1) and (Y, σ_2) are two L-fpts's, the α-closeness degree of f is given by

$$\alpha\mathbf{cl}(f) = \bigwedge_{A \in L^X} \left\{ \mathscr{A}_1^\star(A) \mapsto \mathscr{A}_2^\star(f^\rightarrow(A)) \right\}.$$

Theorem 3.6. *If $f : (X, \sigma_1) \longrightarrow (Y, \sigma_2)$ and $g : (Y, \sigma_2) \longrightarrow (Z, \sigma_3)$ are two functions where (X, σ_1), (Y, σ_2) and (Z, σ_3) are three L-fpts's, then:*

(1) $\alpha\mathbf{i}(f) \wedge \alpha\mathbf{i}(g) \leq \alpha\mathbf{i}(g \circ f)$.
(2) $\alpha\mathbf{o}(f) \wedge \alpha\mathbf{o}(g) \leq \alpha\mathbf{i}(g \circ f)$.
(3) $\alpha\mathbf{cl}(f) \wedge \alpha\mathbf{i}(g) \leq \alpha\mathbf{cl}(g \circ f)$.

Proof. Since the proof of **(2)** and **(3)** is clear, we only prove **(1)**. By using Definition 3.1 and Lemma 2.1 **(4)**, we obtain

$$\begin{aligned}
\alpha\mathbf{i}(f) \wedge \alpha\mathbf{i}(g) &= \bigwedge_{B \in L^Y} \left\{ \mathscr{A}_2(B) \mapsto \mathscr{A}_1(f^\leftarrow(B)) \right\} \wedge \bigwedge_{C \in L^Z} \left\{ \mathscr{A}_3(C) \mapsto \mathscr{A}_2(g^\leftarrow(C)) \right\} \\
&\leq \bigwedge_{C \in L^Z} \left\{ \mathscr{A}_2(g^\leftarrow(C)) \mapsto \mathscr{A}_1(f^\leftarrow(g^\leftarrow(C))) \right\} \wedge \bigwedge_{C \in L^Z} \left\{ \mathscr{A}_3(C) \mapsto \mathscr{A}_2(g^\leftarrow(C)) \right\} \\
&= \bigwedge_{C \in L^Z} \left\{ \left(\mathscr{A}_2(g^\leftarrow(C)) \mapsto \mathscr{A}_1((g \circ g)^\leftarrow(C)) \right) \wedge \left(\mathscr{A}_3(C) \mapsto \mathscr{A}_2(g^\leftarrow(C)) \right) \right\} \\
&\leq \bigwedge_{C \in L^Z} \left\{ \mathscr{A}_3(g^\leftarrow(C)) \mapsto \mathscr{A}_1((g \circ f)^\leftarrow(C)) \right\} \\
&= \alpha\mathbf{i}(g \circ f).
\end{aligned}$$

□

By using Definition 3.2 and Theorem 3.6, we have the following corollary.

Corollary 3.7. *Let (X, σ_1), (Y, σ_2) and (Z, σ_3) be L-fpts's, $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be two bijective functions. Then α-**Hom**$(f) \wedge \alpha$-**Hom**$(g) \leq \alpha$-**Hom**$(g \circ f)$.*

Theorem 3.8. *Let (X, σ_1), (Y, σ_2) and (Z, σ_3) be L-fpts's and $g : Y \longrightarrow Z$ be a surjective function. Then:*

(1) $\alpha\mathbf{o}(g \circ f) \wedge \alpha\mathbf{i}(f) \leq \alpha\mathbf{o}(g)$.
(2) $\alpha\mathbf{cl}(g \circ f) \wedge \alpha\mathbf{i}(f) \leq \alpha\mathbf{cl}(g)$.

Proof. **(1)** Since f is a surjective function, we have $(g \circ f)^\rightarrow(f^\leftarrow(B)) = g^\rightarrow(B)$ for each $B \in L^Y$. By using **(4)** of Lemma 2.1, we get

$$\alpha\mathbf{o}(g \circ f) \wedge \alpha\mathbf{i}(f) = \bigwedge_{A \in L^X} \left\{ \mathscr{A}_1(A) \mapsto \mathscr{A}_3((g \circ f)^\rightarrow(A)) \right\} \wedge \bigwedge_{B \in L^Y} \left\{ \mathscr{A}_2(B) \mapsto \mathscr{A}_1(f^\leftarrow(B)) \right\}$$

$$\leq \bigwedge_{B\in L^Y}\left\{\mathscr{A}_1(f^{\leftarrow}(B)) \mapsto \mathscr{A}_3((g\circ f)^{\rightarrow}(f^{\leftarrow}(B)))\right\} \wedge \bigwedge_{B\in L^Y}\left\{\mathscr{A}_2(B)\mapsto \mathscr{A}_1(g^{\leftarrow}(B))\right\}$$
$$= \bigwedge_{B\in L^Y}\left\{\left(\mathscr{A}_1(f^{\leftarrow}(B)) \mapsto \mathscr{A}_3(g^{\rightarrow}(B))\right) \wedge \left(\mathscr{A}_2(B)\mapsto \mathscr{A}_1(f^{\leftarrow}(B))\right)\right\}$$
$$\leq \bigwedge_{B\in L^Y}\left\{\mathscr{A}_2(B)\mapsto \mathscr{A}_3(g^{\rightarrow}(B))\right\}$$
$$= \alpha\mathbf{o}(g).$$

Analogously, we can prove **(2)**. □

Similarly, the following theorem is true.

