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A note on the structure of a finite group G having a subgroup H maximal in $\langle H, H^g \rangle$

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Abstract: Let G be a finite group and $H \leq G$. The authors study the structure of finite groups G having a subgroup H which is maximal in $\langle H, H^g \rangle$ for some $g \in G$. Some results on the structure of $\langle H, H^g \rangle$ and G are set up. Especially, a generalization of Baer′s theorem is established.

Keywords: finite group, chief factor, solvable, supersolvable

MSC: 20D10, 20D15

1 Introduction

All groups considered are finite. Let $\pi(G)$ be the set of all prime divisors of the order of a group G, $S(G)$ the largest normal solvable subgroup of G. Let $\mathcal{F}$ denote a formation, $\mathcal{U}$ the formation of supersolvable groups. We use $H \lessdot G$ to denote that H is a maximal subgroup of G. $H\ Char\ G$ means that H is a characteristic subgroup of G. The other notations and terminologies are standard (Ref. to [1]).

Let $H \leq G$ and $g \in G$, then $H \leq \langle H, H^g \rangle \leq \langle H, g \rangle$. It is clear that $H = \langle H, H^g \rangle$ for all $g \in G$ if and only if $H \trianglelefteq G$. On the other hand, $\langle H, H^g \rangle = \langle H, g \rangle$ for all $g \in G$ if and only if H abn G ([2, p.247]). In [3], the famous Wielandt′s theorem shows that $H \lhd \lhd \langle H, H^g \rangle$ for all $g \in G$ if and only if $H \lhd \lhd G$. In [4], H is called pronormal in G if H is conjugate to H^g in $\langle H, H^g \rangle$ for all $g \in G$. From these results we can see that the normalities of a subgroup H in G may be reflected from the normalities of H in $\langle H, H^g \rangle$; the closer $\langle H, H^g \rangle$ is to $\langle H, g \rangle$, the more H is non-normal; the closer $\langle H, H^g \rangle$ is to H, the more H is normal. Hence $\langle H, H^g \rangle$ seems to be a "measurement" of normalities of H in G. These properties motivate us to investigate the properties of G from the size of H in $\langle H, H^g \rangle$. In [5], we investigated the structure of G by the index of H in $\langle H, H^g \rangle$. In this paper, we continue this research. We study the structure of a finite group G with a subgroup H which is maximal in $\langle H, H^g \rangle$ for some $g \in G$. We get a series of interesting results on $\langle H, H^g \rangle$ and G.

2 Preliminaries

In this section we include some lemmas from other sources that will be used in the following sections for the convenience of the reader.

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Lemma 2.1. ([6, Theorem 3.6]) Let G be a finite group. Then G is solvable if and only if for every cyclic subgroup H of G, $\langle H, H^g\rangle$ is solvable for all $g \in G$.

Lemma 2.2. ([7, Theorem 4.3.4]) Let G be a finite group. Then G is solvable if and only if for every Sylow subgroup P of G, $\langle P, P^g\rangle$ is solvable for all $g \in G$.

Lemma 2.3. ([8, Theorem 3.1]) Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and G a group with a solvable normal subgroup N such that $G/N \in \mathcal{F}$. Assume that every Sylow subgroup of $F(N)$ is cyclic, then $G \in \mathcal{F}$.

Lemma 2.4. (Wielandt) Let H be a nilpotent Hall subgroup of G and $|\pi(H)| \geq 2$. Suppose that for every prime p dividing $|H|$, $N_G(H_p) = H$, where $H_p \in Syl_p(H)$. Then there exists $K \lhd G$ such that $G = HK$ and $H \cap K = 1$.

Lemma 2.5. ([9]) A finite group is solvable if it has a nilpotent maximal subgroup of odd order.

Lemma 2.6. ([10]) Let G be a finite insolvable group with a nilpotent maximal subgroup. Let $L = F(G)$. Then G/L has a unique minimal normal subgroup K/L, K/L is a direct product of copies of a simple group with dihedral Sylow 2-subgroups, and G/K is a 2-group.

