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Embedding of Supplementary Results in Strong EMT Valuations and Strength

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Abstract: A graph $\wp$ is said to be *edge-magic total (EMT* if there is a bijection $Y : V(\wp) \cup E(\wp) \rightarrow \{1, 2, \ldots, |V(\wp) \cup E(\wp)|\}$ $s.t.$, $Y(\upsilon) + Y(\upsilon v) + Y(v)$ is a constant for every edge $\upsilon v \in E(\wp)$. An EMT graph $\wp$ will be called *strong edge-magic total (SEMT)* if $Y(V(\wp)) = \{1, 2, \ldots, |V(\wp)|\}$. The *SEMT strength*, $sm(\wp)$, of a graph $\wp$ is the minimum of all magic constants $a(Y)$, where the minimum runs over all the SEMT valuations of $\wp$, this minimum is defined only if the graph has at least one such SEMT valuation. Furthermore, the *SEMT deficiency* of a graph $\wp$, $\mu_s(\wp)$, is either the minimum non-negative integer n such that $\wp \cup nK_1$ is SEMT or $+\infty$ if there will be no such integer n. In this paper, we will present the strong edge-magicness and deficiency of disjoint union of 2-sided generalized comb with bistar, path and caterpillar, moreover we will evaluate the SEMT strength for 2-sided generalized comb.

Keywords: EMT, SEMT, SEMT Deficiency, SEMT strength, 2-sided generalized comb

MSC: 05C15

1 Basic Terminologies and Preliminary Results

Let $\wp = (V, E)$ be a simple, finite, planar and undirected graph having $p = |V(\wp)|$ and $q = |E(\wp)|$. The labeling of a graph $\wp$ is a map that carries graph elements (vertices p, edges q or both) to numbers (usually positive integers). If the domain of the given graph $\wp$ is the vertex(edge)-set then the labeling is described as a *vertex*(*edge*) *labeling*. But if domain be the both vertex and edge sets, that labeling will be the *total labeling*.

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Other domains are possible.

An *EMT labeling* of a graph $\wp$ is a one-to-one mapping Y from $V(\wp) \cup E(\wp)$ onto the set of integers $\{1, 2, \ldots, p+q\}$ with the property that, there is an integral constant "a" such that $Y(\upsilon) + Y(\upsilon v) + Y(v) = a$, for any $\upsilon v \in E(\wp)$. If $Y(V(\wp)) = \{1, 2, \ldots, p\}$ then an EMT labeling is called *SEMT labeling*. A graph $\wp$ is called *EMT*(respectively, *SEMT*) if $\exists$ an EMT(respectively, SEMT) labeling of $\wp$. The concept of EMT labeling was first given by Kotzig and Rosa [1, 2], using a different name "magic-valuations". Interest in these labelings has been lately rekindled by the paper on this subject due to Ringle and Lladó [3]. Shortly after this, Enomoto *et al.* [4] defined a more restrictive form of EMT labeling, namely SEMT labeling (which Wallis [5] refers to as Strong EMT labeling) and conjectured that *every tree is SEMT*. Many researchers have considered SEMT labeling for trees to verify this conjecture. Lee and Shah [6] proved this conjecture for trees with upto 17 vertices by the help of a computer. Javed *et al.* [7] provided the SEMT labeling for the 2-sided generalized comb. In [4] Enomoto *et al.* showed that all caterpillars possess SEMT labeling. The combined effort of Figueroa-Centeno, Ichishima and Muntaner-Batle [8] provide a necessary and sufficient condition for a graph to be SEMT i.e.,

Lemma 1. *[8] A (p, q)-graph $\wp$ is SEMT if and only if there exists a bijective function $Y : V(\wp) \to \{1, 2, \ldots, p\}$ such that the set*

$$S = \{Y(\upsilon) + Y(v) : \upsilon v \in E(\wp)\}$$

consists of q consecutive integers. In such a case, $\wp$ extends to a SEMT labeling of $\wp$ with the magic constant $a = p + q + min(S)$, where

$$S = \{a - (p + q), a - (p + q) + 1, \ldots, a - (p + 1)\}.$$

To understand the lemma 1, we consider an example, see fig. 1, where it is shown that if a graph constitutes consecutive edge-sums then its super edge-magicness is assured.

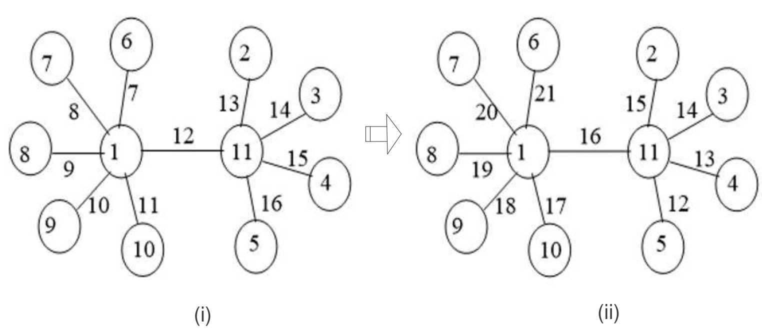

Figure 1: (i) A Bistar $BS(5, 4)$ with consecutive edge-sums, (ii) A SEMT Bistar $BS(5, 4)$ with magic constant $c = 28$

To prove the results in this paper, we will frequently use this Lemma. Conditions given in Lemma 1 will be easier to work with than the original definition.

The *(super) EMT strength* of a graph $\wp$, denoted by $(sm(\wp))$ $m(\wp)$, is defined as the minimum of all magic constants $a(Y)$ where the minimum is taken over all the (super) EMT labelings of $\wp$. This minimum is defined only if the graph has at least one such (super) EMT labeling. One can easily perceive that, because the labels are from the set $\{1, 2, \ldots, p + q\}$,

$$p + q + 3 \leq sm(\wp) \leq 3p.$$

Avadayappan *et al.* first introduced the notions of EMT strength [9] and SEM strength [10] and found EMT strength for path, cycle etc, also the exact values of SEMT strength for some graphs. In [11–13] the SEMT strengths of fire crackers, banana trees, unicyclic graphs, paths, star, bistar, y-tree and the generalized

Petersen graph have been observed.

Kotzig and Rosa [1] verified that for any graph $\wp$ there exists an EMT graph χ s.t. $\chi \cong \wp \cup nK_1$ for some non-negative integer n. This fact leads to the concept of *EMT deficiency* of a graph $\wp$, $\mu(\wp)$, which is the minimum non-negative integer n s.t. $\wp \cup nK_1$ is EMT. In particular,

$$\mu(\wp) = min\{n \geq 0 : \wp \cup nK_1 \ is \ EMT.\}$$

In the same paper, Kotzig and Rosa gave the upper bound for the EMT deficiency of a graph $\wp$ with n vertices i.e.,

$$\mu(\wp) \leq F_{n+2} - 2 - n - \frac{n(n-1)}{2}$$

where F_n is the n^{th} Fibonacci number. Figueroa-Centeno, Ichishima and Muntaner-Batle [14] defined a similar concept for SEMT labeling i.e., the *SEMT deficiency* of a graph $\wp$ denoted by $\mu_s(\wp)$ is the minimum non-negative integer n s.t. $\wp \cup nK_1$ has a SEMT labeling, or $+\infty$ if there is no such n, more precisely,
If $M(\wp) = \{n \geq 0 : \wp \cup nK_1$ *is a SEMT graph*$\}$, then

$$\mu_s(\wp) = \begin{cases} min\, M(\wp) & if\ M(\wp) \neq \phi \\ +\infty & if\ M(\wp) = \phi \end{cases}$$

It can be easily seen that for every graph $\wp$, $\mu(\wp) \leq \mu_s(\wp)$. In [14, 15], Figueroa-Centeno *et al.* provided the exact values of SEMT deficiencies of several classes of graphs. They also proved that *all forests have finite deficiencies*. Ngurah *et al.* [16], Baig *et al.* [17] and Javed *et al.* [18] gave some upper bounds for the SEMT deficiency of various forests. In [19], Figueroa-Centeno *et al.* conjectured that *every forest with two components has SEMT deficiency* ≤ 1. In [20–23], readers can find some related results. The examination of deficiencies in this paper will put evidence on this conjecture. However, this conjecture is still open too. One can go for [24] to review the work that has been done up till now on different types of graph labelings.

In the succeeding section, we will formulate the SEMT labeling and deficiency for forests formed by 2-sided generalized comb, bistar, caterpillar and path, with restricted parameters. For all graph-theoretic terminologies and notions, we refer the reader to [25, 26].

2 Main Results

Two-sided generalized comb denoted by $Cb^2_{\nu,\upsilon}$ is defined in [7] and have proved that it admits SEMT labeling.