Theorem 3.9. *Given three L-fpts's (X,σ_1), (Y,σ_2) and (Z,σ_3). If $f:(X,\sigma_1)\longrightarrow(Y,\sigma_2)$ is an injective function and $g:Y\longrightarrow Z$ is any function, then*

(1) $\alpha\mathbf{o}(g\circ f)\wedge\alpha\mathbf{i}(g)\leq\alpha\mathbf{o}(f)$.
(2) $\alpha\mathbf{cl}(g\circ f)\wedge\alpha\mathbf{i}(g)\leq\alpha\mathbf{cl}(f)$.

Theorem 3.10. *If $f:(X,\sigma_1)\longrightarrow(Y,\sigma_2)$ is a bijective function where (X,σ_1) and (Y,σ_2) are two L-fpts's, then*

(1) $\alpha\mathbf{i}(f)=\bigwedge_{A\in L^X}\left\{\mathscr{A}_2(f^{\rightarrow}(A))\mapsto\mathscr{A}_1(A)\right\}$.
(2) $\alpha\mathbf{o}(f)=\bigwedge_{B\in L^Y}\left\{\mathscr{A}_1(f^{\leftarrow}(B))\mapsto\mathscr{A}_2(B)\right\}$.
(3) $\alpha\mathbf{i}(f^{-1})=\alpha\mathbf{o}(f)=\alpha\mathbf{cl}(f)$.

Proof. The proof of **(2)** is similar to **(1)**, we only prove **(1)** and **(3)**.

(1) From the bijectivity of f, we get $f^{\leftarrow}(f^{\rightarrow}(A))=A$ for any $A\in L^X$, and $f^{\rightarrow}(f^{\leftarrow}(B))=B$ for any $B\in L^Y$. It follows that

$$\bigwedge_{A\in L^X}\left\{\mathscr{A}_2(f^{\rightarrow}(A))\mapsto\mathscr{A}_1(A)\right\} = \bigwedge_{A\in L^X}\left\{\mathscr{A}_2(f^{\rightarrow}(A))\mapsto\mathscr{A}_1(f^{\leftarrow}(f^{\rightarrow}(A)))\right\}$$
$$\geq \bigwedge_{B\in L^Y}\left\{\mathscr{A}_2(B)\mapsto\mathscr{A}_1(f^{\leftarrow}(B))\right\}$$
$$= \bigwedge_{B\in L^Y}\left\{\mathscr{A}_2(f^{\rightarrow}(f^{\leftarrow}(B)))\mapsto\mathscr{A}_1(f^{\leftarrow}(B))\right\}$$
$$\geq \bigwedge_{A\in L^X}\left\{\mathscr{A}_2(f^{\rightarrow}(A))\mapsto\mathscr{A}_1(A)\right\}.$$

Hence

$$\alpha\mathbf{i}(f) = \bigwedge_{B\in L^Y}\left\{\mathscr{A}_2(B)\mapsto\mathscr{A}_1(f^{\leftarrow}(B))\right\}$$
$$= \bigwedge_{A\in L^X}\left\{\mathscr{A}_2(f^{\rightarrow}(A))\mapsto\mathscr{A}_1(A)\right\}.$$

(3) Since f is a bijective function, we get $(f^{-1})^{\leftarrow}(A)=f^{\rightarrow}(A)$ and $f^{\rightarrow}(A')=f^{\rightarrow}(A)'$ for any $A\in L^X$. Therefore

$$\alpha\mathbf{i}(f^{-1}) = \bigwedge_{A\in L^X}\left\{\mathscr{A}_1(A)\mapsto\mathscr{A}_2((f^{-1})^{\leftarrow}(A))\right\}$$
$$= \bigwedge_{A\in L^X}\left\{\mathscr{A}_1(A)\mapsto\mathscr{A}_2(f^{\rightarrow}(A))\right\}$$

$$=\alpha\mathbf{o}(f).$$

and

$$\begin{aligned}\alpha\mathbf{o}(f^{-1}) &= \bigwedge_{A\in L^X}\left\{\mathscr{A}_1(A)\mapsto\mathscr{A}_2(f^{\rightarrow}(A))\right\}\\ &= \bigwedge_{A\in L^X}\left\{\mathscr{A}_1(A')\mapsto\mathscr{A}_2(f^{\rightarrow}(A'))\right\}\\ &= \bigwedge_{A\in L^X}\left\{\mathscr{A}_1(A')\mapsto\mathscr{A}_2(f^{\rightarrow}(A)')\right\}\\ &=\alpha\mathbf{cl}(f).\end{aligned}$$

The proof is completed. □

Corollary 3.11. *Given a bijective function $f:(X,\sigma_1)\longrightarrow(Y,\sigma_2)$ between two L-fpts's (X,σ_1) and (Y,σ_2), then:*

(1) α-$\mathbf{Hom}(f)=\alpha\mathbf{i}(f)\wedge\alpha\mathbf{i}(f^{-1})=\alpha\mathbf{i}(f)\wedge\alpha\mathbf{cl}(f)$.
(2) α-$\mathbf{Hom}(f)=\bigwedge_{A\in L^X}\left\{\mathscr{A}_2(f^{\rightarrow}(A))\leftrightarrow\mathscr{A}_1(A)\right\}$.
(3) α-$\mathbf{Hom}(f)=\bigwedge_{B\in L^Y}\left\{\mathscr{A}_1(f^{\leftarrow}(B))\leftrightarrow\mathscr{A}_2(B)\right\}$.