Lemma 2.7. ([11, Theorem 2]) If G is a simple group with dihedral Sylow 2-subgroups, then either G is isomorphic to $PSL(2, q)$, q odd and $q \geq 5$, or G is isomorphic to A_7.

Lemma 2.8. ([12, Theorem 8.2.3]) Let A be a π'-group that acts on the π-group G, p be a prime divisor of $|G|$ and A or G solvable. Then
(a) There exists an A-invariant Sylow p-subgroup of G.

(b) The A-invariant Sylow p-subgroups of G are conjugate under $C_G(A)$.

3 Main results

Theorem 3.1. If a minimal subgroup A of a finite group G is self-normalizing (or $A < \langle A, A^g\rangle$ for some $g \in G$), then A is a Sylow p-subgroup of G for a prime p and $A^G = G$.

Proof Note that if the order of A is a prime p, then A is contained in a Sylow p-subgroup P and if $A < P$, then A is strictly contained in its normaliser, so $A = P = N_G(P)$. It is well known that $N_G(P)$ is abnormal, so $g \in \langle A, A^g\rangle$. In particular, $G \leq A^G$.

Corollary 3.2. Let G be a non-trivial finite group. Suppose that every minimal subgroup A of G is a proper subgroup of $\langle A, A^g\rangle$ for all $g \in G \setminus A$, then G is a cyclic group of prime order.

Proof Suppose that G is not a simple group, then there exists a normal subgroup N such that $1 < N < G$. Let L be a minimal subgroup of N, of course, it is also a minimal subgroup of G. By the hypothesis and Theorem 3.1, we get $G = L^G \leq N < G$, a contradiction. Thus G is a simple group. Since non-abelian simple groups cannot have all Sylow subgroups of prime order, we have this corollary.

Theorem 3.3. Let H be a solvable subgroup of G and $T = \langle H, H^g\rangle$. If $H \lessdot T$ for some $g \in G \setminus N_G(H)$, then $T = LH$, where $L/\Phi(L)$ is isomorphic to a chief factor of T and $\Phi(L) \leq H$.

Proof If $T = S(T)H$, then T is solvable and $S(T) = T$. Let K/N be a complemented chief factor of H in T such that $N \leq H$, $K \nleq H$, then $T = HK$ and $K \cap H = N$. Let L be a minimal supplement of N in K, then $N \cap L \leq \Phi(L)$ and $T = HK = HLN = LH$. Since $L/N \cap L \cong K/N$ is an elementary abelian group, $\Phi(L) \leq N \cap L$. Thus $N \cap L = \Phi(L)$ and $K/N \cong L/\Phi(L)$. The result is true.

If $T \neq S(T)H$, then $S(T) \leq H$. Let $K/S(T)$ be a minimal normal subgroup of $T/S(T)$, then $T = KH$. Let L be a minimal supplement of $S(T)$ in K, then $K = LS(T)$, where $S(T) \cap L \leq \Phi(L)$. Since $K/S(T) \cong L/S(T) \cap L$ is a direct product of non-abelian simple groups, we have $S(T) \cap L = \Phi(L)$. Thus $K/S(T) \cong L/\Phi(L)$, $T = HL$ and $\Phi(L) \leq H$.

Theorem 3.4. Let H be a non-trivial p-subgroup of a group G. If $|H| = p^{\alpha} (\alpha \in N^{*})$, $H \lessdot \langle H, H^g \rangle$ for some $g \in G \setminus N_G(H)$, then $T = \langle H, H^g \rangle$ satisfies one of the following:
(1) If T is solvable , then
  (a) if $H \notin Syl_p(T)$, then $|T| = p^{\alpha+1}$;
  (b) if $H \in Syl_p(T)$, then $T = HQ$, where Q is a Sylow q-subgroup of T with $p \neq q$ and Q is isomorphic to a chief factor of T.
(2) If T is non-solvable, then $H \in Syl_p(T)$, $S(T) = O_p(T) \leq H$ and $T = HL$, where $L/\Phi(L)$ is a chief factor of T, every simple direct factor of $L/\Phi(L)$ is either isomorphic to $PSL(2, q)$, q odd and $q \geq 5$, or A_7.