Definition 1. A *2-sided generalized comb*, $Cb^2_{\nu,\upsilon}$, deduced from ν paths $x_{\imath,1}, x_{\imath,2}, \ldots,$ $x_{\imath,\nu}$; $1 \leq \imath \leq \upsilon$, $\nu \geq 2$ of length υ, where $\upsilon \geq 3$ is odd, by adding one new vertex $x_{\frac{\upsilon+1}{2},0}$ and ν new edges $x_{\frac{\upsilon+1}{2},\imath}x_{\frac{\upsilon+1}{2},\imath+1}$; $0 \leq \imath \leq \nu - 1$, see Figure 2.

$$V(Cb^2_{\nu,\upsilon}) = \{x_{\imath,\jmath} : 1 \leq \imath \leq \upsilon, 1 \leq \jmath \leq \nu\} \cup \{x_{\frac{\upsilon+1}{2},0}\}$$

$$E(Cb^2_{\nu,\upsilon}) = \{x_{\frac{\upsilon+1}{2},\jmath}x_{\frac{\upsilon+1}{2},\jmath+1} : 0 \leq \jmath \leq \nu - 1\} \cup \{x_{\imath,\jmath}x_{\imath+1,\jmath} : 1 \leq \imath \leq \upsilon - 1, 1 \leq \jmath \leq \nu\}$$

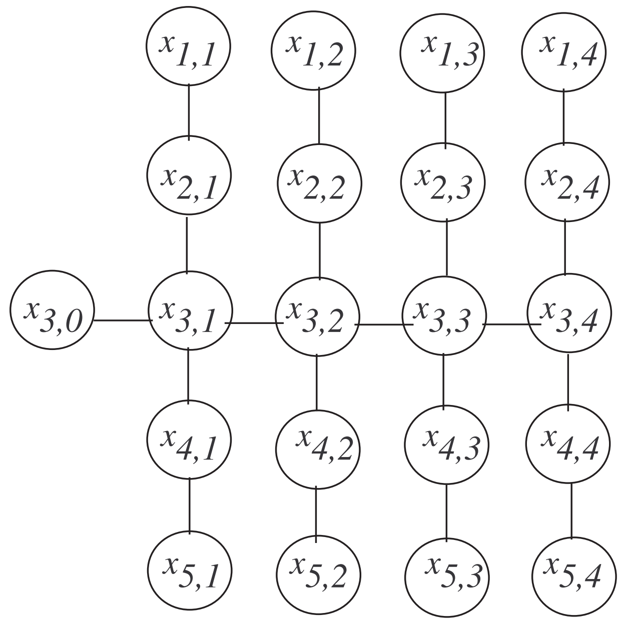

Figure 2: 2-sided Generalized Comb $Cb^2_{4,5}$

Avadayappan *et al.* made a following remark about SEMT graphs i.e.,

Note 2. [10] Let Y be a SEMT labeling of $\wp$ with the magic sum $a(Y)$. Then adding all the magic sums obtained at each edge, we get

$$q\ a(Y) = \sum_{v \in V(\wp)} deg_{\wp}(v)Y(v) + \sum_{e \in E(\wp)} Y(e), \qquad q = |E(\wp)| \tag{1}$$

This condition holds also for EMT labelings. The term $deg(v)$ in above expression is the degree of vertex $v \in \wp$ which can be defined as the set of vertices adjacent to $v \in V(\wp)$, denoted by $N_{\wp}(v)$, and $deg_{\wp}(v) = |N_{\wp}(v)|$ is the *degree* of v in $\wp$

There may exist a variety of SEMT labeling schemes for a single graph- if any graph admits a SEMT labeling then another distinct SEMT labeling will surely exist for the same graph because of the *dual super labeling* detailed in [27]- and of course there will be as many different magic constants as the distinct labeling schemes. Many researchers have found the lower and upper bounds of magic constants for various graphs. In this paper, we will find the bounds for the magic constants of 2-sided generalized comb.

Clearly, $Cb^2_{v,\upsilon}; v \geq 2, \upsilon\ be\ an\ odd\ number\ \geq 3$, has $v\upsilon + 1$ vertices and $v\upsilon$ edges. Among these vertices, $v - 1$ vertices have degree 4, one vertex has degree 3, $v\upsilon - 3v$ vertices have degree 2 and the remaining $2v + 1$ vertices have degree 1, see fig 2. Suppose $Cb^2_{v,\upsilon}$ has an EMT labeling with magic constant *"a"*, then qa where $q = v\upsilon$, can not be smaller than the sum obtained by assigning the smallest $v - 1$ labels to the vertices of degree 4, one next smallest label to the vertex of degree 3, the $q - 3v$ next smallest labels to the vertices of degree 2, and $2v + 1$ next smallest labels to the vertices of degree 1; in other words:

$$\begin{aligned} q\,a &\geq 4\sum_{\imath=1}^{v-1} \imath + 3v + 2\sum_{\imath=v+1}^{q-2v} \imath + \sum_{\imath=q-2v+1}^{q+1} \imath + \sum_{\imath=q+2}^{2q+1} \imath \\ &= \frac{4v(v-1) + 6v + 2(q - v + 1)(q - 3v) + (2q - 2v + 2)(2v + 1) + 3q(q + 1)}{2} \\ &= \frac{5q^2 + 6v^2 - 4qv + 7q - 2v + 2}{2} \end{aligned}$$

An upper bound for qa can be achieved by giving the largest $v-1$ labels to the vertices of degree 4, one next largest label to the vertex of degree 3, the $q-3v$ next largest labels to the vertices of degree 2, and $2v+1$ next largest labels to the vertices of degree 1, in other words:

$$\begin{aligned} q\,a &\le 4\sum_{\imath=2q-v+3}^{2q+1}\imath + 3(2q-v+2) + 2\sum_{\imath=q+2v+2}^{2q-v+1}\imath + \sum_{\imath=q+1}^{q+2v+1}\imath + \sum_{\imath=1}^{q}\imath \\ &= \frac{4(4q-v+4)(v-1)+6(2q-v+2)+2(3q+v+3)(q-3v)}{2} \\ &+ \frac{(2q+2v+2)(2v+1)+q(q+1)}{2} \\ &= \frac{7q^2-6v^2+4qv+5q+2v-2}{2} \end{aligned}$$

Thus, we have the following result,

Lemma 2. *If $Cb^2_{v,\upsilon}$; $v \ge 2$, υ be an odd number ≥ 3, is an EMT graph, then magic constant "a" is in the following interval:*

$$\frac{1}{2q}(5q^2+6v^2-4qv+7q-2v+2) \le a \le \frac{1}{2q}(7q^2-6v^2+4qv+5q+2v-2);\;\; q = v\upsilon$$

By a similar argument, it is easy to verify that the following lemma holds, because in SEMT, vertices receive the smallest labels.

Lemma 3. *If $Cb^2_{v,\upsilon}$; $v \ge 2$, υ be an odd number ≥ 3, is a SEMT graph, then magic constant "a" is in the following interval:*

$$\frac{1}{2q}(5q^2+6v^2-4qv+7q-2v+2) \le a \le \frac{1}{2q}(5q^2-6v^2+4qv+7q+2v-2);\;\; q = v\upsilon$$

2.1 Semt Strength of 2-Sided Generalized Comb

From SEMT labeling for 2-sided generalized comb formulated in [7], we have the magic constant:

$$a = \begin{cases} 2v\upsilon + \lceil \frac{v\upsilon}{2} \rceil + 4\;;\; \frac{\upsilon-1}{2} \equiv 1(mod 2) \\ 2v\upsilon + \lfloor \frac{v\upsilon}{2} \rfloor + 4\;;\; \frac{\upsilon-1}{2} \equiv 0(mod 2) \end{cases}$$

and by given lower bound of magic constants in Lemma 3, we have $a(Y) \ge \frac{5q^2+6v^2-4qv+7q-2v+2}{2q}$, where $q = v\upsilon$, thus we can conclude;

Theorem 1. *The SEMT strength for 2-sided generalized comb $Cb^2_{v,\upsilon}$; $v \ge 2$, υ be an odd number ≥ 3 is, for $q = v\upsilon$:*

$$\frac{5q^2+6v^2-4qv+7q-2v+2}{2q} \le sm(Cb^2_{v,\upsilon}) \le 2q + \lceil \frac{q}{2} \rceil + 4, \qquad \frac{\upsilon-1}{2} \equiv 1(mod 2)$$

$$\frac{5q^2+6v^2-4qv+7q-2v+2}{2q} \le sm(Cb^2_{v,\upsilon}) \le 2q + \lfloor \frac{q}{2} \rfloor + 4, \qquad \frac{\upsilon-1}{2} \equiv 0(mod 2).$$

