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Research Article

Chen Zhuoyu and Zhang Wenpeng*

On the hybrid power mean of two kind different trigonometric sums

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Abstract: The main purpose of this paper is using the analytic method, the properties of trigonometric sums and Gauss sums to study the computational problem of one kind hybrid power mean involving two different trigonometric sums, and give an interesting computational formula for it.

Keywords: The quartic Gauss sums; two-term exponential sums; hybrid power mean; computational formula

MSC: 11L05, 11L07

1 Introduction

Let p be an odd prime. The quartic Gauss sums $C(m,p) = C(m)$ is defined as

$$C(m) = \sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right),$$

where as usual, $e(y) = e^{2\pi iy}$.

Recently, several scholars studied the properties of $C(m,p)$, and obtained some interesting results. For example, Shen Shimeng and Zhang Wenpeng [1] obtained a fourth-order linear recurrence formula for $C(m,p)$.

Li Xiaoxue and Hu Jiayuan [2] studied the computational problem of the hybrid power mean

$$\sum_{b=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ba^4}{p}\right)\right|^2 \cdot \left|\sum_{c=1}^{p-1} e\left(\frac{bc+\overline{c}}{p}\right)\right|^2, \tag{1}$$

and proved an exact computational formula for (1), where $\overline{c}$ denotes the multiplicative inverse of c mod p. That is, $c\cdot\overline{c} \equiv 1 \bmod p$. In the same paper [2], the authors also suggested us to calculate the exact value of the Gauss sums

$$G(k,p) = \tau^k(\psi) + \tau^k\left(\overline{\psi}\right),$$

where $p \equiv 1 \bmod 4$ be a prime, k be any positive integer, ψ denotes a fourth-order character mod p, and $\tau(\psi)$ denotes the classical Gauss sums. That is,

$$\tau(\psi) = \sum_{a=1}^{p-1} \psi(a) e\left(\frac{a}{p}\right).$$

Chen Zhuoyu: School of Mathematics, Northwest University, Xi'an, Shaanxi, P. R. China
***Corresponding Author: Zhang Wenpeng:** School of Mathematics, Northwest University, Xi'an, Shaanxi, P. R. China, E-mail: wpzhang@nwu.edu.cn

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Chen Zhuoyu and Zhang Wenpeng [3] used the analytic method and the properties of the classical Gauss sums to obtain an interesting recurrence formula for $G(k, p)$, which completely solved the computational problem of $G(k, p)$. Some works related to the power mean of the trigonometric sums can also be found in references [4]-[8]. They will not be repeated here.

Inspired by references [1], we will consider the following hybrid power mean

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right)\right|^2 \cdot \left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right|^2. \tag{2}$$

We naturally ask: does there exist a precise computational formula for (2)? The main purpose of this paper is to answer this question. For convenience, we assume that p is a prime with $p \equiv 1 \bmod 4$, $\left(\frac{*}{p}\right) = \chi_2$ denotes the Legendre symbol mod p, and

$$\alpha = \alpha(p) = \sum_{a=1}^{\frac{p-1}{2}} \left(\frac{a+\overline{a}}{p}\right),$$

This α closely related to prime p. In fact, we have the Square Sum Theorem:

$$p = \left(\sum_{a=1}^{\frac{p-1}{2}} \left(\frac{a+\overline{a}}{p}\right)\right)^2 + \left(\sum_{a=1}^{\frac{p-1}{2}} \left(\frac{ra+\overline{a}}{p}\right)\right)^2 \equiv \alpha^2 + \beta^2,$$

where r is any quadratic non-residue mod p (see Theorem 4-11 in [9]).

In this paper, we will use the properties of Gauss sums and Legendre symbol to study the computational problem of (2), and give an interesting computational formula for it. That is, we will prove the following two conclusions.

Theorem 1. If p is a prime with $p \equiv 5 \bmod 8$, then we have the identity

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right)\right|^2 \cdot \left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right|^2 = \begin{cases} 3p^2(p-2) + 2p^{\frac{3}{2}}\alpha & \text{if } 3 \mid (p-1), \\ 3p^3 - 2p^{\frac{3}{2}}\alpha & \text{if } 3 \nmid (p-1). \end{cases}$$

Theorem 2. If p is a prime with $p \equiv 1 \bmod 8$, then we have

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right)\right|^2 \cdot \left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right|^2 = \begin{cases} 3p^2(p-2) - 2p^{\frac{3}{2}}\alpha + 2p^2\left(\tau(\psi) + \tau\left(\overline{\psi}\right)\right) & \text{if } 3 \mid (p-1), \\ 3p^3 + 2p^{\frac{3}{2}}\alpha - 2p^2\left(\tau(\psi) + \tau\left(\overline{\psi}\right)\right) & \text{if } 3 \nmid (p-1), \end{cases}$$

where ψ is any fourth-order character mod p, and $|G(1,p)| = \left|\tau(\psi) + \tau\left(\overline{\psi}\right)\right| = \sqrt{2} \cdot p^{\frac{1}{4}} \cdot (\sqrt{p} + \alpha)^{\frac{1}{2}}$.

