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Research Article

Wenxu Ge*, Weiping Li, and Tianze Wang

On Diophantine approximation by unlike powers of primes

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Abstract: Suppose that $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ are nonzero real numbers, not all of the same sign, λ_1/λ_2 is irrational, λ_2/λ_4 and λ_3/λ_5 are rational. Let η real, and $\varepsilon > 0$. Then there are infinitely many solutions in primes p_j to the inequality $|\lambda_1 p_1 + \lambda_2 p_2^2 + \lambda_3 p_3^3 + \lambda_4 p_4^4 + \lambda_5 p_5^5 + \eta| < (\max p_j^j)^{-1/32+\varepsilon}$. This improves an earlier result under extra conditions of λ_j.

Keywords: Diophantine approximation; primes; Davenport-Heilbronn method

MSC: 11D75; 11P32; 11P55

1 Introduction

Given $k \geq 1$ and non-zero real numbers $\lambda_1, \lambda_2, \cdots, \lambda_s$ (not all in rational ratio, not all in same sign), we write

$$F(\mathbf{p}) = \sum_{j=1}^{s} \lambda_j p_j^k,$$

where $\mathbf{p} = (p_1, p_2, \ldots, p_s)$ with each p_j a prime. Various authors have considered the distribution of values of such forms, see [17, 18] for example.

For $k = 1$, Vaughan [17] first proved that for any real η, there are infinitely many solutions in primes p_j to the inequlity

$$|\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta| < (\max p_j)^{-\xi+\varepsilon}$$

with $\xi = 1/10$. The exponent was subsequently improved by Baker and Harman [1] to $\xi = 1/6$, Harman [6] to $\xi = 1/5$ and Matomäki [14] to $\xi = 2/9$.

For $k = 2$, Baker and Harman [1] and Harman [7] showed that there are infinitely many solutions in primes p_j to the inequality

$$|\lambda_1 p_1^2 + \lambda_2 p_2^2 + \lambda_3 p_3^2 + \lambda_4 p_4^2 + \lambda_5 p_5^2 + \eta| < (\max p_j)^{-1/8+\varepsilon}.$$

In 2011, Li and Wang [11] proved that there are infinitely many solutions in primes p_j to the inequality

$$|\lambda_1 p_1 + \lambda_2 p_2^2 + \lambda_3 p_3^2 + \lambda_4 p_4^2 + \eta| < (\max p_j)^{-1/28+\varepsilon}.$$

***Corresponding Author: Wenxu Ge:** School of Mathematics Statistics, North China University of Water Resources and Electric Power, Zhengzhou 450046, P.R.China, E-mail: gewenxu1982@163.com
Weiping Li: School of Mathematics and Information Sciences, Henan University of Economics and Law, Zhengzhou, 450046, P.R.China, E-mail: wpliyh@163.com
Tianze Wang: School of Mathematics Statistics, North China University of Water Resources and Electric Power, Zhengzhou, 450046, P.R.China, E-mail: wtz@ncwu.edu.cn

Later, Languasco and Zaccagnini [9], Liu and Sun [12], and Wang and Yao [20] replaced $1/28$ with $1/18$, $1/16$ and $1/14$, respectively.

For $k \geq 3$, Vaughan [18] first proved that there are infinitely many solutions in primes p_j to the inequality

$$|\lambda_1 p_1^k + \lambda_2 p_2^k + \cdots + \lambda_s p_s^k + \eta| < (\max_{1 \leq j \leq s} p_j)^{-\sigma+\varepsilon}.$$

In 2006, Cook and Harman [2] improved the exponent σ.

In 2016, The first author and the second author [3] first established that if $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ are nonzero real numbers, not all of the same sign and λ_1/λ_2 is irrational, there are infinitely many solutions in primes p_j to the inequality

$$|\lambda_1 p_1 + \lambda_2 p_2^2 + \lambda_3 p_3^3 + \lambda_4 p_4^4 + \lambda_5 p_5^5 + \eta| < (\max p_j^j)^{-1/720+\varepsilon}. \tag{1.1}$$

Later, Mu [15], Liu [13], Mu and Qu [16] replaced $\frac{1}{720}$ in (1.1) with $1/180$, $5/288$ and $5/252$ respectively.

In this paper, under some extra conditions of λ_j, we get the following result.

Theorem 1.1. *Suppose that $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ are nonzero real numbers, not all of the same sign, λ_1/λ_2 is irrational, λ_2/λ_4 and λ_3/λ_5 are rational. Let η real, and $\varepsilon > 0$. Then there are infinitely many solutions in primes p_j to the inequality*

$$|\lambda_1 p_1 + \lambda_2 p_2^2 + \lambda_3 p_3^3 + \lambda_4 p_4^4 + \lambda_5 p_5^5 + \eta| < (\max p_j^j)^{-1/32+\varepsilon}. \tag{1.2}$$

In the previous arguments, the key of this problem is the estimates for exponential sums over squares of primes (or for certain double sums if sieve methods are invoked). In [13], Liu used $S_2(\lambda_2\alpha) \ll P_2^{1-1/8+\varepsilon}$. In [16], Mu and Qu used sieve method of Harman [7], and got $S_2^\star(\lambda_2\alpha) \ll P_2^{1-1/7+\varepsilon}$. Using the method of Mu and Qu [16], even if one got the best estimation $S_2^\star(\lambda_2\alpha) \ll P_2^{1-1/6+\varepsilon}$, $5/252$ can only be replaced by $5/216$. But in this paper our method don't depend on the estimates of $S_2(\lambda_2\alpha)$.

Notation: Throughout the paper, the letter δ denotes a sufficiently small, fixed positive number. The letter ε denotes an arbitrarily sufficiently small positive real number. Any statement in which ε occurs holds for each fixed $\varepsilon > 0$. c denotes an absolute constant, not necessarily the same in all occurrences. The letter p, with or without subscript, denotes a prime number. Constants, both explicit and implicit, in Vinogradov symbols may depend on $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$. We write $e(x) = \exp(2\pi i x)$.