4 A new extension of α-compactness

Nannan and Ruiying [1] introduced the notion of α-compactness in L-fuzzy topology with the help of L-fuzzy α-open operator. In the following definition, we present the degree of α-compactness based on implication operation as a new generalization of α-compactness.

Definition 4.1. Let (X,σ) be an L-fpts. For any $A\in L^X$, let

$$\begin{aligned}\alpha\mathbf{Com}(A) &= \bigwedge_{\mathcal{U}\subseteq L^X}\left\{\mathscr{A}(\mathcal{U})\mapsto\left(\left[A\tilde{\subset}\bigvee\mathcal{U}\right]\mapsto\bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}}\left[A\tilde{\subset}\bigvee\mathcal{V}\right]\right)\right\}\\ &= \bigwedge_{\mathcal{U}\subseteq L^X}\left\{\bigwedge_{A_1\in\mathcal{U}}\mathscr{A}(A_1)\mapsto\left\{\bigwedge_{x\in X}\left(A'\vee\bigvee_{A_1\in\mathcal{U}}A_1\right)(x)\mapsto\bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}}\bigwedge_{x\in X}\left(A'\vee\bigvee_{A_1\in\mathcal{V}}A_1\right)(x)\right\}\right\}.\end{aligned}$$

Then $\alpha\mathbf{Com}_(A)$ is said to be the degree of α-compactness of A with respect to σ. By using Theorem 2.9, we have $\mathbf{Com}_{\mathscr{A}}(A)=\alpha\mathbf{Com}(A)$ for any $A\in L^X$.

Theorem 4.2. *Let τ be an L-pt on X and $A\in L^X$. An L-subset A is fuzzy α-compact if and only if $\alpha\mathbf{Com}_{\chi_\tau}(A)=\mathbf{1}_L$, where the mapping $\chi_\tau:L^X\longrightarrow L$ is given by*

$$\chi_\tau(A)=\begin{cases}\mathbf{1}_L, & \text{if } A\in\tau;\\ \mathbf{0}_L, & \text{if } A\notin\tau.\end{cases}$$

Proof. Let τ be an L-pt on X. It is clear that χ_τ is L-fpt. An L-subset $A\in L^X$ is α-open set with respect to τ if and only if $\mathscr{A}_{\chi_\tau}(A)=\mathbf{1}_L$. Based on the definition of fuzzy α-compactness, we have an L-subset $A\in L^X$ is fuzzy α-compact such that for any collection $\mathcal{U}\subseteq L^X$, we have that

$$\mathscr{A}_{\chi_\tau}(\mathcal{U})\le\left[\left[A\tilde{\subseteq}\bigvee\mathcal{U}\right]\le\bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}}\left[A\tilde{\subseteq}\bigvee\mathcal{V}\right]\right].$$

By using Lemma 2.1, A is fuzzy α-compact if and only if for any collection $\mathcal{U} \subseteq L^X$, we have

$$\mathscr{A}_{\chi_\tau}(\mathcal{U}) \mapsto \left(\left[A \tilde{\subseteq} \bigvee \mathcal{U} \right] \mapsto \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \left[A \tilde{\subseteq} \bigvee \mathcal{V} \right] \right) = \mathbf{1}_L.$$

This result together with the definition of $\alpha\mathbf{Com}_{\chi_\tau}(A)$ yields to $\alpha\mathbf{Com}_{\chi_\tau}(A) = \mathbf{1}_L$. □

Theorem 4.3. *Let σ be an L-fpt on X and $A \in L^X$. An L-subset A is L-fuzzy α-compact if and only if $\alpha\mathbf{Com}(A) = \mathbf{1}_L$.*

Proof. Based on Definition 4.1 and Lemma 2.1, the conclusion is straightforward. □

Theorem 4.4. *For any L-fpt σ on X and $A \in L^X$, we have $\alpha\mathbf{Com}(A) \leq \mathbf{Com}(A)$.*

Proof. Straightforward. □

Lemma 4.5. *For any L-fpt σ on X and $A \in L^X$, we have $\alpha\mathbf{Com}(A) \geq u$ if and only if*

$$\mathscr{A}(\mathcal{U}) \wedge \left[A \tilde{\subseteq} \bigvee \mathcal{U} \right] \wedge u \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \left[A \tilde{\subseteq} \bigvee \mathcal{V} \right],$$

for any $\mathcal{U} \subseteq L^X$.

Proof. For every $u \in L$, $A \in L^X$ and $\mathcal{U} \subseteq L^X$, we have

$$\begin{aligned}
\alpha\mathbf{Com}(A) \geq u &\Leftrightarrow \bigwedge_{\mathcal{U} \subseteq L^X} \left\{ \mathscr{A}(\mathcal{U}) \mapsto \left\{ \left[A \tilde{\subseteq} \bigvee \mathcal{U} \right] \mapsto \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \left[A \tilde{\subseteq} \bigvee \mathcal{V} \right] \right\} \right\} \geq u \\
&\Leftrightarrow \mathscr{A}(\mathcal{U}) \mapsto \left\{ \left[A \tilde{\subseteq} \bigvee \mathcal{U} \right] \mapsto \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \left[A \tilde{\subseteq} \bigvee \mathcal{V} \right] \right\} \geq u \\
&\Leftrightarrow \left\{ \mathscr{A}(\mathcal{U}) \wedge \left[A \tilde{\subseteq} \bigvee \mathcal{U} \right] \right\} \mapsto \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \left[A \tilde{\subseteq} \bigvee \mathcal{V} \right] \geq u \\
&\Leftrightarrow \mathscr{A}(\mathcal{U}) \wedge \left[A \tilde{\subseteq} \bigvee \mathcal{U} \right] \wedge u \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \left[A \tilde{\subseteq} \bigvee \mathcal{V} \right].
\end{aligned}$$