2.2 Semt Labeling and Deficiency of Forests Formed by 2-Sided Generalized Comb, Bistar and Caterpillar

Definition 2. *Star* on n vertices is denoted by $K_{1,n-1}$, is given by the following set of vertices and edges:

$$V(K_{1,n-1}) = \{v_\imath;\; 1 \le \imath \le n\}$$

$$E(K_{1,n-1}) = \{v_1 v_\imath;\ 2 \le \imath \le n\}$$

Bistar $BS(\zeta, \xi)$ is an acyclic graph on n vertices obtained from two stars $K_{1,\zeta}$ and $K_{1,\xi}$ by joining their central vertices by an edge, where $\zeta, \xi \ge 1$ and $\zeta + \xi = n - 2$,

$$V(BS(\zeta, \xi)) = \{z_{\flat t} : \quad \flat = 1, 2; 0 \le t \le \Lambda\},$$

where

$$\Lambda = \begin{cases} \zeta \ ; \flat = 1 \\ \\ \xi \ ; \flat = 2 \end{cases}$$

$$E(BS(\zeta, \xi)) = \{z_{10} z_{1t}; 1 \le t \le \zeta\} \cup \{z_{10} z_{20}\} \cup \{z_{20} z_{2t}; 1 \le t \le \xi\}$$

Definition 3. [25] A *caterpillar* is a graph derived from a path by hanging any number of leaves from the vertices of the path. The caterpillar can be seen as a sequence of stars $S_1 \cup S_2 \cup \ldots \cup S_m$, where each $S_\imath$ is a star with central vertex $c_\imath$ and $\eta_\imath$ leaves for $\imath = 1, 2, \ldots, m$ and the leaves of $S_\imath$ include $c_{\imath-1}$ and $c_{\imath+1}$, for $\imath = 2, 3, \ldots, m-1$. We denoted the caterpillar as $S_{\eta_1,\eta_2,\ldots,\eta_m}$, where the vertex and edge sets are as respectively, $V(S_{\eta_1,\eta_2,\ldots,\eta_m}) = \{c_\imath : 1 \le \imath \le m\} \cup \bigcup_{\imath=2}^{m-1} \{v_\imath^\jmath : 2 \le \jmath \le \eta_\imath - 1\} \cup \{v_1^\jmath : 1 \le \jmath \le \eta_1 - 1\} \cup \{v_m^\jmath : 2 \le \jmath \le \eta_m\}$, $E(S_{\eta_1,\eta_2,\ldots,\eta_m}) = \{c_\imath c_{\imath+1} : 1 \le \imath \le m - 1\} \cup \bigcup_{\imath=2}^{m-1} \{c_\imath v_\imath^\jmath : 2 \le \jmath \le \eta_\imath - 1\} \cup \{c_1 v_1^\jmath : 1 \le \jmath \le \eta_1 - 1\} \cup \{c_m v_m^\jmath : 2 \le \jmath \le \eta_m\}$ and $|V(S_{\eta_1,\eta_2,\ldots,\eta_m})| = \sum_{\imath=1}^{m} \eta_\imath - m + 2$, $|E(S_{\eta_1,\eta_2,\ldots,\eta_m})| = \sum_{\imath=1}^{m} \eta_\imath - m + 1$.

Theorem 2. *For $\nu \ge 2$, $\upsilon = 2\varepsilon + 1$, $\varepsilon = 1, 2, 3, \ldots$*
(a): $Cb_{\nu,\upsilon}^2 \cup BS(\zeta, \xi)$ is SEMT.
(b): $\mu_s(Cb_{\nu,\upsilon}^2 \cup BS(\zeta, \xi - 1)) \le 1$.
where $\zeta \ge 0$ and

$$\xi = \begin{cases} 1 + \upsilon(\frac{\nu}{2} - 1) + 1\lfloor \frac{\upsilon-1}{4} \rfloor + 2\lfloor \frac{\upsilon-2}{4} \rfloor \ ; \nu \equiv 0(mod2) \\ \\ 2 + \upsilon(\frac{\nu-3}{2}) + 3\lfloor \frac{\upsilon-1}{4} \rfloor + 2\lfloor \frac{\upsilon-2}{4} \rfloor \quad ; \nu \equiv 1(mod2) \end{cases}$$

Proof. **(a):** Consider the graph $\wp \cong Cb_{\nu,\upsilon}^2 \cup BS(\zeta, \xi)$
Let $p = |V(\wp)|$ and $q = |E(\wp)|$, then we get

$$p = \nu\upsilon + \zeta + \xi + 3$$

$$q = \nu\upsilon + \zeta + \xi + 1$$

we define a labeling $Y : V(Cb_{\nu,\upsilon}^2) \to \{1, 2, \ldots, \nu\upsilon + 1\}$ as

$$Y(x_{\frac{\upsilon+1}{2},0}) = \nu\upsilon + \zeta + \xi + 3$$

and consider the labeling $\Psi : V(\wp) \to \{1, 2, \ldots, p\}$.

For $\frac{\upsilon-1}{2}$ odd;

$$Y(x_{\imath,\jmath}) = \begin{cases} \frac{\imath}{2} + \frac{\upsilon(\jmath-1)}{2} \ ; \imath \equiv 0(mod2), \jmath \equiv 1(mod2) \\ \\ \frac{\upsilon\jmath}{2} - \frac{\imath-1}{2} \quad ; \imath \equiv 1(mod2), \jmath \equiv 0(mod2) \end{cases}$$

$$\Psi(z_{\flat t}) = \begin{cases} \lfloor \frac{\nu\upsilon}{2} \rfloor + t \qquad ; \flat = 1, 1 \le t \le \zeta \\ \\ \lfloor \frac{\nu\upsilon}{2} \rfloor + \zeta + 1 \ ; \flat = 2, t = 0 \end{cases}$$

Let $A = \lfloor \frac{\nu\upsilon}{2} \rfloor + \zeta + 1$, then

$$Y(x_{\imath,\jmath}) = \begin{cases} A + \frac{\imath+1}{2} + \frac{\upsilon(\jmath-1)}{2} \ ; \imath, \jmath \equiv 1(mod2) \\ \\ A + \frac{\upsilon\jmath}{2} - \frac{\imath}{2} + 1 \quad ; \imath, \jmath \equiv 0(mod2) \end{cases}$$

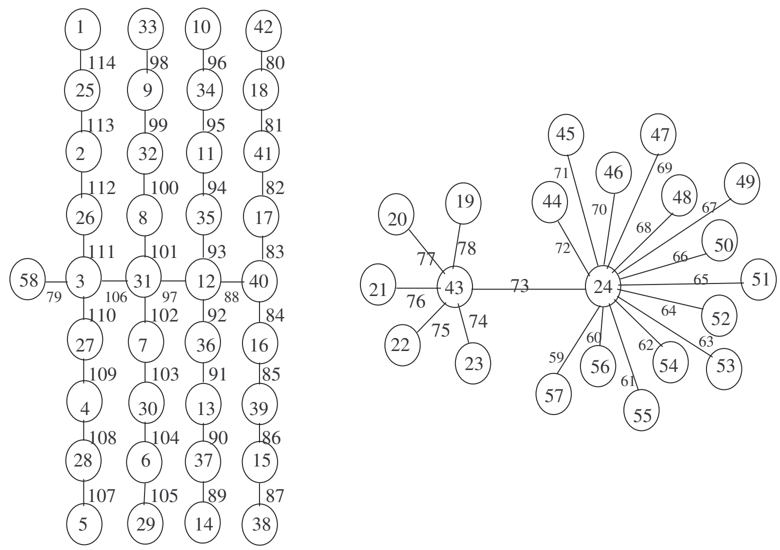

Figure 3: SEMT forest $Cb^2_{4,9} \cup BS(5,14)$

$$\Psi(z_{\flat t}) = \begin{cases} \nu\upsilon + \zeta + 2 & ; \flat = 1, t = 0 \\ \nu\upsilon + \zeta + 2 + t & ; \flat = 2, 1 \le t \le \xi \end{cases}$$

so,

$$Y(x_{\frac{\upsilon+1}{2},0}) = \Psi(x_{\frac{\upsilon+1}{2},0})$$

and

$$Y(x_{\imath,\jmath}) = \Psi(x_{\imath,\jmath}); \quad 1 \le \imath \le \upsilon, 1 \le \jmath \le \nu$$

The edge-sums generated by the above labeling "Ψ" are the set of consecutive positive integers $S = \{\hbar+1, \hbar+2, \ldots, \hbar+q\}$, where $\hbar = \lfloor \frac{\nu\upsilon}{2} \rfloor + \zeta + 2$. Thus by Lemma 1, "$\Psi$" can be extended to a SEMT labeling of $\wp$ and we obtain the magic constant $a = p + q + \hbar + 1$, where $\hbar + 1 = min(S)$.