2 Outline of the method

We use the Hardy-Littlewood circle method which first stated by Davenport-Heilbronn. Note that λ_1/λ_2 is irrational and λ_2/λ_4 is rational. Without loss of generality, we assume that $|\lambda_2/\lambda_4| \leq 1$. Let a/q be a continued fraction convergent to λ_1/λ_2 and put $X = q^{12/5}$. Then $(\lambda_2 a)/(\lambda_4 q) = a'/q'$ is a continued fraction convergent to λ_1/λ_4, where $(a', q') = 1$. Thus we have $q \asymp q'$. Suppose that $0 < \tau < 1$, and write $P_j = X^{1/j}$ and $\mathcal{I}_j = [\delta P_j, P_j]$ for $1 \leq j \leq 5$. We define

$$K_\tau(\alpha) = \left(\frac{\sin \pi\tau\alpha}{\pi\alpha}\right)^2, \quad S_j(\alpha) = \sum_{p \in \mathcal{I}_j} (\log p) e(\alpha p^j).$$

Then we can easily get

$$K_\tau(\alpha) \ll \min(\tau^2, |\alpha|^{-2}), \quad \int_{\mathbb{R}} K_\tau(\alpha) e(\alpha x) \mathrm{d}\alpha = \max(0, \tau - |x|). \tag{2.1}$$

For any measurable subset $\mathfrak{X}$ of $\mathbb{R}$, we define

$$\mathcal{J}(\mathfrak{X}) := \int_{\mathfrak{X}} S_1(\lambda_1\alpha) S_2(\lambda_2\alpha) S_3(\lambda_3\alpha) S_4(\lambda_4\alpha) S_5(\lambda_5\alpha) K_\tau(\alpha) e(\eta\alpha) d\alpha. \tag{2.2}$$

Then by (2.1), we have

$$\begin{aligned}\mathcal{J}(\mathbb{R}) &= \sum_{p_j \in \mathcal{I}_j} (\log p_1)\cdots(\log p_5) \int_{\mathbb{R}} e(\alpha(\lambda_1 p_1 + \cdots + \lambda_4 p_4^4 + \lambda_5 p_5^5 + \eta))K_\tau(\alpha)d\alpha \\ &\le (\log X)^5 \sum_{p_j \in \mathcal{I}_j} \max(0, \tau - |\lambda_1 p_1 + \cdots + \lambda_4 p_4^4 + \lambda_5 p_5^5 + \eta|) \\ &\le \tau(\log X)^5 \mathcal{N}(\eta, X), \end{aligned} \tag{2.3}$$

where $\mathcal{N}(\eta, X)$ is the number of solutions to the inequality

$$|\lambda_1 p_1 + \cdots + \lambda_4 p_4^4 + \lambda_5 p_5^5 + \eta| < \tau, \quad p_j \in \mathcal{I}_j.$$

To estimate the integral $\mathcal{J}(\mathbb{R})$, we divide the real line into three parts: the major arc $\mathfrak{M}$, the minor arc $\mathfrak{m}$ and the trivial arc $\mathfrak{t}$, which are defined by

$$\mathfrak{M} = \{\alpha : |\alpha| \le 1\}, \quad \mathfrak{m} = \{\alpha : 1 < |\alpha| \le \xi\}, \quad \mathfrak{t} = \{\alpha : |\alpha| > \xi\},$$

where $\xi = \tau^{-2}X^{1/80+\varepsilon}$. By the arguments of section 5 in [15], we have

$$\mathcal{J}(\mathfrak{t}) = o(\tau^2 X^{77/60}). \tag{2.4}$$

3 Preliminary lemmas

Lemma 3.1. *[19, Theorem 3.1] Suppose that $N \ge 2$ and α satisfies*

$$|q\alpha - a| \le q^{-1}, \quad (a, q) = 1, \quad q \in \mathbb{N}, \quad a \in \mathbb{Z}.$$

Then we have

$$\sum_{p \le N} (\log p)e(\alpha p) \ll (\log N)^4 (N^{\frac{1}{2}} q^{\frac{1}{2}} + N^{\frac{4}{5}} + Nq^{-\frac{1}{2}}).$$

Corollary 3.2. *Suppose that $X \ge Z \ge X^{\frac{4}{5}+\varepsilon}$ and $|S_1(\alpha)| > Z$. Then there are coprime integers a, q satisfying*

$$1 \le q \ll (X/Z)^2 X^{\varepsilon}, \quad |q\alpha - a| \ll (X/Z)^2 X^{\varepsilon - 1}.$$

Proof. This follows from Lemma 3.1 immediately. □

Lemma 3.3. *[8, Theorem 3] Let $k \ge 3$ and $\sigma(k) = 1/(3 \cdot 2^{k-1})$. Suppose that $N \ge 2$ and α satisfies*

$$|q\alpha - a| \le Q^{-1}, \quad (a, q) = 1, \quad q \in \mathbb{N}, \quad q \le Q, \quad a \in \mathbb{Z},$$

where $Q = N^{(k^2 - 2k\sigma(k))/(2k-1)}$. Then, for any $\varepsilon > 0$,

$$\sum_{p \le N} (\log p)e(\alpha p^k) \ll N^{1-\sigma(k)+\varepsilon} + \frac{N^{1+\varepsilon}}{(q + N^k |q\alpha - a|)^{1/2}}.$$

Corollary 3.4. *Suppose that $P_4 \ge Z \ge P_4^{1-1/24+\varepsilon}$ and $|S_4(\alpha)| > Z$. Then there are coprime integers a, q satisfying*

$$1 \le q \ll (P_4/Z)^2 P_4^{\varepsilon}, \quad |q\alpha - a| \ll (P_4/Z)^2 P_4^{\varepsilon - 4}.$$

Proof. This follows from Lemma 3.3 immediately. □

Lemma 3.5. *[7, Lemma 3] Suppose that $N \geq 2$ and α satisfies*

$$|q\alpha - a| \leq q^{-1}, \quad (a, q) = 1, \quad q \in \mathbb{N}, \quad a \in \mathbb{Z}.$$

Then, for any $\varepsilon > 0$,

$$\sum_{p \leq N} (\log p) e(\alpha p^2) \ll N^{1+\varepsilon} \left(\frac{1}{q} + \frac{1}{N^{1/2}} + \frac{q}{N^2} \right)^{1/4}.$$