□

Theorem 4.6. *For any L-fpt σ on X and $A \in L^X$, we have $\alpha\mathbf{Com}(A) \geq u$ if and only if*

$$\bigvee_{B \in \mathcal{M}} \mathscr{A}^\star(B)' \vee \left\{ \bigvee_{x \in X} \left\{ A(x) \wedge \bigwedge_{B \in \mathcal{M}} B(x) \right\} \right\} \vee u' \geq \bigwedge_{\mathcal{N} \in 2^{(\mathcal{M})}} \bigvee_{x \in X} \left(A(x) \wedge \bigwedge_{B \in \mathcal{N}} B(x) \right),$$

for each $\mathcal{M} \subseteq L^X$.

Proof. Based on the definition of $\mathscr{A}^\star$ and Lemma 2.1, the proof is clear. □

Theorem 4.7. *For any L-fpt σ on X and $A \in L^X$, we have*

$$\alpha\mathbf{Com}(A) = \bigvee \left\{ u \in L | \bigwedge_{B \in \mathcal{U}} \mathscr{A}(B) \wedge \left[A \tilde{\subseteq} \bigvee \mathcal{U} \right] \wedge u \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \left[A \tilde{\subseteq} \bigvee \mathcal{V} \right], \ \forall \mathcal{U} \subseteq L^X \right\}.$$

Proof. By using Lemma 2.1, we have $\alpha\mathbf{Com}(A)$ as the upper bound of

$$\left\{ u \in L | \bigwedge_{B \in \mathcal{U}} \mathscr{A}(B) \wedge \left[A \tilde{\subseteq} \bigvee \mathcal{U} \right] \wedge u \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \left[A \tilde{\subseteq} \bigvee \mathcal{V} \right], \ \forall \mathcal{U} \subseteq L^X \right\}.$$

By using the Definition 4.1, we have

$$\begin{aligned}\alpha\mathbf{Com}(A) &\leq \bigwedge_{B\in\mathcal{U}} \mathscr{A}(B) \mapsto \left(\left[A\tilde{\subseteq}\bigvee\mathcal{U}\right] \mapsto \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \left[A\tilde{\subseteq}\bigvee\mathcal{V}\right]\right)\\ &= \left(\bigwedge_{B\in\mathcal{U}} \mathscr{A}(B) \wedge \left[A\tilde{\subseteq}\bigvee\mathcal{U}\right]\right) \mapsto \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \left[A\tilde{\subseteq}\bigvee\mathcal{V}\right],\end{aligned}$$

for each $\mathcal{U} \subseteq L^X$. By applying the properties of the operation "$\mapsto$", we have

$$\bigwedge_{B\in\mathcal{U}} \mathscr{A}(B) \wedge \left[A\tilde{\subseteq}\bigvee\mathcal{U}\right] \wedge \alpha\mathbf{Com}(A) \leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \left[A\tilde{\subseteq}\bigvee\mathcal{V}\right],$$

and hence

$$\alpha\mathbf{Com}(A) \in \bigvee \left\{u \in L \middle| \bigwedge_{B\in\mathcal{U}} \mathscr{A}(B) \wedge \left[A\tilde{\subseteq}\bigvee\mathcal{U}\right] \wedge u \leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \left[A\tilde{\subseteq}\bigvee\mathcal{V}\right],\ \forall\mathcal{U} \subseteq L^X\right\}.$$

Therefore, we completed the proof. □

Theorem 4.8. *For any L-fpt σ on X and $A_1, A_2 \in L^X$, we have*

$$\alpha\mathbf{Com}(A_1 \vee A_2) \geq \alpha\mathbf{Com}(A_1) \wedge \alpha\mathbf{Com}(A_2).$$

Proof. We can prove the theorem by using the next inequality:

$$\begin{aligned}\alpha\mathbf{Com}(A_1 \vee A_2) &= \bigvee \left\{u \in L \middle| \bigwedge_{B\in\mathcal{U}} \mathscr{A}(B) \wedge \left[A_1 \vee A_2\tilde{\subseteq}\bigvee\mathcal{U}\right] \wedge u \right.\\ &\left.\leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \left[A_1 \vee A_2\tilde{\subseteq}\bigvee\mathcal{V}\right],\ \forall\mathcal{U} \subseteq L^X\right\}\\ &= \bigvee \left\{u \in L \middle| \bigwedge_{B\in\mathcal{U}} \mathscr{A}(B) \wedge \left[A_1\tilde{\subseteq}\bigvee\mathcal{U}\right] \wedge \left[A_2\tilde{\subseteq}\bigvee\mathcal{U}\right]\right.\\ &\left.\wedge\ u \leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \left[A_1\tilde{\subseteq}\bigvee\mathcal{V}\right] \wedge \left[A_2\tilde{\subseteq}\bigvee\mathcal{V}\right],\ \forall\mathcal{U} \subseteq L^X\right\}\\ &\geq \bigvee \left\{u \in L \middle| \bigwedge_{B\in\mathcal{U}} \mathscr{A}(B) \wedge \left[A_1\tilde{\subseteq}\bigvee\mathcal{U}\right]\right.\\ &\left.\wedge\ u \leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \left[A_1\tilde{\subseteq}\bigvee\mathcal{V}\right],\ \forall\mathcal{U} \subseteq L^X\right\}\\ &\wedge \bigvee \left\{u \in L \middle| \bigwedge_{B\in\mathcal{U}} \mathscr{A}(B) \wedge \left[A_2\tilde{\subseteq}\bigvee\mathcal{U}\right]\right.\\ &\left.\wedge\ u \leq \bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}} \left[A_2\tilde{\subseteq}\bigvee\mathcal{V}\right],\ \forall\mathcal{U} \subseteq L^X\right\}\\ &= \alpha\mathbf{Com}(A_1) \wedge \alpha\mathbf{Com}(A_2).\end{aligned}$$

□

Theorem 4.9. *For any L-fpt σ on X and $A_1, A_2 \in L^X$, we have*

$$\alpha\mathbf{Com}(A_1 \wedge A_2) \geq \alpha\mathbf{Com}(A_1) \wedge \mathscr{A}(A_2').$$

Proof. We can prove the theorem by using the next inequality

$$\begin{aligned}\alpha\mathbf{Com}(A_1\wedge A_2) &= \bigvee\Big\{u\in L|\bigwedge_{B\in\mathcal{U}}\mathscr{A}(B)\wedge\Big[A_1\wedge A_2\tilde{\subseteq}\bigvee\mathcal{U}\Big]\\ &\wedge\ u\le\bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}}\Big[A_1\wedge A_2\tilde{\subseteq}\bigvee\mathcal{V}\Big],\ \forall\mathcal{U}\subseteq L^X\Big\}\\ &= \bigvee\Big\{u\in L|\bigwedge_{B\in\mathcal{U}}\mathscr{A}(B)\wedge\Big[A_1\tilde{\subseteq}A_2'\vee\bigvee\mathcal{U}\Big]\\ &\wedge\ u\le\bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}}\Big[A_1\tilde{\subseteq}A_2'\vee\bigvee\mathcal{V}\Big],\ \forall\mathcal{U}\subseteq L^X\Big\}\\ &\ge \alpha\mathbf{Com}(A_1)\wedge\mathscr{A}(A_2').\end{aligned}$$

□

Corollary 4.10. *For any L-fpt σ on X and $A\in L^X$, we have*

$$\alpha\mathbf{Com}(A)\ge\alpha\mathbf{Com}(\mathbf{1}_L)\wedge\mathscr{A}(A').$$

Theorem 4.11. *For any L-fts's (X,σ_1) and (X,σ_2) such that $\sigma_1\le\sigma_2$ and for any $A\in L^X$, we have $\alpha\mathbf{Com}_{\sigma_2}(A)\le\alpha\mathbf{Com}_{\sigma_1}(A)$.*

Corollary 4.12. *For any L-fpts (X,σ) with the base or the subbase $\mathcal{B}$, we have $\alpha\mathbf{Com}(A)\le\alpha\mathbf{Com}_{\mathcal{B}}(A)$, for any $A\in L^X$.*

Theorem 4.13. *For any L-fpts's (X,σ_1) and (Y,σ_2), if $f:(X,\sigma_1)\longrightarrow(Y,\sigma_2)$ is an L-fuzzy α-irresolute function, then*

$$\alpha\mathbf{Com}_{\sigma_2}(f_L^{\rightarrow}(C))\ge\alpha\mathbf{Com}_{\sigma_1}(C),$$

for every $C\in L^X$.

Proof. For each $C\in L^X$, we have

$$\begin{aligned}\alpha\mathbf{Com}_{\sigma_2}(f_L^{\rightarrow}(C)) &= \bigvee\Big\{u\in L|\mathscr{A}_2(\mathcal{U})\wedge\Big[f_L^{\rightarrow}(C)\tilde{\subseteq}\bigvee\mathcal{U}\Big]\\ &\wedge\ u\le\bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}}\Big[f_L^{\rightarrow}(C)\tilde{\subseteq}\bigvee\mathcal{V}\Big],\ \forall\mathcal{U}\subseteq L^Y\Big\}\\ &\ge \bigvee\Big\{u\in L|\mathscr{A}_1(f_L^{\leftarrow}(\mathcal{U}))\wedge\Big[C\tilde{\subseteq}\bigvee f_L^{\leftarrow}(\mathcal{U})\Big]\\ &\wedge\ u\le\bigvee_{\mathcal{V}\in 2^{(\mathcal{U})}}\Big[C\tilde{\subseteq}\bigvee f_L^{\leftarrow}(\mathcal{V})\Big],\ \forall\mathcal{U}\subseteq L^Y\Big\}\\ &\ge \bigvee\Big\{u\in L|\mathscr{A}_1(\mathcal{P})\wedge\Big[C\tilde{\subseteq}\bigvee\mathcal{P}\Big]\\ &\wedge\ u\le\bigvee_{\mathcal{R}\in 2^{(\mathcal{P})}}\Big[C\tilde{\subseteq}\bigvee\mathcal{R}\Big],\ \forall\mathcal{R}\subseteq L^X\Big\}\\ &= \alpha\mathbf{Com}_{\sigma_1}(C).\end{aligned}$$

□

Theorem 4.14. *For any L-fpts's (X,σ_1) and (Y,σ_2), if $f:(X,\sigma_1)\longrightarrow(Y,\sigma_2)$ is an L-fuzzy α-continuous function, then*

$$\mathbf{Com}_{\sigma_2}(f_L^{\rightarrow}(C))\ge\alpha\mathbf{Com}_{\sigma_1}(C),$$

for every $C \in L^X$.