Proof Assume that T is solvable. If $|\pi(T)| \geq 3$, then there exists a $Q \in Syl_q(T)$ such that HQ is a subgroup of T, where $p \neq q$. So $H < HQ < T$ contrary to the maximality of H in T. Therefore $|\pi(T)| \leq 2$. If $|\pi(T)| = 1$, then 1 (a) is true. If $|\pi(T)| = 2$, then $T = HQ$, where Q is a Sylow q-subgroup of T with $p \neq q$. By Theorem 3.3, $Q/\Phi(Q)$ is isomorphic to a chief factor of T and $\Phi(Q) \leq H$, thus $|\Phi(Q)| \mid |H| = p^{\alpha}$. Hence $\Phi(Q) = 1$ and so 1 (b) is true.

Now we suppose that T is non-solvable. Then $S(T) \leq H$ and $S(T) = O_p(T)$. Let $\overline{T} = T/S(T)$, $\overline{H} = H/S(T)$. It is clear that $\overline{H}$ is a Sylow p-subgroup of the group $T/O_p(T)$ and $\overline{H} \lessdot \overline{T}$. Let $\overline{L}$ be a minimal normal subgroup of $\overline{T}$, then $\overline{T} = \overline{HL}$. By Lemma 2.5, $\overline{H}$ is a Sylow 2-subgroup of $\overline{T}$. By Lemma 2.6, $\overline{L} = N_1 \times N_2 \times \cdots \times N_t$, where N_i are non-abelian simple groups that are isomorphic to each other and $\overline{H} \cap \overline{L}$ is the direct product of dihedral 2-subgroups of $\overline{L}$, which are Sylow 2-subgroups of each N_i. Hence by Lemma 2.7, we have that $N_i \cong PSL(2, q)$, q odd and $q \geq 5$, or A_7. Thus our conclusion is true.
By Theorem 3.4, we have the following Corollary.

Corollary 3.5. Let H be a p-subgroup of a group G with $|H| = p^{\alpha}$ and G be dihedral free. If $H \lessdot \langle H, H^g \rangle$ for some $g \in G \setminus N_G(H)$, then $\langle H, H^g \rangle$ is solvable and satisfies one of the following:
(a) if $H \notin Syl_p(\langle H, H^g \rangle)$, then $|\langle H, H^g \rangle| = p^{\alpha+1}$;
(b) if $H \in Syl_p(\langle H, H^g \rangle)$, then $\langle H, H^g \rangle = HQ$ where Q is a Sylow q-subgroup of $\langle H, H^g \rangle$ with $p \neq q$ and Q is isomorphic to a chief factor of $\langle H, H^g \rangle$.

Theorem 3.6. Let G be dihedral free. If for every Sylow subgroup P of G, $P \lessdot \langle P, P^g \rangle$ for all $g \in G \setminus N_G(P)$, then G is solvable.

Proof By Corollary 3.5, we can get that $\langle P, P^g \rangle$ is solvable for all $P \in Syl_p(G)$ and $g \in G$, then G is solvable by Lemma 2.2.

Theorem 3.7. Let H be a p-subgroup of G. If $H \lessdot \langle H, H^g \rangle$ and $|\pi(\langle H, H^g \rangle)| \geq 2$ for all $g \in G \setminus N_G(H)$, then H is pronormal in G.