Corollary 3.6. *[7, Corollary 1] Suppose that $P_2 \geq Z \geq P_2^{7/8+\varepsilon}$, and that $|S_2(\alpha)| > Z$. Then there are coprime integers a, q satisfying*

$$1 \ll q \ll (P_2/Z)^4 P_2^{\varepsilon}, \quad |q\alpha - a| \ll (P_2/Z)^4 P_2^{\varepsilon-2}.$$

Lemma 3.7. *[16, Lemma 3.7] Suppose that*

$$f(\alpha) \in \{S_1(\lambda_1\alpha)^2, S_3(\lambda_3\alpha)^8, S_4(\lambda_4\alpha)^{16}, S_2(\lambda_2\alpha)^2 S_3(\lambda_3\alpha)^2 S_5(\lambda_5\alpha)^2,$$
$$S_2(\lambda_2\alpha)^2 S_4(\lambda_4\alpha)^4, S_2(\lambda_2\alpha)^2 S_5(\lambda_5\alpha)^6\}.$$

Then we have

$$\int_{-1}^{1} |f(\alpha)| d\alpha \ll f(0) X^{-1+\varepsilon}; \tag{3.1}$$

$$\int_{\mathbb{R}} |f(\alpha)| K_\tau(\alpha) d\alpha \ll \tau f(0) X^{-1+\varepsilon}. \tag{3.2}$$

We define the multiplicative function $w_3(q)$ by taking

$$w_3(p^{3u+v}) = \begin{cases} 3p^{-u-1/2}, & \text{when } u \geq 0 \text{ and } v = 1; \\ p^{-u-1}, & \text{when } u \geq 0 \text{ and } 2 \leq v \leq 3. \end{cases} \tag{3.3}$$

Lemma 3.8. *[21, Lemma 2.3] If α is a real number satisfying that there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a, q) = 1$, $1 \leq q \leq P^{3/4}$ and $|q\alpha - a| \leq P^{-9/4}$, then one has*

$$\sum_{P \leq x < 2P} e(x^3\alpha) \ll \frac{w_3(q)P}{1 + P^3|\alpha - a/q|},$$

otherwise, one has $\sum_{P \leq x < 2P} e(x^3\alpha) \ll P^{\frac{3}{4}+\varepsilon}$.

Lemma 3.9. *[21, Lemma 2.1] Let c be a constant. For $Q \geq 2$, one has*

$$\sum_{1 \leq q \leq Q} d(q)^c w_3(q)^2 \ll (\log Q)^A,$$

where A is a positive constant, $d(q)$ is the divisor function.

4 The major arc

In this section, we give a low bound for the integral on the major arc $\mathfrak{M}$. First, we consider the standard major arc $\mathfrak{M}^* = \{\alpha : |\alpha| \leq X^{-1+1/12-\varepsilon}\}$. Using the idea due to Harman [7], we get the following lemma (one can also see section 3 of Mu and Qu [16]). One may improve the standard major arc to $\{\alpha : |\alpha| \leq X^{-1+2/15-\varepsilon}\}$ by using some ideas due to Languasco and Zaccagnini [10] (one can also see [5]). But there is no improvement for our result, because our improvement comes from the minor arc.

Lemma 4.1. *We have*

$$\mathfrak{J}(\mathfrak{M}^\star) \gg \tau^2 X^{77/60}. \tag{4.1}$$

Lemma 4.2. *We have*

$$\mathfrak{J}(\mathfrak{M} \setminus \mathfrak{M}^\star) = o(\tau^2 X^{77/60}). \tag{4.2}$$

Proof. For a given α, by Dirichlet's theorem in Diophantine approximation, there exist integers a_1, a_2, q_1, q_2 depending on α such that

$$|q_1\lambda_1\alpha - a_1| \le X^{-1+1/100}, \quad |q_2\lambda_2\alpha - a_2| \le X^{-1+1/100}$$

with $(a_j, q_j) = 1$ and $1 \le q_j \le X^{1-1/100}$. Since $\alpha \in \mathfrak{M} \setminus \mathfrak{M}^\star$, we see that $a_1a_2 \ne 0$ and $a_j/|\alpha| \ll q_j$. Now we assert that

$$\max(q_1, q_2) \ge X^{1/100}. \tag{4.3}$$

We will reason by absurdity. Suppose both q_1 and q_2 are less that $X^{1/100}$. We have

$$\begin{aligned} |a_2q_1\lambda_1/\lambda_2 - a_1q_2| &= \left|\frac{a_2}{\lambda_2\alpha}(q_1\lambda_1\alpha - a_1) - \frac{a_1}{\lambda_2\alpha}(q_2\lambda_2\alpha - a_2)\right| \\ &\ll X^{-1+1/50}. \end{aligned}$$

Since there is a convergent a/q to λ_1/λ_2 with $q = X^{5/12}$. Thus we have

$$|a_2q_1\lambda_1/\lambda_2 - a_1q_2| = o(q^{-1}). \tag{4.4}$$

But

$$|a_2q_1| \ll q_1q_2 \ll X^{1/50} = o(q). \tag{4.5}$$

This contradicts the definition of q as the denominator of a convergent to λ_1/λ_2 (see Lemma 9 of [1]). Thus one of q_1, q_2 is greater than $X^{1/100}$. Then, by Lemmas 3.1 and 3.5, we have

$$\min\left(|S_1(\lambda_1\alpha)|, |S_2(\lambda_2\alpha)|^2\right) \ll X^{1-1/200+\varepsilon}. \tag{4.6}$$

Hence, by the arguments of Lemma 4.6 of [3], it is easy to get

$$\mathfrak{J}(\mathfrak{M} \setminus \mathfrak{M}^\star) = o(\tau^2 X^{77/60}).$$

□

5 The minor arc

First, we divide the minor arc $\mathfrak{m}$ into four parts. Let $\mathfrak{m}' = \mathfrak{m}_1 \cup \mathfrak{m}_2 \cup \mathfrak{m}_3$, and $\mathfrak{m}_4 = \mathfrak{m} \setminus \mathfrak{m}'$, where

$$\begin{aligned} \mathfrak{m}_1 &= \{\alpha \in \mathfrak{m} : |S_1(\lambda_1\alpha)| \le X^{1-1/6+\varepsilon}\}, \\ \mathfrak{m}_2 &= \{\alpha \in \mathfrak{m} : |S_1(\lambda_1\alpha)| > X^{1-1/6+\varepsilon}; |S_2(\lambda_2\alpha)| > X^{1/2-1/16+\varepsilon}\}, \\ \mathfrak{m}_3 &= \{\alpha \in \mathfrak{m} : |S_1(\lambda_1\alpha)| > X^{1-1/6+\varepsilon}; |S_4(\lambda_4\alpha)| > X^{1/4-1/96+\varepsilon}\}. \end{aligned}$$