Proof. For each $C \in L^X$, we have

$$\begin{aligned}
\mathbf{Com}_{\sigma_2}(f_L^{\rightarrow}(C)) &= \bigvee \Big\{ u \in L | \sigma_2(\mathcal{U}) \wedge \Big[f_L^{\rightarrow}(C) \tilde{\subseteq} \bigvee \mathcal{U} \Big] \\
&\quad \wedge\ u \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \Big[f_L^{\rightarrow}(C) \tilde{\subseteq} \bigvee \mathcal{V} \Big],\ \forall \mathcal{U} \subseteq L^Y \Big\} \\
&\geq \bigvee \Big\{ u \in L | \mathscr{A}_1(f_L^{\leftarrow}(\mathcal{U})) \wedge \Big[C \tilde{\subseteq} \bigvee f_L^{\leftarrow}(\mathcal{U}) \Big] \\
&\quad \wedge\ u \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \Big[C \tilde{\subseteq} \bigvee f_L^{\leftarrow}(\mathcal{V}) \Big],\ \forall \mathcal{U} \subseteq L^Y \Big\} \\
&\geq \bigvee \Big\{ u \in L | \mathscr{A}_1(\mathcal{P}) \wedge \Big[C \tilde{\subseteq} \bigvee \mathcal{P} \Big] \\
&\quad \wedge\ u \leq \bigvee_{\mathcal{R} \in 2^{(\mathcal{P})}} \Big[C \tilde{\subseteq} \bigvee \mathcal{R} \Big],\ \forall \mathcal{R} \subseteq L^X \Big\} \\
&= \alpha \mathbf{Com}_{\sigma_1}(C).
\end{aligned}$$

□

Theorem 4.15. *For any L-fpt σ on X, $A \in L^X$ and $u \in L \setminus \{\mathbf{0}_L\}$, the next statements are equivalent:*

(1) $\alpha\mathbf{Com}(A) \geq u$.
(2) *For each $v \in P(L)$, $v \not\geq u$, every strong v-shading $\mathcal{U}$ of A with $\mathscr{A}(\mathcal{U}) \not\leq v$ has a finite sub-collection $\mathcal{V}$ which is a strong v-shading of A.*
(3) *For each $v \in P(L)$, $v \not\geq u$, every strong v-shading $\mathcal{U}$ of A with $\mathscr{A}(\mathcal{U}) \not\leq v$, there exists a finite sub-collection $\mathcal{V}$ of $\mathcal{U}$ and $w \in \beta^*(v)$ such that $\mathcal{V}$ is a w-shading of A.*
(4) *For each $v \in P(L)$, $v \not\geq u$, every strong v-shading $\mathcal{U}$ of A with $\mathscr{A}(\mathcal{U}) \not\leq v$, there exists a finite sub-collection $\mathcal{V}$ of $\mathcal{U}$ and $w \in \beta^*(v)$ such that $\mathcal{V}$ is a strong w-shading of A.*
(5) *For each $v \in J(L)$, $v \not\leq u'$, every strong v-remote collection $\mathcal{W}$ of A with $\mathscr{A}^*(\mathcal{W}) \not\leq v'$ has a finite sub-collection $\mathcal{R}$ which is a strong v-remote collection of A.*
(6) *For each $v \in J(L)$, $v \not\leq u'$, every strong v-remote collection $\mathcal{W}$ of A with $\mathscr{A}^*(\mathcal{W}) \not\leq v'$, there exist a finite sub-collection $\mathcal{R}$ of $\mathcal{W}$ and $w \in \alpha^*(v)$ such that $\mathcal{R}$ is an w-remote collection of A.*
(7) *For each $v \in J(L)$, $v \not\leq u'$, every strong v-remote collection $\mathcal{W}$ of A with $\mathscr{A}^*(\mathcal{W}) \not\leq v'$, there exist a finite sub-collection $\mathcal{R}$ of $\mathcal{W}$ and $w \in \alpha^*(v)$ such that $\mathcal{R}$ is a strong w-remote collection of A.*
(8) *For each $v \leq u$, $u \in \alpha(v)$, $v, w \neq \mathbf{0}_L$, every Q_v-cover $\mathcal{U} \subseteq (\mathscr{A})_v$ of A has a finite sub-collection $\mathcal{V}$ which is a Q_w-cover of A.*
(9) *For each $v \leq u$, $w \in \alpha(v)$, $v, w \neq \mathbf{0}_L$, every Q_v-cover $\mathcal{U} \subseteq (\mathscr{A})_v$ of A has a finite sub-collection $\mathcal{V}$ which is a strong α_w-cover of A.*
(10) *For each $v \leq u$, $w \in \alpha(v)$, $v, w \neq \mathbf{0}_L$, every Q_v-cover $\mathcal{U} \subseteq (\mathscr{A})_v$ of A has a finite sub-collection $\mathcal{V}$ which is a α_w-cover of A.*
(11) *For each $v \leq u$, $w \in \alpha(v)$, $v, u \neq \mathbf{0}_L$, every strong α_v-cover $\mathcal{U} \subseteq (\mathscr{A})_v$ of A has a finite sub-collection $\mathcal{V}$ which is a Q_w-cover of A.*
(12) *For each $v \leq u$, $w \in \alpha(v)$, $v, w \neq \mathbf{0}_L$, every strong α_v-cover $\mathcal{U} \subseteq (\mathscr{A})_v$ of A has a finite sub-collection $\mathcal{V}$ which is a strong α_w-cover of A.*
(13) *For each $v \leq u$, $w \in \alpha(v)$, $v, w \neq \mathbf{0}_L$, every strong α_v-cover $\mathcal{U} \subseteq (\mathscr{A})_v$ of A has a finite sub-collection $\mathcal{V}$ which is a α_w-cover of A.*

