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Research Article

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A General Version of the Nullstellensatz for Arbitrary Fields

Abstract: We prove a general version of Bezout's form of the Nullstellensatz for arbitrary fields. The corresponding sufficient and necessary condition only involves the local existence of multi-valued roots for each of the polynomials belonging to the ideal in consideration. Finally, this version implies the standard Nullstellensatz when the coefficient field is algebraically closed.

1 Introduction

A fundamental result in algebraic geometry is the well-known Hilbert's Nullstellensatz, which describes the interrelations between an ideal I in a polynomial ring over an algebraically closed field k and the corresponding ideal of all polynomials vanishing on the zeroes of I, i.e. $I(V(I)) = Rad(I)$ [1, Ch. 1]. Besides, the Nullstellensatz has also a 'weak' form (or Bezout version) which states that under the former hypothesis an ideal I in $k[x_1, \cdots, x_n]$ contains 1 if and only if there is no common zero for all the polynomials of I in k^n. In addition, the standard version of the Nullstellensatz can be easily deduced from the weak form using the Rabinowitsch's trick [2].

In the literature one can find several kinds of generalizations of both forms of this seminal result, for example a noncommutative version due to S. A. Amitsur [3]; the work of W. D. Brownawell describing a corresponding "pure power product version", which relates in a sophisticated way the exponents emerging from the 'radical' condition stated in 'homogeneous' forms of the Nullstellensatz [4]. In addition, the main result of T. Krick et al. in [5] offers sharp bounds for the degree and the height of the polynomials involved in the arithmetic (weak) form of the Nullstellensatz, and the work of L. Ein and R. Lazarsfeld proves more sophisticated geometric versions of it involving ideal sheaves, among others [6]. Finally, from the Artin-Tate lemma a more general form of the Nullstellensatz for arbitrary fields can be derived, i.e., the quotient of a polynomial ring in finitely many variables over a field L by a maximal ideal m, is a finite field extension of L. This is an elementary consequence of the Artin-Tate lemma and the Steinitz theorem (see for example [7]). Now, let us assume that I is an arbitrary ideal. Then, what will be the natural condition for I characterizing the fact that $V(I)$ is non-empty?

Surprisingly, none of the results above offers an answer to this elementary question, whose answer can be considered genuinely as a formal generalization of the (weak) Nullstellensatz for arbitrary fields.

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So, in this short communication we prove in a completely elementary way that the global non-emptiness of the zero-locus of an ideal of polynomials over any field is equivalent to the local non-emptiness of the zero-locus of any of its elements, which would seem to be, in general, a strictly weaker condition. This result would offer a natural extension of a general condition characterizing the global non-emptiness of the zero-locus of ideals independently of the coefficient field k.

2 Main Result

Theorem 1. *(Bezout's Form of the Nullstellensatz for arbitrary fields) Let k be a field (not necessarily alg. closed) and $I \subset k[X_1, \ldots, X_n]$ an ideal. Then for each polynomial $f(X) \in I$, where $X := (X_1, \ldots, X_n)$, there exists $a = (a_1, \ldots, a_n) \in k^n$ such that $f(a) = 0$, if and only if the algebraic set determined by I, $V(I)$, is non-empty.*

Proof. Case I) k is algebraically closed: this corresponds to the classic version of the (weak) Nullstellensatz.

Case II) k is not algebraically closed. In this case we state that given polynomials f_1 and $f_2 \in I$, there is another polynomial $p(f_1, f_2) \in I$ such that $p(f_1, f_2)(a) = 0 \Leftrightarrow f_1(a) = f_2(a) = 0$.

To see this, let $l(T) = T^m + a_1 T^{m-1} + \cdots + a_m \in k[T]$ be any monic non constant polynomial without roots in k. We define

$$p(f_1, f_2)(X) = f_2^m(X) \left(\left(\frac{f_1(X)}{f_2(X)} \right)^m + a_1 \left(\frac{f_1(X)}{f_2(X)} \right)^{m-1} + \cdots + a_m \right) = f_1^m + a_1 f_2 f_1^{m-1} + \cdots + a_k f_1^{m-k} f_2^k + \cdots + a_m f_2^m$$

It is clear that $p(f_1, f_2) \in I$. Let us prove that $p(f_1, f_2)(a) = 0 \Leftrightarrow f_1(a) = f_2(a) = 0$. Suppose $p(f_1, f_2)(a) = 0$. Then $f_2(a) = 0$, since otherwise $f_1(a)/f_2(a)$ would be a root of $l(T)$. Since $p(f_1, f_2)(a) = 0$, it follows that $f_1^m(a) = f_1(a) = 0$.

The reciprocal is clear, since $f_1(a) = f_2(a) = 0$ implies $p(f_1, f_2)(a) = 0$. A version of the former fact for finitely many polynomials can be found as an exercise in [8, Ch. 4, §1, Ex. 8].

Finally, let $f_1, \ldots, f_r$ be arbitrary generators of I. We inductively define $p_1 = p(f_1, f_2), \ldots, p_r = p(f_r, p_{r-1})$. Clearly $p_r(a) = 0$ if and only if $f_1(a) = \cdots = f_r(a) = 0$. Since $p_r \in I$, the hypothesis guarantees the existence of $a \in k^n$ such that $p_r(a) = 0$. Thus, $f_1(a) = \cdots = f_r(a) = 0$, whereas it follows clearly that $g(a) = 0$, for all $g \in I$. □

Example. *An enlighten example that illustrates very well the way in which the core argument of the former proof works is given when $k = \mathbb{R}$. Effectively, given two polynomials $f_1(X), f_2(X) \in \mathbb{R}[X]$, is it a straightforward computation to verify that one can choose $p(f_1, f_2)$ to be $f_1(X)^2 + f_2(X)^2$.*

Remark. *If k is an algebraically closed field, the hypothesis of the theorem are satisfied under the standard assumption that $I \neq R := k[X_1, \ldots, X_n]$ and consequently it generalizes the classic (weak) Nullstellensatz. Effectively, if f is a non-constant polynomial in I, let us see that the zero-locus of f should be non-empty. So, after a standard change of coordinates it is possible to write f as a monic polynomial in one of the variables, lets say X_n. Hence we may assume that $f(X_1, \ldots, X_n)$ can be written in the form*

$$X_n^r + a_1(X_1, \ldots, X_{n-1})X_n^{r-1} + \ldots + a_r(X_1, \ldots, X_{n-1}).$$

Furthermore, substituting $X_1 = X_2 = \cdots = X_{n-1} = 0$ we obtain the polynomial $X_n^r + a_1(0)X_n^{r-1} + \cdots + a_r(0)$. Since we are assuming k is algebraically closed, it has a zero $c \in k$. Thus, in these new coordinates f has a cero $(0, \ldots 0, c) \in k^n$.

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