Now, we begin to estimate the integral on $\mathfrak{m}_j$ respectively. First, it is easy to see that

$$\mathfrak{J}(\mathfrak{m}_1) \ll \left(\max_{\alpha\in\mathfrak{m}_1} S_1(\lambda_1\alpha)\right)^{3/16} \left(\int_{\mathbb{R}} |S_1(\lambda_1\alpha)|^2 K_\tau(\alpha)d\alpha\right)^{13/32} \left(\int_{\mathbb{R}} |S_3(\lambda_3\alpha)|^8 K_\tau(\alpha)d\alpha\right)^{3/32} \tag{5.1}$$

$$\times\left(\int_{\mathbb{R}}|S_2(\lambda_2\alpha)^2S_4(\lambda_4\alpha)^4|K_\tau(\alpha)d\alpha\right)^{1/4}\left(\int_{\mathbb{R}}|S_2(\lambda_2\alpha)^2S_5(\lambda_5\alpha)^6|K_\tau(\alpha)d\alpha\right)^{1/8}$$
$$\times\left(\int_{\mathbb{R}}|S_2(\lambda_2\alpha)^2S_3(\lambda_4\alpha)^2S_5(\lambda_5\alpha)^2|K_\tau(\alpha)d\alpha\right)^{1/8}$$
$$\ll (X^{1-1/6+\varepsilon})^{3/16}(\tau X^{1+\varepsilon})^{13/32}(\tau X^{5/3+\varepsilon})^{3/32}(\tau X^{1+\varepsilon})^{1/4}(\tau X^{6/5+\varepsilon}\tau X^{16/15+\varepsilon})^{1/8}$$
$$\ll \tau X^{77/60-1/32+2\varepsilon}.$$

Lemma 5.1. *We have*

$$\mathcal{J}(\mathfrak{m}_2)\ll \tau X^{77/60-1/32+\varepsilon}. \tag{5.2}$$

Proof. We use the method of Harman [7]. We divide $\mathfrak{m}_2$ into disjoint sets such that for $\alpha\in\mathcal{A}(Z_1,Z_2,y)$, we have

$$Z_1\le|S_1(\lambda_1\alpha)|<2Z_1\ \text{ or }\ Z_2\le|S_2(\lambda_2\alpha)|<2Z_2\ \text{ or }\ y\le|\alpha|<2y,$$

where $Z_1 = X^{1-1/6+\varepsilon}2^{t_1}$, $Z_2 = X^{1/2-1/16+\varepsilon}2^{t_2}$, $y = 2^s$ for some positive integers t_1, t_2, s. Thus, by Corollaries 3.2 and 3.6, there exist two pairs of coprime integers (a_1,q_1), (a_2,q_2) with $a_1a_2\ne 0$ and

$$1\le q_1\ll (X/Z_1)^2X^{\varepsilon},\ \ |q_1\lambda_1\alpha-a_1|\ll (X/Z_1)^2X^{\varepsilon-1};$$
$$1\le q_2\ll (X^{1/2}/Z_2)^4X^{\varepsilon},\ \ |q_2\lambda_2\alpha-a_2|\ll (X^{1/2}/Z_2)^4X^{\varepsilon-1}.$$

Then for any $\alpha\in\mathcal{A}(Z_1,Z_2,y)$, we have $|a_j/\alpha|\ll q_j$.

Let $\mathcal{A}'=\mathcal{A}(Z_1,Z_2,y,Q_1,Q_2)$ be the subset of $\mathcal{A}(Z_1,Z_2,y)$ for which $q_j\sim Q_j$. Then, by a familiar argument (see P. 147 of [17] for example),

$$\begin{aligned}\left|a_2q_1\frac{\lambda_1}{\lambda_2}-a_1q_2\right| &= \left|\frac{a_2(q_1\lambda_1\alpha-a_1)+a_1(a_2-q_2\lambda_2\alpha)}{\lambda_2\alpha}\right|\\ &\ll Q_2(X/Z_1)^2X^{\varepsilon-1}+Q_1(X^{1/2}/Z_2)^4X^{\varepsilon-1}\\ &\ll \frac{X^{3+2\varepsilon}}{Z_1^2Z_2^4}\ll X^{-5/12-4\varepsilon}.\end{aligned}$$

Also

$$|a_2q_1|\ll yQ_1Q_2.$$

Note that $q=X^{5/12}$. We have

$$\left\|a_2q_1\frac{\lambda_1}{\lambda_2}\right\|\le\frac{1}{4q},\ \ q_1\sim Q_1,\ \ a_2\asymp yQ_2, \tag{5.3}$$

since X is sufficiently large. Then by the pigeon-hole principle and the Legendres law of best approximation for continued fractions, the above inequality (5.7) have $\ll yQ_1Q_2q^{-1}$ solutions of $|a_2q_1|$ (see Lemma 9 of [1]). Clearly, each value of $|a_2q_1|$ corresponds to $\ll X^{\varepsilon}$ values of a_1,a_2,q_1,q_2 by the well-known bound on the divisor function. Hence, we conclude that

$$\begin{aligned}\mu(\mathcal{A}') &\ll X^{\varepsilon}\frac{yQ_1Q_2}{q}\min\left((X/Z_1)^2X^{\varepsilon-1}Q_1^{-1},(X^{1/2}/Z_2)^4X^{\varepsilon-1}Q_2^{-1}\right)\\ &\ll X^{\varepsilon}\frac{yQ_1Q_2}{q}\frac{X^{1+\varepsilon}}{Z_1Z_2^2Q_1^{1/2}Q_2^{1/2}}\ll\frac{X^{1+2\varepsilon}yQ_1^{1/2}Q_2^{1/2}}{qZ_1Z_2^2}\ll\frac{X^{3+3\varepsilon}y}{qZ_1^2Z_2^4},\end{aligned} \tag{5.4}$$

where $\mu(\mathcal{A}')$ is the Lebesgue measure of $\mathcal{A}'$. Thus we have

$$\mathcal{J}(\mathcal{A}')\ll Z_1Z_2X^{1/3+1/4+1/5}\mu(\mathcal{A}')\min(\tau^2,y^{-2})$$

$$\ll \tau \frac{X^{227/60+3\varepsilon}}{qZ_1Z_2^3} \ll \tau X^{77/60-1/16+\varepsilon}.$$