For $\frac{\upsilon-1}{2}$ even;

$$Y(x_{\imath,\jmath}) = \begin{cases} \frac{\imath+1}{2} + \frac{\upsilon(\jmath-1)}{2} & ; \imath, \jmath \equiv 1(mod2) \\ \frac{\upsilon\jmath}{2} - \frac{\imath}{2} + 1 & ; \imath, \jmath \equiv 0(mod2) \end{cases}$$

$$\Psi(z_{\flat t}) = \begin{cases} \lceil \frac{\nu\upsilon}{2} \rceil + t & ; \flat = 1, 1 \le t \le \zeta \\ \lceil \frac{\nu\upsilon}{2} \rceil + \zeta + 1 & ; \flat = 2, t = 0 \end{cases}$$

Let $A' = \lceil \frac{\nu\upsilon}{2} \rceil + \zeta + 1$, then

$$Y(x_{\imath,\jmath}) = \begin{cases} A' + \frac{\upsilon(\jmath-1)}{2} + \frac{\imath}{2} & ; \imath \equiv 0(mod2), \jmath \equiv 1(mod2) \\ A' + \frac{\upsilon\jmath}{2} - \frac{\imath-1}{2} & ; \imath \equiv 1(mod2), \jmath \equiv 0(mod2) \end{cases}$$

$$\Psi(z_{\flat t}) = \begin{cases} \nu\upsilon + \zeta + 2 & ; \flat = 1, t = 0 \\ \nu\upsilon + \zeta + 2 + t & ; \flat = 2, 1 \le t \le \xi \end{cases}$$

so,

$$Y(x_{\frac{\upsilon+1}{2},0}) = \Psi(x_{\frac{\upsilon+1}{2},0})$$

and

$$Y(x_{\imath,\jmath}) = \Psi(x_{\imath,\jmath}); \quad 1 \le \imath \le \upsilon, 1 \le \jmath \le \nu$$

The edge-sums generated by the above labeling "Ψ" are the set of consecutive positive integers $S = \{\hbar' + 1, \hbar' + 2, \ldots, \hbar' + q\}$, where $\hbar' = \lceil \frac{\nu\upsilon}{2} \rceil + \zeta + 2$. Thus by Lemma 1, "$\Psi$" can be extended to a SEMT labeling of $\wp$ and we obtain the magic constant $a = p + q + \hbar' + 1$, where $\hbar' + 1 = min(S)$.

(b): Let $\mho \cong Cb^2_{\nu,\upsilon} \cup BS(\zeta, \xi - 1) \cup K_1$

Here

$$V(\mho) = V(Cb^2_{\nu,\upsilon}) \cup V(BS(\zeta, \xi - 1)) \cup \{z\}$$

and

$$V(BS(\zeta, \xi - 1) = \{z_{\flat t} : \quad \flat = 1, 2; 0 \le t \le \Lambda\},$$

where

$$\Lambda = \begin{cases} \zeta & ; \flat = 1 \\ \xi - 1 & ; \flat = 2 \end{cases}$$

$$E(BS(\zeta, \xi - 1)) = \{z_{10}z_{1t}; 1 \le t \le \zeta\} \cup \{z_{10}z_{20}\} \cup \{z_{20}z_{2t}; 1 \le t \le \xi - 1\}$$

Let $p' = |V(\mho)|$ and $q' = |E(\mho)|$, so we get

$$p' = \nu\upsilon + \zeta + \xi + 3$$

$$q' = \nu\upsilon + \zeta + \xi$$

Before formulating the labeling $\Psi' : V(\mho) \to \{1, 2, \ldots, p'\}$, keep in view the labeling Y defined in **(a)**. We define the labeling Ψ' as follows:

For $\frac{\upsilon-1}{2}$ odd;

$$Y(x_{\imath,\jmath}) = \Psi(x_{\imath,\jmath}) = \Psi'(x_{\imath,\jmath}); \quad 1 \le \imath \le \upsilon, 1 \le \jmath \le \nu$$

with $A = \Psi(z_{20}) = \Psi'(z_{20})$

$$\Psi'(z_{1t}) = \Psi(z_{1t}), \quad 0 \le t \le \zeta$$

$$\Psi'(z_{2t}) = \Psi(z_{2t}), \quad 0 \le t \le \xi - 1$$

$$\Psi'(z) = \nu\upsilon + \zeta + \xi + 2$$

$$\Psi'(x_{\frac{\upsilon+1}{2},0}) = \Psi(x_{\frac{\upsilon+1}{2},0}) = Y(x_{\frac{\upsilon+1}{2},0})$$

The edge-sums generated by the above labeling "Ψ'" are the set of consecutive positive integers $S = \{\hbar + 1, \hbar + 2, \ldots, \hbar + q'\}$, where $\hbar = \lfloor \frac{\nu\upsilon}{2} \rfloor + \zeta + 2$. Thus by Lemma 1, "$\Psi'$" can be extended to a SEMT labeling of $\mho$ and we obtain the magic constant $a = p' + q' + \hbar + 1$, where $\hbar + 1 = min(S)$.

For $\frac{\upsilon-1}{2}$ even;

$$Y(x_{\imath,\jmath}) = \Psi(x_{\imath,\jmath}) = \Psi'(x_{\imath,\jmath}); \quad 1 \le \imath \le \upsilon, 1 \le \jmath \le \nu$$

with $A' = \Psi(z_{20}) = \Psi'(z_{20})$

$$\Psi'(z_{1t}) = \Psi(z_{1t}); \quad 0 \le t \le \zeta$$

$$\Psi'(z_{2t}) = \Psi(z_{2t}); \quad 0 \le t \le \xi - 1$$

$$\Psi'(z) = \nu\upsilon + \zeta + \xi + 2$$

$$\Psi'(x_{\frac{\upsilon+1}{2},0}) = \Psi(x_{\frac{\upsilon+1}{2},0}) = Y(x_{\frac{\upsilon+1}{2},0})$$

The edge-sums generated by the above labeling "Ψ'" are the set of consecutive positive integers $S = \{\hbar' + 1, \hbar' + 2, \ldots, \hbar' + q'\}$, where $\hbar' = \lceil \frac{\nu\upsilon}{2} \rceil + \zeta + 2$. Thus by Lemma 1, "$\Psi'$" can be extended to a SEMT labeling of $\mho$ and we obtain the magic constant $a = p' + q' + \hbar' + 1$, where $\hbar' + 1 = min(S)$. □

All the remaining results of this paper will use the same labeling Y as defined in the part (a) of Theorem 2.

Theorem 3. *For $\nu \geq 2$, $\upsilon = 2\varepsilon + 1$, $\varepsilon = 1, 2, 3, \ldots$*
(a): $Cb^2_{\nu,\upsilon} \cup S_{\zeta,\xi,\eta}$ is SEMT.
(b): $\mu_s(Cb^2_{\nu,\upsilon} \cup S_{\zeta,\xi-1,\eta}) \leq 1$; $(\nu, \upsilon) \neq (2, 3)$
where $\zeta, \eta \geq 2$ and

$$\xi = \begin{cases} 2 + \upsilon(\frac{\nu}{2} - 1) + 1\lfloor\frac{\upsilon-1}{4}\rfloor + 2\lfloor\frac{\upsilon-2}{4}\rfloor & ; \nu \equiv 0 (mod 2) \\ 3 + \upsilon(\frac{\nu-3}{2}) + 3\lfloor\frac{\upsilon-1}{4}\rfloor + 2\lfloor\frac{\upsilon-2}{4}\rfloor & ; \nu \equiv 1 (mod 2) \end{cases}$$

Proof. **(a):** Consider the graph $\wp \cong Cb^2_{\nu,\upsilon} \cup S_{\zeta,\xi,\eta}$, where

$$V(S_{\zeta,\xi,\eta}) = \{c_\imath : 1 \leq \imath \leq 3\} \cup \{x_\imath; 1 \leq \imath \leq \zeta - 1\} \cup \{y_\imath; 2 \leq \imath \leq \xi - 1\} \cup \{z_\imath; 2 \leq \imath \leq \eta\}$$

and

$$E(S_{\zeta,\xi,\eta}) = \{c_\imath c_{\imath+1} : 1 \leq \imath \leq 2\} \cup \{c_1 x_\imath; 1 \leq \imath \leq \zeta - 1\} \cup \{c_2 y_\imath; 2 \leq \imath \leq \xi - 1\} \cup \{c_3 z_\imath; 2 \leq \imath \leq \eta\}$$

Let $p = |V(\wp)|$ and $q = |E(\wp)|$, so we get

$$p = \nu\upsilon + \zeta + \xi + \eta$$
$$q = \nu\upsilon + \zeta + \xi + \eta - 2$$

Before formulating the labeling $\Psi : V(\wp) \to \{1, 2, \ldots, p\}$, keep in view the labeling Y defined in Theorem 2.

For $\frac{\upsilon-1}{2}$ odd;
Take the previously mentioned labeling Y (for $\frac{\upsilon-1}{2}$ odd) with $A = \lfloor\frac{\nu\upsilon}{2}\rfloor + \zeta + \eta - 1$. We define the labeling Ψ as follows:

$$\Psi(x_\imath) = \lfloor\frac{\nu\upsilon}{2}\rfloor + \imath; \quad 1 \leq \imath \leq \zeta - 1$$
$$\Psi(c_2) = \lfloor\frac{\nu\upsilon}{2}\rfloor + \zeta$$
$$\Psi(z_\imath) = \lfloor\frac{\nu\upsilon}{2}\rfloor + \zeta + \imath; \quad 1 \leq \imath \leq \eta - 1$$
$$\Psi(c_1) = \nu\upsilon + \zeta + \eta$$
$$\Psi(y_\imath) = \nu\upsilon + \zeta + \eta + \imath; \quad 1 \leq \imath \leq \xi - 2$$
$$\Psi(c_3) = \nu\upsilon + \zeta + \xi + \eta - 1$$
$$\Psi(x_{\frac{\upsilon+1}{2},0}) = \nu\upsilon + \zeta + \xi + \eta$$

$\Psi(x_{\imath,\jmath}) = Y(x_{\imath,\jmath})$; $1 \leq \imath \leq \upsilon, 1 \leq \jmath \leq \nu$

The edge-sums generated by the above labeling "Ψ" are the set of consecutive positive integers $S = \{\hbar + 1, \hbar + 2, \ldots, \hbar + q\}$, where $\hbar = \lfloor\frac{\nu\upsilon}{2}\rfloor + \zeta + \eta$. Thus by Lemma 1, "$\Psi$" can be extended to a SEMT labeling of $\wp$ and we obtain the magic constant $a = p + q + \hbar + 1$, where $\hbar + 1 = min(S)$.