Note that the estimations $|G(1,p)| \le 2\sqrt{p}$ and $|\alpha| \le \sqrt{p}$, from our theorems we may immediately deduce the following two corollaries:

Corollary 1. Let p be an odd prime with $p \equiv 5 \bmod 8$, then we have the asymptotic formula

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right)\right|^2 \cdot \left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right|^2 = 3p^3 + O\left(p^2\right).$$

Corollary 2. Let p be an odd prime with $p \equiv 1 \bmod 8$, then we have

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right)\right|^2 \cdot \left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right|^2 = 3p^3 + O\left(p^{\frac{5}{2}}\right).$$

Some notes: If $p = 4k+3$, then $\left(\frac{-1}{p}\right) = -1$. In this case, for any integer m with $(m, p) = 1$, from Theorem 7.5.4 of [10] we have

$$\sum_{a=0}^{p-1} e\left(\frac{ma^2}{p}\right) = i\left(\frac{m}{p}\right)\sqrt{p}, \quad i^2 = -1.$$

$$\sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right) = 1 + \sum_{a=1}^{p-1}\left(1+\left(\frac{a}{p}\right)\right) e\left(\frac{ma^2}{p}\right) = \left(\frac{m}{p}\right) i\sqrt{p}.$$

Therefore, for any positive integer k, we have the identity

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right)\right|^{2k} \cdot \left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right|^2 = p^k \sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right|^2 = \begin{cases} p^{k+1}(p-2) & \text{if } 3 \mid (p-1), \\ p^{k+2} & \text{if } 3 \nmid (p-1). \end{cases}$$

For any positive integer k, $N_k(p)$ is defined as follows:

$$N_k(p) = \sum_{m=1}^{p-1}\left(\sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right)\right)^k \cdot \left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right|^2.$$

Then from the recurrence formula for $C(m) = \sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right)$ (see Lemma 3 in [1]), we can also give a fourth-order linear recurrence formula for $N_k(p)$.

If $p \equiv 1 \bmod 8$, then $G(1,p) = \tau(\psi) + \tau\left(\overline{\psi}\right)$ is a real number. How to determine its positive or negative sign is an interesting open problem.

If $p \equiv 5 \bmod 8$, then using the method of proving our Theorem 1 we can give an exact computational formula for the hybrid power mean

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right)\right|^{2k} \cdot \left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right|^2,$$

where k is any positive integer. Only the calculation is more complex, if k is large enough. So we have not given a general conclusion here.

2 Several Lemmas

To complete the proofs of our theorems we need four simple lemmas. Here we will use many properties of the classical Gauss sums and Legendre's symbol mod p, all of them can be found in many elementary number theory books, such as reference [11], so the related contents will not be repeated here. First we have the following:

Lemma 1. If p is a prime with $p \equiv 1 \bmod 4$, then for any fourth-order character ψ mod p, we have the identity

$$\sum_{m=1}^{p-1} \psi(m)\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right|^2 = \psi(-1)\chi_2(6)\, p^{\frac{3}{2}},$$

where $\chi_2 = \left(\frac{*}{p}\right)$ denotes the Legendre's symbol mod p.

Proof. First applying trigonometric identity

$$\sum_{m=1}^{q} e\left(\frac{nm}{q}\right) = \begin{cases} q \text{ if } q \mid n, \\ 0 \text{ if } q \nmid n \end{cases} \tag{3}$$

and note that $\psi^3 = \overline{\psi}$ we have

$$\sum_{m=1}^{p-1} \psi(m)\left(\sum_{a=1}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right) = \sum_{a=1}^{p-1} e\left(\frac{a}{p}\right) \sum_{m=1}^{p-1} \psi(m) e\left(\frac{ma^3}{p}\right)$$
$$= \tau(\psi) \sum_{a=1}^{p-1} \overline{\psi}\left(a^3\right) e\left(\frac{a}{p}\right) = \tau(\psi) \sum_{a=1}^{p-1} \psi(a) e\left(\frac{a}{p}\right) = \tau^2(\psi). \tag{4}$$

Similarly, we can also deduce that

$$\sum_{m=1}^{p-1} \psi(m) \left(\sum_{a=1}^{p-1} e\left(\frac{-ma^3 - a}{p} \right) \right) = \tau^2(\psi). \tag{5}$$