Summing over all possible values of Z_1, Z_2, y, Q_1, Q_2, we conclude that

$$\mathcal{J}(\mathfrak{m}_2) \ll \tau X^{77/60-1/32+\varepsilon}. \tag{5.5}$$

□

Lemma 5.2. *We have*

$$\mathcal{J}(\mathfrak{m}_3) \ll \tau X^{77/60-1/32+\varepsilon}. \tag{5.6}$$

Proof. The proof is similar to that of lemma 5.1, we only give a brief proof. We divide $\mathfrak{m}_3$ into disjoint sets such that for $\alpha \in \mathcal{A}(Z_1, Z_2, y)$, we have

$$Z_1 \le |S_1(\lambda_1\alpha)| < 2Z_1 \ \text{ or } \ Z_2 \le |S_4(\lambda_4\alpha)| < 2Z_2 \ \text{ or } \ y \le |\alpha| < 2y,$$

where $Z_1 = X^{1-1/6+\varepsilon}2^{t_1}$, $Z_2 = X^{1/4-1/96+\varepsilon}2^{t_2}$, $y = 2^s$ for some positive integers t_1, t_2, s. Thus, by Corollaries 3.2 and 3.4, there exist two pairs of coprime integers (a_1, q_1), (a_2, q_2) with $a_1a_2 \ne 0$ and

$$1 \le q_1 \ll (X/Z_1)^2X^{\varepsilon}, \ \ |q_1\lambda_1\alpha - a_1| \ll (X/Z_1)^2X^{\varepsilon-1};$$
$$1 \le q_2 \ll (X^{1/4}/Z_2)^2X^{\varepsilon}, \ \ |q_2\lambda_4\alpha - a_2| \ll (X^{1/4}/Z_2)^2X^{\varepsilon-1}.$$

Let $\mathcal{A}' = \mathcal{A}(Z_1, Z_2, y, Q_1, Q_2)$ be the subset of $\mathcal{A}(Z_1, Z_2, y)$ for which $q_j \sim Q_j$. Then,

$$\left| a_2q_1\frac{\lambda_1}{\lambda_4} - a_1q_2 \right| \ll \frac{X^{3/2+2\varepsilon}}{Z_1^2Z_2^2} \ll X^{-31/48-2\varepsilon}.$$

Also

$$|a_2q_1| \ll yQ_1Q_2.$$

Since $q' \asymp q = X^{5/12}$, we have

$$\left\| a_2q_1\frac{\lambda_1}{\lambda_4} \right\| \le \frac{1}{4q'}, \ \ q_1 \sim Q_1, \ \ a_2 \asymp yQ_2. \tag{5.7}$$

Hence, we conclude that

$$\begin{aligned} \mu(\mathcal{A}') &\ll X^{\varepsilon}\frac{yQ_1Q_2}{q'} \min\left((X/Z_1)^2X^{\varepsilon-1}Q_1^{-1}, (X^{1/4}/Z_2)^2X^{\varepsilon-1}Q_2^{-1}\right) \\ &\ll X^{\varepsilon}\frac{yQ_1Q_2}{q'}\frac{X^{1/4+\varepsilon}}{Z_1Z_2Q_1^{1/2}Q_2^{1/2}} \ll \frac{X^{1/4+2\varepsilon}yQ_1^{1/2}Q_2^{1/2}}{q'Z_1Z_2} \ll \frac{X^{3/2+3\varepsilon}y}{q'Z_1^2Z_2^2}. \end{aligned} \tag{5.8}$$

Thus by Lemma 3.7, we have

$$\begin{aligned} \mathcal{J}(\mathcal{A}') &\ll \left(\int_{\mathcal{A}'} |S_1(\lambda_1\alpha)S_4(\lambda_4\alpha)|^2K_\tau(\alpha)d\alpha\right)^{1/2} \left(\int_{\mathbb{R}} |S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)S_5(\lambda_5\alpha)|^2K_\tau(\alpha)d\alpha\right)^{1/2} \\ &\ll \left(\tau X^{16/15+\varepsilon}\right)^{1/2}\left(\min(\tau^2, y^{-2})Z_1^2Z_2^2\frac{X^{3/2+3\varepsilon}y}{q'Z_1^2Z_2^2}\right)^{1/2} \\ &\ll \tau\frac{X^{77/60+2\varepsilon}}{(q')^{1/2}} \ll \tau X^{77/60-5/24+2\varepsilon}. \end{aligned}$$

Summing over all possible values of Z_1, Z_2, y, Q_1, Q_2, we conclude that

$$\mathcal{J}(\mathfrak{m}_3) \ll \tau X^{77/60-1/32+\varepsilon}.$$

□

Lemma 5.3. *We have*

$$\mathcal{J}(\mathfrak{m}_4) \ll \tau X^{77/60-1/32+\varepsilon}. \tag{5.9}$$

Proof. We use the method of the first author and Zhao [4]. First, by Cauchy's inequality, we get

$$\mathcal{J}(\mathfrak{m}_4) \ll \left(\int_{\mathbb{R}} |S_1(\lambda_1\alpha)|^2 K_\tau(\alpha)d\alpha\right)^{1/2} \mathfrak{I}(2)^{1/2} \ll (\tau X^{1+\varepsilon})^{1/2}\mathfrak{I}(2)^{1/2}, \tag{5.10}$$

where

$$\mathfrak{I}(t) = \int_{\mathfrak{m}_4} |S_2(\lambda_2\alpha)^2 S_3(\lambda_3\alpha)^t S_4(\lambda_4\alpha)^2 S_5(\lambda_5\alpha)^2|K_\tau(\alpha)d\alpha. \tag{5.11}$$