For $\frac{\upsilon-1}{2}$ even;
Take the previously mentioned labeling Y (for $\frac{\upsilon-1}{2}$ odd) with $A' = \lceil\frac{\nu\upsilon}{2}\rceil + \zeta + \eta - 1$. We define the labeling Ψ as follows:

$$\Psi(x_\imath) = \lceil\frac{\nu\upsilon}{2}\rceil + \imath; \quad 1 \leq \imath \leq \zeta - 1$$
$$\Psi(c_2) = \lceil\frac{\nu\upsilon}{2}\rceil + \zeta$$
$$\Psi(z_\imath) = \lceil\frac{\nu\upsilon}{2}\rceil + \zeta + \imath; \quad 1 \leq \imath \leq \eta - 1$$

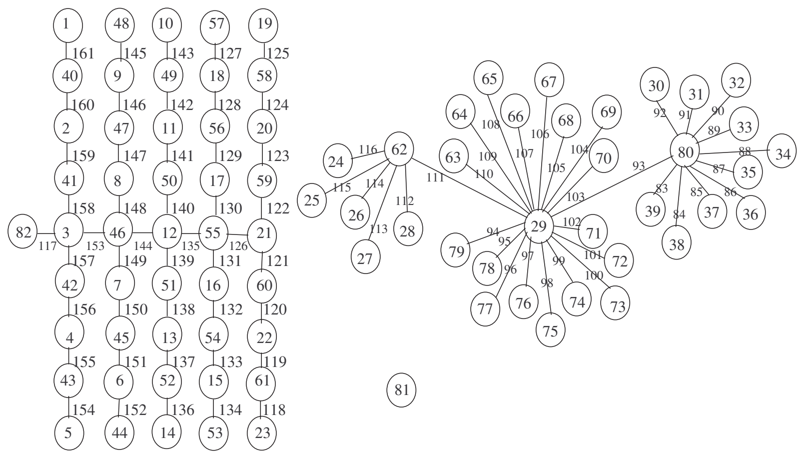

Figure 4: SEMT forest $Cb^2_{5,9} \cup S_{6,19,11} \cup K_1$

$$\Psi(c_1) = \nu\upsilon + \zeta + \eta$$

$$\Psi(y_\imath) = \nu\upsilon + \zeta + \eta + \imath; \quad 1 \le \imath \le \xi - 2$$

$$\Psi(c_3) = \nu\upsilon + \zeta + \xi + \eta - 1$$

$$\Psi(x_{\frac{\upsilon+1}{2},0}) = \nu\upsilon + \zeta + \xi + \eta$$

$\Psi(x_{\imath,\jmath}) = Y(x_{\imath,\jmath}) \quad ; 1 \le \imath \le \upsilon, 1 \le \jmath \le \nu$

The edge-sums generated by the above labeling "Ψ" are the set of consecutive positive integers $S = \{\hbar' + 1, \hbar' + 2, \ldots, \hbar' + q\}$, where $\hbar' = \lceil \frac{\nu\upsilon}{2} \rceil + \zeta + \eta$. Thus by Lemma 1, "$\Psi$" can be extended to a SEMT labeling of $\wp$ and we obtain the magic constant $a = p + q + \hbar' + 1$, where $\hbar' + 1 = min(S)$.

(b): Let $\mho \cong Cb^2_{\nu,\upsilon} \cup S_{\zeta,\xi-1,\eta} \cup K_1$, where

$$V(\mho) = V(Cb^2_{\nu,\upsilon}) \cup V(S_{\zeta,\xi-1,\eta}) \cup \{d\}$$

$$V(S_{\zeta,\xi-1,\eta}) = \{c_\imath : 1 \le \imath \le 3\} \cup \{x_\imath; 1 \le \imath \le \zeta - 1\} \cup \{y_\imath; 2 \le \imath \le \xi - 2\} \cup \{z_\imath; 2 \le \imath \le \eta\}$$

and

$$E(S_{\zeta,\xi-1,\eta}) = \{c_\imath c_{\imath+1} : 1 \le \imath \le 2\} \cup \{c_1 x_\imath; 1 \le \imath \le \zeta - 1\} \cup \{c_2 y_\imath; 2 \le \imath \le \xi - 2\} \cup \{c_3 z_\imath; 2 \le \imath \le \eta\}$$

Let $p' = |V(\mho)|$ and $q' = |E(\mho)|$, so we get

$$p' = \nu\upsilon + \zeta + \xi + \eta$$

$$q' = \nu\upsilon + \zeta + \xi + \eta - 3$$

Before formulating the labeling $\Psi' : V(\mho) \to \{1, 2, \ldots, p'\}$, keep in view the labeling Y defined in Theorem 2.

For $\frac{\upsilon-1}{2}$ odd;

$$Y(x_{\imath,\jmath}) = \Psi(x_{\imath,\jmath}) = \Psi'(x_{\imath,\jmath}) \quad ; 1 \le \imath \le \upsilon, 1 \le \jmath \le \nu$$

with $A = \lfloor \frac{\nu\upsilon}{2} \rfloor + \zeta + \eta - 1$

$$\Psi'(x_\imath) = \Psi(x_\imath); \quad 1 \le \imath \le \zeta - 1$$

$$\Psi'(c_2) = \Psi(c_2)$$
$$\Psi'(z_\imath) = \Psi(z_\imath); \quad 1 \le \imath \le \eta - 1$$
$$\Psi'(c_1) = \Psi(c_1)$$
$$\Psi'(y_\imath) = \nu\upsilon + \zeta + \eta + \imath; \quad 1 \le \imath \le \xi - 3$$
$$\Psi'(c_3) = \nu\upsilon + \zeta + \xi + \eta - 2$$
$$\Psi'(d) = \nu\upsilon + \zeta + \xi + \eta - 1$$
$$\Psi'(x_{\frac{\upsilon+1}{2},0}) = \Psi(x_{\frac{\upsilon+1}{2},0})$$

The edge-sums generated by the above labeling "Ψ'" are the set of consecutive positive integers $S = \{\hbar + 1, \hbar + 2, \dots, \hbar + q'\}$, where $\hbar = \lfloor \frac{\nu\upsilon}{2} \rfloor + \zeta + \eta$. Thus by Lemma 1, "$\Psi'$" can be extended to a SEMT labeling of $\mho$ and we obtain the magic constant $a = p' + q' + \hbar + 1$, where $\hbar + 1 = min(S)$.

For $\frac{\upsilon-1}{2}$ even;

$$Y(x_{\imath,\jmath}) = \Psi(x_{\imath,\jmath}) = \Psi'(x_{\imath,\jmath}) \quad ; 1 \le \imath \le \upsilon, 1 \le \jmath \le \nu$$

with $A' = \lceil \frac{\nu\upsilon}{2} \rceil + \zeta + \eta - 1$

$$\Psi'(x_\imath) = \Psi(x_\imath); \quad 1 \le \imath \le \zeta - 1$$
$$\Psi'(c_2) = \Psi(c_2)$$
$$\Psi'(z_\imath) = \Psi(z_\imath); \quad 1 \le \imath \le \eta - 1$$
$$\Psi'(c_1) = \Psi(c_1)$$
$$\Psi'(y_\imath) = \nu\upsilon + \zeta + \eta + \imath; \quad 1 \le \imath \le \xi - 3$$
$$\Psi'(c_3) = \nu\upsilon + \zeta + \xi + \eta - 2$$
$$\Psi'(d) = \nu\upsilon + \zeta + \xi + \eta - 1$$
$$\Psi'(x_{\frac{\upsilon+1}{2},0}) = \Psi(x_{\frac{\upsilon+1}{2},0})$$

The edge-sums generated by the above labeling "Ψ'" are the set of consecutive positive integers $S = \{\hbar' + 1, \hbar' + 2, \dots, \hbar' + q'\}$, where $\hbar' = \lceil \frac{\nu\upsilon}{2} \rceil + \zeta + \eta$. Thus by Lemma 1, "$\Psi'$" can be extended to a SEMT labeling of $\mho$ and we obtain the magic constant $a = p' + q' + \hbar' + 1$, where $\hbar' + 1 = min(S)$. □

2.3 Semt Labeling and Deficiency of Forests Formed by 2-Sided Generalized Comb and Path

Definition 4. P_n be a path of order n and length $n-1$, with vertices labelled from v_1 to v_n along P_n and $E(P_n) = \{v_\imath v_{\imath+1};\ 1 \le \imath \le n-1\}$.