Theorem 4.16. *For any L-fpt σ on X, $A \in L^X$, and $u \in L \setminus \{\mathbf{0}_L\}$, if $\alpha(w \wedge s) = \alpha(w) \wedge \alpha(s)$ for each $w, s \in L$, then the next statements will be equivalent:*

(1) $\alpha\mathbf{Com}(A) \geq u$.
(2) *For each* $v \in \alpha(u)$, $v \neq \mathbf{0}_L$, *every strong* α_v*-cover* $\mathcal{U}$ *of* A *with* $v \in \alpha(\mathscr{A}(\mathcal{U}))$ *has a finite sub-collection* $\mathcal{V}$ *which is a* Q_v*-cover of* A.
(3) *For each* $v \in \alpha(u)$, $v \neq \mathbf{0}_L$, *every strong* α_v*-cover* $\mathcal{U}$ *of* A *with* $v \in \alpha(\mathscr{A}(\mathcal{U}))$ *has a finite sub-collection* $\mathcal{V}$ *which is a strong* α_v*-cover of* A.
(4) *For each* $v \in \alpha(u)$, $v \neq \mathbf{0}_L$, *every strong* α_v*-cover* $\mathcal{U}$ *of* A *with* $v \in \alpha(\mathscr{A}(\mathcal{U}))$ *has a finite sub-collection* $\mathcal{V}$ *which is a* α_v*-cover of* A.

The following theorem and its corollary verify the relationship between α-irresoluteness degree and α-compactness degree.

Theorem 4.17. *If* $f : (X, \sigma_1) \longrightarrow (Y, \sigma_2)$ *is a function between two L-fpts's* (X, σ_1) *and* (Y, σ_2), *then*

$$\alpha\mathbf{Com}_{\mathscr{A}_1}(A) \wedge \alpha\mathbf{i}(f) \leq \alpha\mathbf{Com}_{\mathscr{A}_2}(f^{\rightarrow}(A))$$

for any $A \in L^X$.

Proof. Suppose that $u_1 \in L$ with $u_1 \lhd \alpha\mathbf{Com}_{\mathscr{A}_1}(A) \wedge \alpha\mathbf{i}(f)$. Then

$$u_1 \lhd \alpha\mathbf{i}(f) = \bigwedge_{B \in L^Y} \left\{\mathscr{A}_2(B) \mapsto \mathscr{A}_1(f^{\leftarrow}(B))\right\},$$

and

$$\begin{aligned} u_1 &\lhd \alpha\mathbf{Com}_{\mathscr{A}_1}(A) \\ &= \bigwedge_{\mathcal{U} \in L^X} \left\{\left\{\bigwedge_{A_1 \in \mathcal{U}} \mathscr{A}_1(A_1) \wedge \bigwedge_{x \in X}\left(A' \vee \bigvee_{A_1 \in \mathcal{U}} A_1\right)(x)\right\} \mapsto \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \bigwedge_{x \in X}\left(A' \vee \bigvee_{A_1 \in \mathcal{V}} A_1\right)(x)\right\} \end{aligned}$$

Then for any $B \in L^Y$ and $\mathcal{U} \subseteq L^X$, we have $u_1 \leq \mathscr{A}_2(B) \mapsto \mathscr{A}_1(f^{\leftarrow}(B))$ and

$$u_1 \leq \left\{\bigwedge_{A_1 \in \mathcal{U}} \mathscr{A}_1(A_1) \wedge \bigwedge_{x \in X}\left(A' \vee \bigvee_{A_1 \in \mathcal{U}} A_1\right)(x)\right\} \mapsto \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \bigwedge_{x \in X}\left(A' \vee \bigvee_{A_1 \in \mathcal{V}} A_1\right)(x).$$

By Lemma 2.1 **(1)**, we have $u_1 \wedge \mathscr{A}_2(B) \leq \mathscr{A}_1(f^{\leftarrow}(B))$ for any $B \in L^Y$, and

$$u_1 \wedge \bigwedge_{W \in \mathcal{U}} \mathscr{A}_1(W) \wedge \bigwedge_{x \in X}\left(A' \vee \bigvee_{W \in \mathcal{U}} W\right)(x) \leq \bigvee_{\mathcal{V} \in 2^{(\mathcal{U})}} \bigwedge_{x \in X}\left(A' \vee \bigvee_{W \in \mathcal{U}} W\right)(x).$$