Then we have

$$\begin{aligned}\mathfrak{I}(2) &= \sum_{p\in\mathfrak{I}_3}(\log p)\int_{\mathfrak{m}_4} e(\alpha\lambda_3 p^3)S_3(-\lambda_3\alpha)|S_2(\lambda_2\alpha)^2 S_4(\lambda_4\alpha)^2 S_5(\lambda_5\alpha)^2|K_\tau(\alpha)d\alpha\\ &\le (\log X)\sum_{n\in\mathfrak{I}_3}\left|\int_{\mathfrak{m}_4} e(\alpha\lambda_3 n^3)S_3(-\lambda_3\alpha)|S_2(\lambda_2\alpha)^2 S_4(\lambda_4\alpha)^2 S_5(\lambda_5\alpha)^2|K_\tau(\alpha)d\alpha\right|.\end{aligned}$$

Then, by Cauchy's inequality, we get

$$\mathfrak{I}(2) \ll P_3^{1/2}(\log X)\mathcal{L}^{1/2}, \tag{5.12}$$

where

$$\mathcal{L} = \sum_{n\in\mathfrak{I}_3}\left|\int_{\mathfrak{m}_4} e(\alpha\lambda_3 n^3)S_3(-\lambda_3\alpha)|S_2(\lambda_2\alpha)^2 S_4(\lambda_4\alpha)^2 S_5(\lambda_5\alpha)^2|K_\tau(\alpha)d\alpha\right|^2$$

For the sum $\mathcal{L}$, we have

$$\begin{aligned}\mathcal{L} &= \sum_{n\in\mathfrak{I}_3}\int_{\mathfrak{m}_4}\int_{\mathfrak{m}_4} |S_2(\lambda_2\alpha)^2 S_4(\lambda_4\alpha)^2 S_5(\lambda_5\alpha)^2 S_2(\lambda_2\beta)^2 S_4(\lambda_4\beta)^2 S_5(\lambda_5\beta)^2|\\ &\qquad S_3(-\lambda_3\alpha)S_3(\lambda_3\beta)e(\lambda_3 n^3(\alpha-\beta))K_\tau(\alpha)K_\tau(\beta)d\alpha d\beta\\ &\le \int_{\mathfrak{m}_4} |S_2(\lambda_2\beta)^2 S_4(\lambda_4\beta)^2 S_5(\lambda_5\beta)^2 S_3(\lambda_3\beta)F(\beta)|K_\tau(\beta)d\beta,\end{aligned} \tag{5.13}$$

where

$$F(\beta) = \int_{\mathfrak{m}_4} |S_2(\lambda_2\alpha)^2 S_4(\lambda_4\alpha)^2 S_5(\lambda_5\alpha)^2 S_3(-\lambda_3\alpha)T(\lambda_3(\alpha-\beta))|K_\tau(\alpha)d\alpha \tag{5.14}$$

and

$$T(x) = \sum_{n\in\mathfrak{I}_3} e(xn^3).$$

Let $\mathcal{M}_\beta(r,b) = \{\alpha\in\mathfrak{m}_4 : |r\lambda_3(\alpha-\beta)-b| \le P_3^{-9/4}\}$. Then the set $\mathcal{M}_\beta(r,b)\neq\emptyset$ forces that

$$|b+r\lambda_3\beta| \le |r\lambda_3(\alpha-\beta)-b| + |r\lambda_3\alpha| \le P_3^{-9/4} + r|\lambda_3|\xi.$$

Let $\mathcal{B} = \{b\in\mathbb{Z} : |b+r\lambda_3\beta| \le P_3^{-9/4} + r|\lambda_3|\xi\}$. We divide the set $\mathcal{B}$ into two sets $\mathcal{B}_1 = \{b\in\mathbb{Z} : |b+r\lambda_3\beta| \le r|\lambda_3|\tau^{-1}\}$ and $\mathcal{B}_2 = \mathcal{B}\setminus\mathcal{B}_1$. Let

$$\mathcal{M}_\beta = \bigcup_{1\le r\le P_3^{3/4}}\ \bigcup_{\substack{b\in\mathcal{B}\\(b,r)=1}} \mathcal{M}_\beta(r,b).$$

Then by Lemma 3.8, we have

$$F(\beta) \ll P_3 \int_{\mathcal{M}_\beta} \frac{|S_2(\lambda_2\alpha)^2 S_4(\lambda_4\alpha)^2 S_5(\lambda_5\alpha)^2 S_3(-\lambda_3\alpha)|w_3(r)K_\tau(\alpha)}{1+P_3^3|\lambda_3(\alpha-\beta)-b/r|}\,d\alpha + P_3^{3/4+\varepsilon}\mathfrak{I}(1), \tag{5.15}$$

where $w_3(r)$ is defined as in (3.3). Note that $|S_2(\lambda_2\alpha)| \le P_2 X^{-1/16+\varepsilon}$ and $|S_4(\lambda_4\alpha)| \le P_4 X^{-1/96+\varepsilon}$ for $\alpha \in \mathfrak{m}_4$. Then, by Cauchy's inequality, we get

$$\begin{aligned}
&\int_{\mathcal{M}_\beta} \frac{|S_2(\lambda_2\alpha)^2 S_4(\lambda_4\alpha)^2 S_5(\lambda_5\alpha)^2 S_3(-\lambda_3\alpha)|w_3(r)K_\tau(\alpha)}{1+P_3^3|\lambda_3(\alpha-\beta)-b/r|}\,d\alpha \\
&\ll \left(\int_{\mathfrak{m}_4} |S_2(\lambda_2\alpha)^4 S_3(\lambda_3\alpha)^2 S_4(\lambda_4\alpha)^4 S_5(\lambda_5\alpha)^2|K_\tau(\alpha)\,d\alpha\right)^{1/2} \mathfrak{J}(\beta)^{1/2} \\
&\ll P_2 P_4 X^{-7/96+\varepsilon}\mathfrak{I}(2)^{1/2}\mathfrak{J}(\beta)^{1/2},
\end{aligned} \tag{5.16}$$

where

$$\mathfrak{J}(\beta) = \int_{\mathcal{M}_\beta} \frac{|S_5(\lambda_5\alpha)^2|w_3(r)^2 K_\tau(\alpha)}{(1+P_3^3|\lambda_3(\alpha-\beta)-b/r|)^2}\,d\alpha. \tag{5.17}$$