In the next two theorems, we will present two distinct SEMT labelings- which are non-dual of each other- for the same forest be composed of the disjoint union of path P_m and 2-sided Generalized Comb.

Theorem 4. *For $\nu \ge 2$, $\upsilon = 2\varepsilon + 1$, $\varepsilon = 1, 2, 3, \dots$*
(a)(i): $Cb^2_{\nu,\upsilon} \cup P_r$ is SEMT.
(a)(ii): $Cb^2_{\nu,\upsilon} \cup P_{r-1}$ is SEMT.
(b)(i): $\mu_s(Cb^2_{\nu,\upsilon} \cup P_{r-2}) \le 1$.
(b)(ii): $\mu_s(Cb^2_{\nu,\upsilon} \cup P_{r-3}) \le 1$; $(\nu, \upsilon) \ne (2, 3)$
where

$$r = \begin{cases} 4 + 2\upsilon(\frac{\nu}{2} - 1) + 2\lfloor \frac{\upsilon-1}{4} \rfloor + 4\lfloor \frac{\upsilon-2}{4} \rfloor & ; \nu \equiv 0(mod 2) \\ 6 + 2\upsilon(\frac{\nu-3}{2}) + 6\lfloor \frac{\upsilon-1}{4} \rfloor + 4\lfloor \frac{\upsilon-2}{4} \rfloor & ; \nu \equiv 1(mod 2) \end{cases}$$

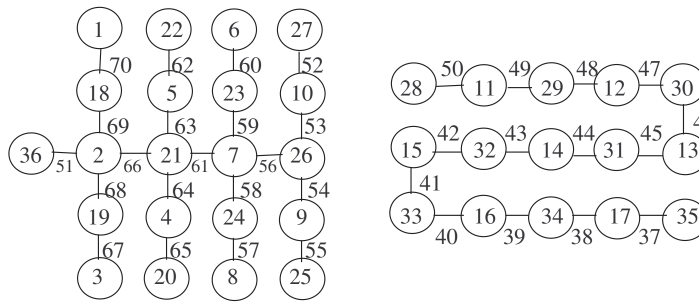

Figure 5: SEMT forest $Cb^2_{4,5} \cup P_{15}$

Proof. **(a):** Consider the graph $\wp \cong Cb^2_{v,\upsilon} \cup P_\varrho$
Let $p = |V(\wp)|$ and $q = |E(\wp)|$, so we get

$$p = v\upsilon + \varrho + 1$$

$$q = v\upsilon + \varrho - 1$$

where

$$\varrho = \begin{cases} r & ;for\ \ a(i) \\ r-1 & ;for\ \ a(ii) \end{cases}$$

Before formulating the labeling $\Psi : V(\wp) \rightarrow \{1, 2, \ldots, p\}$, keep in view the labeling Y defined in Theorem 2.

For $\frac{\upsilon-1}{2}$ odd;
Take the previously mentioned labeling Y (for $\frac{\upsilon-1}{2}$ odd) with $A = \lfloor \frac{v\upsilon}{2} \rfloor + \lceil \frac{\varrho-1}{2} \rceil$. We define the labeling Ψ as follows:

$$\Psi(x_t) = \begin{cases} \lfloor \frac{v\upsilon}{2} \rfloor + k & ; t = 2k,\ 1 \le k \le \lceil \frac{\varrho-1}{2} \rceil \\ v\upsilon + \lfloor \frac{v\upsilon}{2} \rfloor + k - \upsilon + 3(\frac{\upsilon-3}{4}) + 2 & ; t = 2k-1,\ 1 \le k \le \frac{\varrho}{2}, for\ \ a(i) \\ v\upsilon + \lfloor \frac{v\upsilon}{2} \rfloor + k - \upsilon + 3(\frac{\upsilon-3}{4}) + 1 & ; t = 2k-1,\ 1 \le k \le \lceil \frac{\varrho}{2} \rceil, for\ \ a(ii) \end{cases}$$

$\Psi(x_{\imath,\jmath}) = Y(x_{\imath,\jmath})\quad ; 1 \le \imath \le \upsilon,\ 1 \le \jmath \le v$

$$\Psi(x_{\frac{\upsilon+1}{2},0}) = \begin{cases} v\upsilon + \lfloor \frac{v\upsilon}{2} \rfloor + \frac{\varrho}{2} - \upsilon + 3(\frac{\upsilon-3}{4}) + 3 & ; for\ \ a(i) \\ v\upsilon + \lfloor \frac{v\upsilon}{2} \rfloor + \lceil \frac{\varrho}{2} \rceil - \upsilon + 3(\frac{\upsilon-3}{4}) + 2 & ; for\ \ a(ii) \end{cases}$$

The edge-sums generated by the above labeling "Ψ" are the set of consecutive positive integers $S = \{\hbar+1, \hbar+2, \ldots, \hbar+q\}$, where $\hbar = \lfloor \frac{v\upsilon}{2} \rfloor + \lceil \frac{\varrho-1}{2} \rceil + 1$. Thus by Lemma 1, "$\Psi$" can be extended to a SEMT labeling of $\wp$ and we obtain the magic constant $a = p + q + min(S)$, where $min(S) = \hbar + 1$.

For $\frac{\upsilon-1}{2}$ even;
Take the labeling Y which we have defined in Theorem 2 (for $\frac{\upsilon-1}{2}$ even) with $A' = \lceil \frac{v\upsilon}{2} \rceil + \lceil \frac{\varrho-1}{2} \rceil$. We define the labeling Ψ as follows:

$$\Psi(x_t) = \begin{cases} \lceil \frac{v\upsilon}{2} \rceil + k & ; t = 2k,\ 1 \le k \le \lceil \frac{\varrho-1}{2} \rceil \\ v\upsilon + \lceil \frac{v\upsilon}{2} \rceil + k - \upsilon + 3(\frac{\upsilon-5}{4}) + 3 & ; t = 2k-1,\ 1 \le k \le \frac{\varrho}{2}, for\ \ a(i) \\ v\upsilon + \lceil \frac{v\upsilon}{2} \rceil + k - \upsilon + 3(\frac{\upsilon-5}{4}) + 2 & ; t = 2k-1,\ 1 \le k \le \lceil \frac{\varrho}{2} \rceil, for\ \ a(ii) \end{cases}$$

$\Psi(x_{\imath,\jmath}) = Y(x_{\imath,\jmath}); \quad 1 \le \imath \le \upsilon, 1 \le \jmath \le \nu$

$$\Psi(x_{\frac{\upsilon+1}{2},0}) = \begin{cases} \nu\upsilon + \lceil \frac{\nu\upsilon}{2} \rceil + \frac{\varrho}{2} - \upsilon + 3(\frac{\upsilon-5}{4}) + 4 & ; for \;\; a(i) \\ \nu\upsilon + \lceil \frac{\nu\upsilon}{2} \rceil + \lceil \frac{\varrho}{2} \rceil - \upsilon + 3(\frac{\upsilon-5}{4}) + 3 & ; for \;\; a(ii) \end{cases}$$

The edge-sums generated by the above labeling "Ψ" are the set of consecutive positive integers $S = \{\hbar' + 1, \hbar' + 2, \dots, \hbar' + q\}$, where $\hbar' = \lceil \frac{\nu\upsilon}{2} \rceil + \lceil \frac{\varrho-1}{2} \rceil + 1$. Thus by Lemma 1, "$\Psi$" can be extended to a SEMT labeling of $\wp$ and we obtain the magic constant $a = p + q + min(S)$, where $min(S) = \hbar' + 1$.

(b): Let $\mho \cong Cb^2_{\nu,\upsilon} \cup P_\varrho \cup K_1$, where

$$V(K_1) = \{z\}$$

Let $p' = |V(\mho)|$ and $q' = |E(\mho)|$, so we get

$$p' = \nu\upsilon + \varrho + 2$$

$$q' = \nu\upsilon + \varrho - 1$$

where

$$\varrho = \begin{cases} r - 2 \; ; for \;\; b(i) \\ r - 3 \; ; for \;\; b(ii) \end{cases}$$

Before formulating the labeling $\Psi' : V(\mho) \to \{1, 2, \dots, p'\}$, keep in view the labeling Y defined in Theorem 2.