Proof If $g \in N_G(H)$, for any $x \in \langle H, H^g \rangle$, we have $H^x = H^g$. If $g \in G \setminus N_G(H)$, then $H \in Syl_p(\langle H, H^g \rangle)$ by Theorem 3.4 and $|\pi(\langle H, H^g \rangle)| \geq 2$. Hence there exists some $x \in \langle H, H^g \rangle$ such that $H^x = H^g$. In a word, for every $g \in G$, there exists a $x \in \langle H, H^g \rangle$ such that $H^x = H^g$. Therefore, H is pronormal in G.

Theorem 3.8. Let H be an abelian p-subgroup of G with $|H| = p^{\alpha}$. If $H \lessdot \langle H, H^g \rangle$ for some $g \in G \setminus N_G(H)$, then $\langle H, H^g \rangle$ satisfies one of the following:
(a) $\langle H, H^g \rangle$ is a non-cyclic p-subgroup of order $p^{\alpha+1}$;
(b) $\langle H, H^g \rangle = HL$, where L is a minimal normal elementary abelian q-subgroup of $\langle H, H^g \rangle$ and $q \neq p$.

Proof Let $T = \langle H, H^g\rangle$. Since H is abelian, we have $H \le C_T(H) \le N_T(H) \le T$. Suppose that $C_T(H) < N_T(H)$, then $H = C_T(H)$ and $N_T(H) = T$. If $H \in Syl_p(T)$, then $H^g \in Syl_p(T)$, so $H = H^g$ and $H = \langle H, H^g\rangle = T$, a contradiction. Hence $H \notin Syl_p(T)$, thus T is a p-group and $|T| = p^{a+1}$. If T is a cyclic p-subgroup, then it has a unique maximal subgroup, so $H = H^g$, we also have $H = T$, a contradiction. Hence T is a non-cyclic p-subgroup. Therefore, we get claim (a).

Suppose that $C_T(H) = N_T(H)$. It is clear that $H \in Syl_p(T)$. By the Burnside′s theorem, we get that T is a p-nilpotent subgroup. Then there exists a normal subgroup K of T such that $T = HK$ and $H \cap K = 1$. Now consider the action of the p-subgroup H on the p'-subgroup K. By Lemma 2.8 (a), there exists an H-invariant Sylow q-subgroup K_q of K and we conclude by maximality that $T = HK_q$ and $K_q \trianglelefteq T$. By the maximality of H in T, K_q is a minimal normal subgroup of T and $\Phi(K_q) = 1$, then K_q is an elementary abelian subgroup. Thus we get (b).

Corollary 3.9. Let P be an abelian Sylow p-subgroup of G. If $P \lessdot \langle P, P^g\rangle$ for some $g \in G \setminus N_G(P)$, then $\langle P, P^g\rangle = PL$, where L is a minimal normal elementary abelian q-subgroup of $\langle P, P^g\rangle$ and $q \neq p$.

Proof It follows straight from Theorem 3.8.

Theorem 3.10. Let H be a cyclic subgroup of G. If $H \lessdot \langle H, H^g\rangle$ for some $g \in G \setminus N_G(H)$, then there exists a supersolvable normal subgroup K such that $\langle H, H^g\rangle = KH$.

Proof Call $T = \langle H, H^g\rangle$. Then T has a cyclic maximal subgroup H. If $H \trianglelefteq T$, then T/H is a cyclic group of prime order. Hence T itself is supersolvable, the theorem follows. If $H \ntrianglelefteq T$, then there exists a $g \in T \setminus N_T(H)$ such that $T = \langle H, H^g\rangle$. Thus we may assume that $g \in T$. Let T be a counterexample with minimal order. If there exists $p \in \pi(H)$ such that $1 \neq H_p < T_p$, where $H_p \in Syl_p(H)$ and $T_p \in Syl_p(T)$, then $H < N_T(H_p)$. By the maximality of H in T, we have $N_T(H_p) = T$, so $H_p \lhd T$. It is clear that H/H_p is a cyclic subgroup of T/H_p and $H/H_p \lessdot T/H_p$. Then the hypothesis is still true for $(T/H_p, H/H_p)$, so T/H_p has a supersolvable normal subgroup L/H_p such that $T/H_p = (L/H_p)(H/H_p)$. Obviously, $L \lhd T$ and $T = LH$. Since H_p is a cyclic subgroup, we have L is supersolvable by Lemma 2.3, a contradiction. Thus $H_p = T_p$ for every $p \in \pi(H)$.