Now we begin to estimate the integral $\mathfrak{J}(\beta)$. First, we divide it into two parts.

$$\begin{aligned}
\mathfrak{J}(\beta) &= \sum_{1\le r\le P_3^{3/4}} \sum_{\substack{b\in\mathcal{B}\\(b,r)=1}} \int_{\mathcal{M}_\beta(r,b)} \frac{|S_5(\lambda_5\alpha)^2|w_3(r)^2 K_\tau(\alpha)}{(1+P_3^3|\lambda_3(\alpha-\beta)-b/r|)^2}\,d\alpha \\
&= \mathfrak{J}_1(\beta) + \mathfrak{J}_2(\beta),
\end{aligned} \tag{5.18}$$

where

$$\mathfrak{J}_j(\beta) = \sum_{1\le r\le P_3^{3/4}} \sum_{\substack{b\in\mathcal{B}_j\\(b,r)=1}} \int_{\mathcal{M}_\beta(r,b)} \frac{|S_5(\lambda_5\alpha)^2|w_3(r)^2 K_\tau(\alpha)}{(1+P_3^3|\lambda_3(\alpha-\beta)-b/r|)^2}\,d\alpha. \tag{5.19}$$

For the first part, we have

$$\begin{aligned}
\mathfrak{J}_1(\beta) &\ll \tau^2 \sum_{1\le r\le P_3^{3/4}} w_3(r)^2 \sum_{\substack{b\in\mathcal{B}_1\\(b,r)=1}} \int_{|r\lambda_3\gamma|<P_3^{-9/4}} \frac{|S_5(\lambda_5(\beta+\gamma)+b\lambda_5/(r\lambda_3))|^2}{(1+P_3^3|\lambda_3\gamma|)^2}\,d\gamma \\
&\ll \tau^2 \sum_{1\le r\le P_3^{3/4}} w_3(r)^2 \int_{|r\lambda_3\gamma|<P_3^{-9/4}} \frac{U(\mathcal{B}_1^\star)}{(1+P_3^3|\lambda_3\gamma|)^2}\,d\gamma,
\end{aligned}$$

where

$$U(\mathcal{B}_1^\star) = \sum_{b\in\mathcal{B}_1^\star} |S_5(\lambda_5(\beta+\gamma)+b\lambda_5/(r\lambda_3))|^2,$$

and

$$\mathcal{B}_1^\star = \{b\in\mathbb{Z} : -rv([|\lambda_3|v^{-1}\tau^{-1}]+1) < b + r\lambda_3\beta \le rv([|\lambda_3|v^{-1}\tau^{-1}]+1)\}.$$

Since λ_5/λ_3 is rational, we take $\lambda_5/\lambda_3 = u/v$ with $u, v \in \mathbb{Z}$ and $(u,v)=1$. We take $r_1 = \frac{r}{(u,r)}$. Then we have

$$U(\mathcal{B}_1^\star) = \sum_{p_1,p_2\in\mathfrak{I}_5} \sum_{b\in\mathcal{B}_1^\star} e((\lambda_5(\beta+\gamma)+b\lambda_5/(r\lambda_3))(p_1^5-p_2^5))$$

$$
\begin{aligned}
&\le \sum_{p_1,p_2\in\mathcal{I}_5}\left|\sum_{b\in\mathcal{B}_1^\star} e\left(\frac{bu}{rv}(p_1^5-p_2^5)\right)\right| \\
&= 2rv([|\lambda_3|v^{-1}\tau^{-1}]+1)\sum_{\substack{p_1,p_2\in\mathcal{I}_5\\ u(p_1^5-p_2^5)\equiv 0(\mathrm{mod}\, rv)}} 1 \\
&\ll r\tau^{-1}\sum_{\substack{p_1,p_2\in\mathcal{I}_5\\ p_1^5\equiv p_2^5(\mathrm{mod}\, r_1v)}} 1 \\
&\ll r\tau^{-1}P_5^2(r_1v)^{-2}\sum_{\substack{1\le b_1,b_2\le r_1v;(b_1b_2,r_1v)=1\\ b_1^5\equiv b_2^5(\mathrm{mod}\, r_1v)}} 1 \\
&\ll r\tau^{-1}P_5^2(r_1v)^{-1}\sum_{\substack{1\le b\le r_1v\\ b^5\equiv 1(\mathrm{mod}\, r_1v)}} 1 \\
&\ll \tau^{-1}P_5^2 d(r)^c.
\end{aligned}
$$

Thus, by Lemma 3.9, we have

$$
\begin{aligned}
\mathfrak{J}_1(\beta) &\ll \tau P_5^2\sum_{1\le r\le P_3^{3/4}} w_3(r)^2 d(r)^c \int\limits_{|r\lambda_3\gamma|<P_3^{-9/4}} \frac{1}{(1+P_3^3|\lambda_3\gamma|)^2}\,d\gamma \\
&\ll \tau P_5^2X^{-1}\sum_{1\le r\le P_3^{3/4}} w_3(r)^2 d(r)^c \ll \tau P_5^2 X^{-1+\varepsilon}.
\end{aligned}
\tag{5.20}
$$

Now, we begin to estimate $\mathfrak{J}_2(\beta)$. First, without loss of generality we need only consider the set

$$
\mathcal{B}_2' = \{b\in\mathbb{Z} : r|\lambda_3|\tau^{-1} < b + r\lambda_3\beta \le P_3^{-9/4} + r|\lambda_3|\xi\}
$$

which falls in the set

$$
\mathcal{B}_2^\star = \{b\in\mathbb{Z} : rv\kappa_1 < b + r\lambda_3\beta \le rv\kappa_2\},
$$

where $\kappa_1 = [|\lambda_3|v^{-1}\tau^{-1}]$ and $\kappa_2 = [|\lambda_3|v^{-1}\xi]+2$. Then we have