On the other hand, from the properties of the fourth-order character ψ mod p we have

$$\begin{aligned}
\sum_{m=1}^{p-1} \psi(m) \left| \sum_{a=1}^{p-1} e\left(\frac{ma^3 + a}{p} \right) \right|^2 &= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{m=1}^{p-1} \psi(m) e\left(\frac{mb^3\left(a^3-1\right) + b(a-1)}{p} \right) \\
&= \tau(\psi) \sum_{a=1}^{p-1} \overline{\psi}\left(a^3-1\right) \sum_{b=1}^{p-1} \psi(m) e\left(\frac{b(a-1)}{p} \right) \\
&= \tau^2(\psi) \sum_{a=1}^{p-1} \overline{\psi}\left(a^3-1\right) \overline{\psi}(a-1) \\
&= \tau^2(\psi) \sum_{a=1}^{p-2} \overline{\psi}\left(a^3+3a^2+3a\right) \overline{\psi}(a) \\
&= -\tau^2(\psi) + \tau^2(\psi) \sum_{a=1}^{p-1} \overline{\psi}\left(3\overline{a}^2 + 3\overline{a} + 1\right) \\
&= -2\tau^2(\psi) + \tau^2(\psi) \sum_{a=0}^{p-1} \overline{\psi}\left(3a^2 + 3a + 1\right) \\
&= -2\tau^2(\psi) + \psi(12)\tau^2(\psi) \sum_{a=0}^{p-1} \overline{\psi}\left((6a+3)^2 + 3\right) \\
&= -2\tau^2(\psi) + \psi(12)\tau^2(\psi) \sum_{a=0}^{p-1} \overline{\psi}\left(a^2 + 3\right).
\end{aligned} \tag{6}$$

where we have used identity $\psi^4(a) = 1$ for any integer a with $(a, p) = 1$.

Since $p \equiv 1 \bmod 4$, for any integer b with $(b, p) = 1$, from [10] we know that

$$\sum_{a=0}^{p-1} e\left(\frac{ba^2}{p} \right) = \chi_2(b)\sqrt{p}. \tag{7}$$

Note that $\psi\chi_2 = \overline{\psi}$, from (7) and the properties of the classical Gauss sums we have

$$\sum_{a=0}^{p-1} \overline{\psi}\left(a^2+3\right) = \frac{1}{\tau(\psi)} \sum_{b=1}^{p-1} \psi(b) \sum_{a=0}^{p-1} e\left(\frac{b\left(a^2+3\right)}{p} \right) = \frac{1}{\tau(\psi)} \sum_{b=1}^{p-1} \psi(b) e\left(\frac{3b}{p} \right) \chi_2(b)\sqrt{p} = \frac{\psi(3)\sqrt{p}\,\tau\left(\overline{\psi}\right)}{\tau(\psi)}. \tag{8}$$

From (6) and (8) we deduce that

$$\sum_{m=1}^{p-1} \psi(m) \left| \sum_{a=1}^{p-1} e\left(\frac{ma^3 + a}{p} \right) \right|^2 = -2\tau^2(\psi) + \psi(36)\sqrt{p}\,\tau(\psi)\tau\left(\overline{\psi}\right). \tag{9}$$

Combining (4), (5), (9) and note that $\tau(\psi)\tau\left(\overline{\psi}\right) = \psi(-1)p$ and $\psi(36) = \chi_2(6)$ we have the identity

$$\begin{aligned}
\sum_{m=1}^{p-1} \psi(m) \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p} \right) \right|^2 &= \sum_{m=1}^{p-1} \psi(m) \left(\sum_{a=1}^{p-1} e\left(\frac{ma^3 + a}{p} \right) + \sum_{a=1}^{p-1} e\left(\frac{-ma^3 - a}{p} \right) + \left| \sum_{a=1}^{p-1} e\left(\frac{ma^3 + a}{p} \right) \right|^2 \right) \\
&= 2\tau^2(\psi) - 2\tau^2(\psi) + \psi(36)\sqrt{p}\,\tau(\psi)\tau\left(\overline{\psi}\right)
\end{aligned}$$

$$= \chi_2(6)\sqrt{p}\,\tau(\psi)\tau\left(\overline{\psi}\right) = \psi(-1)\chi_2(6)\,p^{\frac{3}{2}}.$$

This proves Lemma 1. □

Lemma 2. Let p be an odd prime with $p \equiv 1 \bmod 4$. Then we have the identity

$$\sum_{m=1}^{p-1}\chi_2(m)\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right|^2 = -\chi_2(3)\,p.$$