For $\frac{\upsilon-1}{2}$ odd;

$Y(x_{\imath,\jmath}) = \Psi(x_{\imath,\jmath}) = \Psi'(x_{\imath,\jmath}); \quad 1 \le \imath \le \upsilon, 1 \le \jmath \le \nu, for \;\; b(i) \;\; and \;\; b(ii) both$

with $A = \lfloor \frac{\nu\upsilon}{2} \rfloor + \lceil \frac{\varrho-1}{2} \rceil$

$$\Psi'(x_t) = \Psi(x_t), \quad t \equiv 0(mod 2)$$

$$\Psi'(x_t) = \begin{cases} \nu\upsilon + \lfloor \frac{\nu\upsilon}{2} \rfloor + k - \upsilon + 3(\frac{\upsilon-3}{4}) + 1 \; ; t = 2k-1, 1 \le k \le \lfloor \frac{\varrho+1}{2} \rfloor, for \;\; b(i) \\ \nu\upsilon + \lfloor \frac{\nu\upsilon}{2} \rfloor + k - \upsilon + 3(\frac{\upsilon-3}{4}) \quad\quad ; t = 2k-1, 1 \le k \le \frac{\varrho+1}{2}, for \;\; b(ii) \end{cases}$$

Let $B = \nu\upsilon + \lfloor \frac{\nu\upsilon}{2} \rfloor + \lfloor \frac{\varrho+1}{2} \rfloor - \upsilon + 3(\frac{\upsilon-3}{4}) + 1$
and $C = \nu\upsilon + \lfloor \frac{\nu\upsilon}{2} \rfloor + \frac{\varrho+1}{2} - \upsilon + 3(\frac{\upsilon-3}{4})$, then

$$\Psi'(z) = \begin{cases} B + 1 \; ; for \;\; b(i) \\ C + 1 \; ; for \;\; b(ii) \end{cases}$$

$$\Psi'(x_{\frac{\upsilon+1}{2},0}) = \begin{cases} B + 2 \; ; for \;\; b(i) \\ C + 2 \; ; for \;\; b(ii) \end{cases}$$

The edge-sums generated by the above labeling "Ψ'" are the set of consecutive positive integers $S = \{\hbar + 1, \hbar + 2, \dots, \hbar + q'\}$, where $\hbar = \lfloor \frac{\nu\upsilon}{2} \rfloor + \lceil \frac{\varrho-1}{2} \rceil + 1$. Thus by Lemma 1, "$\Psi'$" can be extended to a SEMT labeling of $\mho$ and we obtain the magic constant $a = p' + q' + min(S)$, where $min(S) = \hbar + 1$.

For $\frac{\upsilon-1}{2}$ even;

$Y(x_{\imath,\jmath}) = \Psi(x_{\imath,\jmath}) = \Psi'(x_{\imath,\jmath}); \quad 1 \le \imath \le \upsilon, 1 \le \jmath \le \nu, for \;\; b(i) \;\; and \;\; b(ii) both$

with $A' = \lceil \frac{\nu\upsilon}{2} \rceil + \lceil \frac{\varrho-1}{2} \rceil$

$$\Psi'(x_t) = \Psi(x_t), \quad t \equiv 0(mod 2)$$

$$\Psi'(x_t)=\begin{cases} \nu\upsilon+\lceil\frac{\nu\upsilon}{2}\rceil+k-\upsilon+3(\frac{\upsilon-5}{4})+2\ ;t=2k-1,1\le k\le\lfloor\frac{\varrho+1}{2}\rfloor, for\ \ b(i)\\ \nu\upsilon+\lceil\frac{\nu\upsilon}{2}\rceil+k-\upsilon+3(\frac{\upsilon-5}{4})+1\ ;t=2k-1,1\le k\le\frac{\varrho+1}{2}, for\ \ b(ii)\end{cases}$$

Let $B=\nu\upsilon+\lceil\frac{\nu\upsilon}{2}\rceil+\lfloor\frac{\varrho+1}{2}\rfloor-\upsilon+3(\frac{\upsilon-5}{4})+2$
and $C=\nu\upsilon+\lceil\frac{\nu\upsilon}{2}\rceil+\frac{\varrho+1}{2}-\upsilon+3(\frac{\upsilon-5}{4})+1$, then

$$\Psi'(z)=\begin{cases} B+1\ ;for\ \ b(i)\\ C+1\ ;for\ \ b(ii)\end{cases}$$

$$\Psi'(x_{\frac{\upsilon+1}{2},0})=\begin{cases} B+2\ ;for\ \ b(i)\\ C+2\ ;for\ \ b(ii)\end{cases}$$

The edge-sums generated by the above labeling "Ψ'" are the set of consecutive positive integers $S=\{\hbar'+1,\hbar'+2,\dots,\hbar'+q'\}$, where $\hbar'=\lceil\frac{\nu\upsilon}{2}\rceil+\lceil\frac{\varrho-1}{2}\rceil+1$. Thus by Lemma 1, "$\Psi'$" can be extended to a SEMT labeling of $\mho$ and we obtain the magic constant $a=p'+q'+min(S)$, where $min(S)=\hbar'+1$. □

Theorem 5. *For $\nu\ge2$, $\upsilon=2\varepsilon+1$, $\varepsilon=1,2,3,\dots$*
(a)(i): $Cb^2_{\nu,\upsilon}\cup P_r$ is SEMT.
(a)(ii): $Cb^2_{\nu,\upsilon}\cup P_{r-1}$ is SEMT.
(b)(i): $\mu_s(Cb^2_{\nu,\upsilon}\cup P_{r-2})\le1$.
(b)(ii): $\mu_s(Cb^2_{\nu,\upsilon}\cup P_{r-3})\le1$.
where

$$r=\begin{cases} 5+2\upsilon(\frac{\nu}{2}-1)+2\lfloor\frac{\upsilon-1}{4}\rfloor+4\lfloor\frac{\upsilon-2}{4}\rfloor\ ;\nu\equiv0(mod2)\\ 7+2\upsilon(\frac{\nu-3}{2})+6\lfloor\frac{\upsilon-1}{4}\rfloor+4\lfloor\frac{\upsilon-2}{4}\rfloor\quad;\nu\equiv1(mod2)\end{cases}$$

Proof. **(a):** Consider the graph $\wp\cong Cb^2_{\nu,\upsilon}\cup P_\varrho$
Let $p=|V(\wp)|$ and $q=|E(\wp)|$, so we get

$$p=\nu\upsilon+\varrho+1$$

$$q=\nu\upsilon+\varrho-1$$

where

$$\varrho=\begin{cases} r\qquad;for\ \ a(i)\\ r-1\ ;for\ \ a(ii)\end{cases}$$

Before formulating the labeling $\Psi:V(\wp)\to\{1,2,\dots,p\}$, keep in view the labeling Y defined in Theorem 2.

For $\frac{\upsilon-1}{2}$ odd;

Take the previously mentioned labeling Y (for $\frac{\upsilon-1}{2}$ odd) with $A=\lfloor\frac{\nu\upsilon}{2}\rfloor+\lfloor\frac{\varrho+1}{2}\rfloor$. We define the labeling Ψ as follows:

$$\Psi(x_t)=\begin{cases} \lfloor\frac{\nu\upsilon}{2}\rfloor+k & ;t=2k-1,1\le k\le\lfloor\frac{\varrho+1}{2}\rfloor\\ \nu\upsilon+\lfloor\frac{\nu\upsilon}{2}\rfloor+k-\upsilon+3(\frac{\upsilon+1}{4}) & ;t=2k,1\le k\le\lfloor\frac{\varrho}{2}\rfloor, for\ \ a(i)\\ \nu\upsilon+\lfloor\frac{\nu\upsilon}{2}\rfloor+k-\upsilon+3(\frac{\upsilon+1}{4})-1 & ;t=2k,1\le k\le\frac{\varrho}{2}, for\ \ a(ii)\end{cases}$$

$\Psi(x_{\imath,\jmath})=Y(x_{\imath,\jmath})\quad;1\le\imath\le\upsilon,1\le\jmath\le\nu$

$$\Psi(x_{\frac{\upsilon+1}{2},0})=\begin{cases} \nu\upsilon+\lfloor\frac{\nu\upsilon}{2}\rfloor+\lfloor\frac{\varrho}{2}\rfloor-\upsilon+3(\frac{\upsilon+1}{4})+1\ ;for\ \ a(i)\\ \nu\upsilon+\lfloor\frac{\nu\upsilon}{2}\rfloor+\frac{\varrho}{2}-\upsilon+3(\frac{\upsilon+1}{4})\qquad;for\ \ a(ii)\end{cases}$$

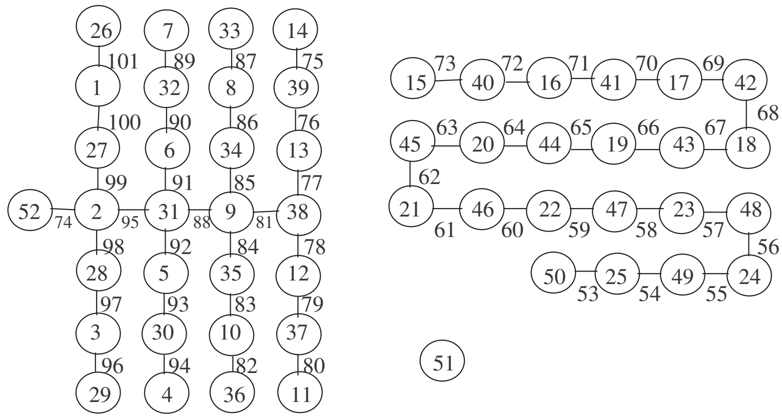

Figure 6: SEMT forest $Cb^2_{4,7} \cup P_{22} \cup K_1$

The edge-sums generated by the above labeling "Ψ" are the set of consecutive positive integers $S = \{\hbar+1, \hbar+2, \dots, \hbar+q\}$, where $\hbar = \lfloor \frac{\nu\upsilon}{2} \rfloor + \lfloor \frac{\varrho+1}{2} \rfloor + 1$. Thus by Lemma 1, "$\Psi$" can be extended to a SEMT labeling of $\wp$ and we obtain the magic constant $a = p + q + min(S)$, where $min(S) = \hbar + 1$.