$$
\begin{aligned}
\mathfrak{J}_2(\beta) &\ll \sum_{1\le r\le P_3^{3/4}}\sum_{b\in\mathcal{B}_2^\star}\int_{\mathcal{M}_\beta(r,b)} \frac{|S_5(\lambda_5\alpha)|^2 w_3(r)^2 K_\tau(\alpha)}{(1+P_3^3|\lambda_3(\alpha-\beta)-b/r|)^2}\,d\alpha \\
&\ll \sum_{1\le r\le P_3^{3/4}} w_3(r)^2 \sum_{b\in\mathcal{B}_2^\star}\int_{\mathcal{M}_\beta(r,b)} \frac{|S_5(\lambda_5\alpha)|^2|\alpha|^{-2}}{(1+P_3^3|\lambda_3(\alpha-\beta)-b/r|)^2}\,d\alpha \\
&\ll \sum_{1\le r\le P_3^{3/4}} w_3(r)^2 \sum_{\kappa_1\le k<\kappa_2}\frac{1}{(k-1)^2}\sum_{rvk<b+r\lambda_3\beta\le rv(k+1)}\int_{\mathcal{M}_\beta(r,b)} \frac{|S_5(\lambda_5\alpha)|^2}{(1+P_3^3|\lambda_3(\alpha-\beta)-b/r|)^2}\,d\alpha \\
&\ll \sum_{1\le r\le P_3^{3/4}} w_3(r)^2 \sum_{\kappa_1\le k<\kappa_2}\frac{1}{(k-1)^2}\int\limits_{|r\lambda_3\gamma|<P_3^{-9/4}} \frac{U(\mathcal{C}_k)}{(1+P_3^3|\lambda_3\gamma|)^2}\,d\gamma,
\end{aligned}
$$

where $\mathcal{C}_k = \{b\in\mathbb{Z} : rvk < b + r\lambda_3\beta \le rv(k+1)\}$. On the other hand, similar to the above estimate of $U(\mathcal{B}_1^\star)$, we have $U(\mathcal{C}_k)\ll P_5^2 d(r)^c$. Thus we have

$$
\mathfrak{J}_2(\beta) \ll P_2^5X^{-1}\sum_{1\le r\le P_3^{3/4}} w_3(r)^2 d(r)^c \sum_{\kappa_1\le k<\kappa_2}\frac{1}{(k-1)^2} \ll \tau P_2^5 X^{-1+\varepsilon}.
\tag{5.21}
$$

Combining (5.15)-(5.21), we have

$$
F(\beta) \ll \tau^{1/2}P_2P_3P_4P_5X^{-1/2-7/96+\varepsilon}\mathfrak{J}(2)^{1/2} + P_3^{3/4+\varepsilon}\mathfrak{J}(1)
\tag{5.22}
$$

uniformly for $\beta \in \mathbb{R}$.

Hence, by (5.12), (5.13) and (5.22), we have

$$\mathfrak{I}(2) \ll P_3^{7/8}\mathfrak{I}(1) + \tau^{1/4}(P_2P_3^2P_4P_5)^{1/2}X^{-1/4-7/192+\varepsilon}\mathfrak{I}(1)^{1/2}\mathfrak{I}(2)^{1/4}. \tag{5.23}$$

By Hölder's inequality and Lemma 3.7, we have

$$\begin{aligned}\mathfrak{I}(1) &\le \mathfrak{I}(2)^{1/3}\left(\int_{\mathbb{R}} |S_2(\lambda_2\alpha)^2 S_4(\lambda_4\alpha)^4|K_\tau(\alpha)d\alpha\right)^{1/3}\\ &\left(\int_{\mathbb{R}} |S_2(\lambda_2\alpha)^2 S_3(\lambda_3\alpha)^2 S_5(\lambda_5\alpha)^2|K_\tau(\alpha)d\alpha\right)^{1/6}\left(\int_{\mathbb{R}} |S_2(\lambda_2\alpha)^2 S_5(\lambda_5\alpha)^6|K_\tau(\alpha)d\alpha\right)^{1/6}\\ &\ll \mathfrak{I}(2)^{1/3}(\tau P_2^2P_4^4X^{-1+\varepsilon})^{1/3}(\tau P_2^2P_3^2P_5^2X^{-1+\varepsilon})^{1/6}(\tau P_2^2P_5^6X^{-1+\varepsilon})^{1/6}\\ &\ll \mathfrak{I}(2)^{1/3}(\tau P_2^2P_3^{1/2}P_4^2P_5^2X^{-1+\varepsilon})^{2/3}.\end{aligned}$$

Thus we have

$$\mathfrak{I}(2) \ll P_3^{7/8}\mathfrak{I}(2)^{1/3}(\tau P_2^2P_3^{1/2}P_4^2P_5^2X^{-1+\varepsilon})^{2/3} + \mathfrak{I}(2)^{5/12}\tau^{7/12}(P_2P_3P_4P_5)^{7/6}X^{-7/12-7/192+\varepsilon}.$$

Then this implies

$$\mathfrak{I}(2) \ll \tau P_2^2P_3^{1/2}P_4^2P_5^2X^{-1+\varepsilon}P_3^{21/16} + \tau(P_2P_3P_4P_5)^2X^{-1-1/16+\varepsilon} \ll \tau X^{47/30-1/16+\varepsilon}. \tag{5.24}$$

Then (5.9) follows from (5.10) and (5.24) immediately. □

6 Completion of the proof of Theorem 1.1

We take $\tau = X^{1/32+2\varepsilon}$. Combining (2.4), (5.1) and Lemmas 4.1, 4.2, 5.1, 5.2, 5.3, we deduce that $\mathcal{J}(\mathbb{R}) \gg \tau^2X^{77/60}$. Thus by (2.3), we have

$$\mathcal{N}(\eta, X) \gg \tau X^{77/60}(\log X)^{-5}.$$

Note that $\max(p_j^j) \asymp X$, so $\tau \asymp \max(p_j^j)^{-1/32+2\varepsilon}$. Then we see that the following inequality

$$|\lambda_1p_1 + \cdots + \lambda_4p_4^4 + \lambda_5p_5^5 + \eta| < \max(p_j^j)^{-1/32+2\varepsilon}$$

has $\tau X^{77/60}(\log X)^{-5}$ solutions in primes p_j. Since $X = q^{12/5}$ and λ_1/λ_2 is irrational, there are infinitely many pairs of integers q, a. This implies that the last inequality has infinitely many solutions in primes p_j.

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