To prove

$$\begin{aligned} u_1 \leq &\alpha\mathbf{Com}_{\mathscr{A}_2}(f^{\rightarrow}(A)) \\ &= \bigwedge_{\mathcal{W} \in L^Y} \left\{\left(\bigwedge_{B_1 \in \mathcal{W}} \mathscr{A}_2(B_1) \wedge \bigwedge_{y \in Y}\left(f^{\rightarrow}(A)' \vee \bigvee_{B_1 \in \mathcal{W}} B_1\right)(y)\right)\right. \\ &\left. \mapsto \bigvee_{\mathcal{D} \in 2^{(\mathcal{W})}} \bigwedge_{y \in Y}\left(f^{\rightarrow}(A)' \vee \bigvee_{B_1 \in \mathcal{W}} B_1\right)(y)\right\}, \end{aligned}$$

for all $\mathcal{W} \subseteq L^Y$, let $f^{\leftarrow}(\mathcal{W}) = \{f^{\leftarrow}(B_1) | B_1 \in \mathcal{W}\} \subseteq L^X$. Then, we have

$$\begin{aligned} u_1 \wedge &\bigwedge_{B_1 \in \mathcal{W}} \mathscr{A}_2(B_1) \wedge \bigwedge_{y \in Y}\left(f^{\rightarrow}(A)' \vee \bigvee_{B_1 \in \mathcal{W}} B_1\right)(y) \\ &\leq u_1 \wedge \bigwedge_{B_1 \in \mathcal{W}} \mathscr{A}_1(f^{\leftarrow}(B_1)) \wedge \bigwedge_{y \in Y}\left(f^{\rightarrow}(A)' \vee \bigvee_{B_1 \in \mathcal{W}} B_1\right)(y) \end{aligned}$$

$$=u_1 \wedge \bigwedge_{B_1\in\mathcal{W}} \mathscr{A}_1(f^{\leftarrow}(B_1)) \bigwedge_{x\in X}\Big(A' \vee \bigvee_{B_1\in\mathcal{W}} f^{\leftarrow}(B_1)\Big)(x)$$
$$=u_1 \wedge \bigwedge_{A_1\in f^{\leftarrow}(\mathcal{W})} \mathscr{A}_1(A_1) \wedge \bigwedge_{x\in X}\Big(A' \vee \bigvee_{A_1\in f^{\leftarrow}(\mathcal{W})} (A_1)\Big)(x)$$
$$\le \bigvee_{\mathcal{V}\in 2^{(f^{\leftarrow}(\mathcal{W}))}} \bigwedge_{x\in X}\Big(A' \vee \bigvee_{A_1\in\mathcal{W}} (A_1))(x)$$
$$= \bigvee_{\mathcal{D}\in 2^{(\mathcal{W})}} \bigwedge_{x\in X}\Big(A' \vee \bigvee_{B_1\in\mathcal{D}} f^{\leftarrow}(B_1)\Big)(x)$$
$$= \bigvee_{\mathcal{D}\in 2^{(\mathcal{W})}} \bigwedge_{y\in Y}\Big(f^{\rightarrow}(A)' \vee \bigvee_{B_1\in\mathcal{D}} B_1\Big)(y).$$

By using Lemma 2.1 **(1)**, we know

$$u_1 \le \Big(\bigwedge_{B_1\in\mathcal{W}} \mathscr{A}_2(B_1) \wedge \bigwedge_{y\in Y}\Big(f^{\rightarrow}(A)' \vee \bigvee_{B_1\in\mathcal{W}} B_1\Big)(y)\Big)$$
$$\mapsto \bigvee_{\mathcal{D}\in 2^{(\mathcal{W})}} \bigwedge_{y\in Y}\Big(f^{\rightarrow}(A)' \vee \bigvee_{B_1\in\mathcal{D}} B_1\Big)(y).$$

Thus

$$u_1 \le \bigwedge_{\mathcal{W}\subseteq L^Y}\Big\{\Big(\bigwedge_{B_1\in\mathcal{W}} \mathscr{A}_2(B_1) \wedge \bigwedge_{y\in Y}\Big(f^{\rightarrow}(A)' \vee \bigvee_{B_1\in\mathcal{W}} B_1\Big)(y)\Big)$$
$$\mapsto \bigvee_{\mathcal{D}\in 2^{(\mathcal{W})}} \bigwedge_{y\in Y}\Big(f^{\rightarrow}(A)' \vee \bigvee_{B_1\in\mathcal{D}} B_1\Big)(y)\Big\} = \alpha\mathbf{Com}_{\mathscr{A}_2}(f^{\rightarrow}(A)).$$

Since u_1 is arbitrary, we have $\alpha\mathbf{Com}_{\mathscr{A}_1}(A) \wedge \alpha\mathbf{i}(f) \le \alpha\mathbf{Com}_{\mathscr{A}_2}(f^{\rightarrow}(A))$. The proof is completed. □

Corollary 4.18. *For any surjective function $f : (X, \sigma_1) \longrightarrow (Y, \sigma_2)$ where (X, σ_1) and (Y, σ_2) are L-fpts's , we have*

$$\alpha\mathbf{Com}_{\mathscr{A}_1}(\underline{\mathbf{1}}_{L^X}) \wedge \alpha\mathbf{i}(f) \le \alpha\mathbf{Com}_{\mathscr{A}_2}(\underline{\mathbf{1}}_{L^Y}).$$

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