For $\frac{\upsilon-1}{2}$ even;

Take the labeling Y which we have defined in Theorem 2 (for $\frac{\upsilon-1}{2}$ even) with $A' = \lceil \frac{\nu\upsilon}{2} \rceil + \lfloor \frac{\varrho+1}{2} \rfloor$. We define the labeling Ψ as follows:

$$\Psi(x_t) = \begin{cases} \lceil \frac{\nu\upsilon}{2} \rceil + k & ; t = 2k-1, 1 \le k \le \lfloor \frac{\varrho+1}{2} \rfloor \\ \nu\upsilon + \lceil \frac{\nu\upsilon}{2} \rceil + k - \upsilon + 3(\frac{\upsilon-1}{4}) + 1 & ; t = 2k, 1 \le k \le \lfloor \frac{\varrho}{2} \rfloor, for\ \ a(i) \\ \nu\upsilon + \lceil \frac{\nu\upsilon}{2} \rceil + k - \upsilon + 3(\frac{\upsilon-1}{4}) & ; t = 2k, 1 \le k \le \frac{\varrho}{2}, for\ \ a(ii) \end{cases}$$

$\Psi(x_{\imath,\jmath}) = Y(x_{\imath,\jmath});\quad 1 \le \imath \le \upsilon, 1 \le \jmath \le \nu$

$$\Psi(x_{\frac{\upsilon+1}{2},0}) = \begin{cases} \nu\upsilon + \lceil \frac{\nu\upsilon}{2} \rceil + \lfloor \frac{\varrho}{2} \rfloor - \upsilon + 3(\frac{\upsilon-1}{4}) + 2 & ; for\ \ a(i) \\ \nu\upsilon + \lceil \frac{\nu\upsilon}{2} \rceil + \frac{\varrho}{2} - \upsilon + 3(\frac{\upsilon-1}{4}) + 1 & ; for\ \ a(ii) \end{cases}$$

The edge-sums generated by the above labeling "Ψ" are the set of consecutive positive integers $S = \{\hbar'+1, \hbar'+2, \dots, \hbar'+q\}$, where $\hbar' = \lceil \frac{\nu\upsilon}{2} \rceil + \lfloor \frac{\varrho+1}{2} \rfloor + 1$. Thus by Lemma 1, "$\Psi$" can be extended to a SEMT labeling of $\wp$ and we obtain the magic constant $a = p + q + min(S)$, where $min(S) = \hbar' + 1$.

(b): Let $\mho \cong Cb^2_{\nu,\upsilon} \cup P_\varrho \cup K_1$, where

$$V(K_1) = \{z\}$$

Let $p' = |V(\mho)|$ and $q' = |E(\mho)|$, so we get

$$p' = \nu\upsilon + \varrho + 2$$
$$q' = \nu\upsilon + \varrho - 1$$

where

$$\varrho = \begin{cases} r - 2 & ; for\ \ b(i) \\ r - 3 & ; for\ \ b(ii) \end{cases}$$

Before formulating the labeling $\Psi' : V(\mho) \to \{1, 2, \dots, p'\}$, keep in view the labeling Y defined in Theorem 2.

For $\frac{\upsilon-1}{2}$ odd;
$Y(x_{\imath,\jmath}) = \Psi(x_{\imath,\jmath}) = \Psi'(x_{\imath,\jmath}); \quad 1 \le \imath \le \upsilon, 1 \le \jmath \le \nu, for\ \ b(i)\ \ and\ \ b(ii) both$
with $A = \lfloor \frac{\nu\upsilon}{2} \rfloor + \lfloor \frac{\varrho+1}{2} \rfloor$

$$\Psi'(x_t) = \Psi(x_t), \quad t \equiv 1(mod 2)$$

$$\Psi'(x_t) = \begin{cases} \nu\upsilon + \lfloor \frac{\nu\upsilon}{2} \rfloor + k - \upsilon + 3(\frac{\upsilon+1}{4}) - 1 \ ; t = 2k, 1 \le k \le \frac{\varrho-1}{2}, for\ \ b(i) \\ \\ \nu\upsilon + \lfloor \frac{\nu\upsilon}{2} \rfloor + k - \upsilon + 3(\frac{\upsilon+1}{4}) - 2 \ ; t = 2k, 1 \le k \le \lceil \frac{\varrho-1}{2} \rceil, for\ \ b(ii) \end{cases}$$

Let $B = \nu\upsilon + \lfloor \frac{\nu\upsilon}{2} \rfloor + \frac{\varrho-1}{2} - \upsilon + 3(\frac{\upsilon+1}{4}) - 1$
and $C = \nu\upsilon + \lfloor \frac{\nu\upsilon}{2} \rfloor + \lceil \frac{\varrho-1}{2} \rceil - \upsilon + 3(\frac{\upsilon+1}{4}) - 2$, then

$$\Psi'(z) = \begin{cases} B + 1 \ ; for\ \ b(i) \\ \\ C + 1 \ ; for\ \ b(ii) \end{cases}$$

$$\Psi'(x_{\frac{\upsilon+1}{2},0}) = \begin{cases} B + 2 \ ; for\ \ b(i) \\ \\ C + 2 \ ; for\ \ b(ii) \end{cases}$$

The edge-sums generated by the above labeling "Ψ'" are the set of consecutive positive integers $S = \{\hbar+1, \hbar+2, \ldots, \hbar + q'\}$, where $\hbar = \lfloor \frac{\nu\upsilon}{2} \rfloor + \lfloor \frac{\varrho+1}{2} \rfloor + 1$. Thus by Lemma 1, "$\Psi'$" can be extended to a SEMT labeling of $\mho$ and we obtain the magic constant $a = p' + q' + min(S)$, where $min(S) = \hbar + 1$.

For $\frac{\upsilon-1}{2}$ even;
$Y(x_{\imath,\jmath}) = \Psi(x_{\imath,\jmath}) = \Psi'(x_{\imath,\jmath}); \quad 1 \le \imath \le \upsilon, 1 \le \jmath \le \nu, for\ \ b(i)\ \ and\ \ b(ii) both$
with $A' = \lceil \frac{\nu\upsilon}{2} \rceil + \lfloor \frac{\varrho+1}{2} \rfloor$

$$\Psi'(x_t) = \Psi(x_t), \quad t \equiv 1(mod 2)$$

$$\Psi'(x_t) = \begin{cases} \nu\upsilon + \lceil \frac{\nu\upsilon}{2} \rceil + k - \upsilon + 3(\frac{\upsilon-1}{4}) \qquad ; t = 2k, 1 \le k \le \frac{\varrho-1}{2}, for\ \ b(i) \\ \\ \nu\upsilon + \lceil \frac{\nu\upsilon}{2} \rceil + k - \upsilon + 3(\frac{\upsilon-1}{4}) - 1 \ ; t = 2k, 1 \le k \le \lceil \frac{\varrho-1}{2} \rceil, for\ \ b(ii) \end{cases}$$

Let $B = \nu\upsilon + \lceil \frac{\nu\upsilon}{2} \rceil + \frac{\varrho-1}{2} - \upsilon + 3(\frac{\upsilon-1}{4})$
and $C = \nu\upsilon + \lceil \frac{\nu\upsilon}{2} \rceil + \lceil \frac{\varrho-1}{2} \rceil - \upsilon + 3(\frac{\upsilon-1}{4}) - 1$, then

$$\Psi'(z) = \begin{cases} B + 1 \ ; for\ \ b(i) \\ \\ C + 1 \ ; for\ \ b(ii) \end{cases}$$

$$\Psi'(x_{\frac{\upsilon+1}{2},0}) = \begin{cases} B + 2 \ ; for\ \ b(i) \\ \\ C + 2 \ ; for\ \ b(ii) \end{cases}$$

The edge-sums generated by the above labeling "Ψ'" are the set of consecutive positive integers $S = \{\hbar' + 1, \hbar' + 2, \ldots, \hbar' + q'\}$, where $\hbar' = \lceil \frac{\nu\upsilon}{2} \rceil + \lfloor \frac{\varrho+1}{2} \rfloor + 1$. Thus by Lemma 1, "$\Psi'$" can be extended to a SEMT labeling of $\mho$ and we obtain the magic constant $a = p' + q' + min(S)$, where $min(S) = \hbar' + 1$. □

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