Hence H is a π-Hall subgroup of T, where $\pi = \pi(H)$. Assume that $|\pi(H)| = 1$. By Theorem 3.8, the result is true. Assume that $|\pi(H)| \ge 2$. Since H is a cyclic subgroup, we have $H \le N_T(H_p) \le T$ for every $p \in \pi(H)$. By the maximality of H in T, we get $N_T(H_p) = T$ or $N_T(H_p) = H$. Assume that $N_T(H_p) = T$, by the proof of the first paragraph, we also get a contradiction. Thus $N_T(H_p) = H$ for every $p \in \pi(H)$. By Lemma 2.4, there exists a normal subgroup K of T such that $T = HK$ and $H \cap K = 1$. Now the π-subgroup H acts on the π'-subgroup K, by Lemma 2.8 (a), there exists an H-invariant Sylow q-subgroup K_q of K. By the maximality of H in T, we have $T = HK_q$, contrary to the fact that H has no supersolvable normal supplement in T. This contradiction completes the proof of this theorem.

Theorem 3.11. Let G be a group. If for every cyclic subgroup H of G, $H \lessdot \langle H, H^g\rangle$ for all $g \in G \setminus N_G(H)$, then G is solvable.

Proof By Theorem 3.10, for every cyclic subgroup H of G, we have that $\langle H, H^g\rangle$ is solvable for all $g \in G$. Then G is solvable by Lemma 2.1.
Now we give a very interesting result, which is a generalization of famous Baer′s theorem.

Theorem 3.12. Let H be a p-subgroup of G. Then $H \le O_p(\langle H, H^g\rangle)$ for all $g \in G$ if and only if $H \le O_p(G)$.

Proof Suppose that $H \le O_p(G)$, then $\langle H, H^g\rangle \le O_p(G)$ for all $g \in G$, so $H \le O_p(\langle H, H^g\rangle)$ for all $g \in G$. Conversely, for any $g \in G$, suppose that $H \le O_p(\langle H, H^g\rangle)$, then $H \lhd \lhd \langle H, H^g\rangle$ for all $g \in G$. By [3, Wielandt′s theorem], we have $H \lhd \lhd G$, then there exists a chain of subgroups $U_0, U_1, \ldots, U_r$ of G such that

$$H = U_0 \trianglelefteq U_1 \trianglelefteq \cdots \trianglelefteq U_{r-1} \trianglelefteq U_r = G$$

Then $U_i \trianglelefteq U_{i+1}$. Since $O_p(U_i)$ is characteristic in U_i and U_i is normal in U_{i+1}, then $O_p(U_i)$ is normal in U_{i+1}, hence $O_p(U_i) \leq O_p(U_{i+1})$, furthermore, $O_p(U_i) \trianglelefteq O_p(U_{i+1})$. Hence $H = O_p(U_0) \trianglelefteq O_p(U_1) \trianglelefteq \cdots \trianglelefteq O_p(U_{r-1}) \trianglelefteq O_p(U_r) = O_p(G)$. Therefore, $H \leq O_p(G)$.

Corollary 3.13. ([13, Baer′s theorem]) Let x be a p-element of G. Suppose that $\langle x, x^g \rangle$ is a p-subgroup for every $g \in G$, then $x \in O_p(G)$.

Proof It follows from Theorem 3.12 and $\langle H, H^g \rangle = \langle x, x^g \rangle$ for $H = \langle x \rangle$.

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