Proof. Applying (7) and note that the identity

$$\sum_{a=0}^{p-1}\left(\frac{a^2+3}{p}\right) = \frac{1}{\sqrt{p}}\sum_{b=1}^{p-1}\left(\frac{b}{p}\right)\sum_{a=0}^{p-1} e\left(\frac{ba^2+3b}{p}\right) = \sum_{b=1}^{p-1} e\left(\frac{3b}{p}\right) = -1,$$

from the method of proving Lemma 1 we have

$$\begin{aligned}
\sum_{m=1}^{p-1}\chi_2(m)\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right|^2 &= \sum_{a=1}^{p-1}\sum_{m=1}^{p-1}\chi_2(m)e\left(\frac{ma^3+a}{p}\right)\\
&+ \sum_{a=1}^{p-1}\sum_{m=1}^{p-1}\chi_2(m)e\left(\frac{-ma^3-a}{p}\right) + \sum_{m=1}^{p-1}\chi_2(m)\left|\sum_{a=1}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right|^2\\
&= 2p + \sum_{m=1}^{p-1}\chi_2(m)\sum_{a=1}^{p-1}\sum_{b=1}^{p-1} e\left(\frac{mb^3\left(a^3-1\right)+b(a-1)}{p}\right)\\
&= 2p + p\sum_{a=1}^{p-1}\chi_2\left(a^3-1\right)\chi_2(a-1) = 2p + p\sum_{a=1}^{p-2}\chi_2(a^2+3a+3)\\
&= p + p\sum_{a=1}^{p-1}\chi_2\left(3a^2+3a+1\right) = p\sum_{a=0}^{p-1}\chi_2\left(3a^2+3a+1\right)\\
&= \chi_2(3)p\sum_{a=0}^{p-1}\chi_2\left((6a+3)^2+3\right) = \chi_2(3)p\sum_{a=0}^{p-1}\chi_2\left(a^2+3\right) = -\chi_2(3)p.
\end{aligned}$$

This proves Lemma 2. □

Lemma 3. Let p be an odd prime with $p \equiv 1 \bmod 4$, ψ be any fourth-order character mod p. Then we have the identity

$$\tau^2(\psi) + \tau^2\left(\overline{\psi}\right) = \sqrt{p}\cdot\sum_{a=1}^{p-1}\left(\frac{a+\overline{a}}{p}\right) = 2\sqrt{p}\cdot\alpha.$$

Proof. See Lemma 2.2 in [3]. □

Lemma 4. Let p be an odd prime with $p \equiv 1 \bmod 4$. Then we have the identity

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right|^2 = \begin{cases} p(p-2) & \text{if } 3 \mid (p-1),\\ p^2 & \text{if } 3 \nmid (p-1).\end{cases}$$

Proof. If $3 \mid (p-1)$, then the congruence equation $x^3 \equiv 1 \bmod p$ has 3 different solutions in a reduced residue system mod$p$. So from (3) we have

$$\begin{aligned}
\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right|^2 &= \sum_{m=0}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right|^2\\
&= \sum_{m=0}^{p-1}\left(1 + \sum_{a=1}^{p-1} e\left(\frac{ma^3+a}{p}\right) + \sum_{a=1}^{p-1} e\left(\frac{-ma^3-a}{p}\right) + \left|\sum_{a=1}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right|^2\right)
\end{aligned}$$

$$= p + \sum_{a=1}^{p-1}\sum_{m=0}^{p-1} e\left(\frac{ma^3+a}{p}\right) + \sum_{a=1}^{p-1}\sum_{m=0}^{p-1} e\left(\frac{-ma^3-a}{p}\right) + \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{m=0}^{p-1} e\left(\frac{mb^3\left(a^3-1\right)+b(a-1)}{p}\right)$$
$$= p + p(p-1) + \sum_{a=2}^{p-1}\sum_{b=1}^{p-1}\sum_{m=0}^{p-1} e\left(\frac{m\left(a^3-1\right)+b(a-1)}{p}\right)$$
$$= p + p(p-1) - 2p = p(p-2). \tag{10}$$

If $3 \nmid (p-1)$, then the equation $x^3 \equiv 1 \bmod p$ has only one solution in a reduced residue system mod p. In this case, from the method of proving (10) we have

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right|^2 = p + p(p-1) = p^2. \tag{11}$$

Now Lemma 4 follows from (10) and (11). □

3 Proofs of the theorems

Now we prove our main results. First for convenience, we let

$$C(m) = \sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right).$$

Then for any integer m with $(m,p) = 1$, from (3) and the properties of the classical Gauss sums and the fourth-order character ψ mod p we have

$$C(m) = 1 + \sum_{a=1}^{p-1}\left(1 + \psi(a) + \chi_2(a) + \overline{\psi}(a)\right) e\left(\frac{ma}{p}\right)$$
$$= \sum_{a=0}^{p-1} e\left(\frac{ma}{p}\right) + \sum_{a=1}^{p-1}\psi(a)e\left(\frac{ma}{p}\right) + \sum_{a=1}^{p-1}\chi_2(a)e\left(\frac{ma}{p}\right) + \sum_{a=1}^{p-1}\overline{\psi}(a)e\left(\frac{ma}{p}\right)$$
$$= \chi_2(m)\sqrt{p} + \overline{\psi}(m)\tau(\psi) + \psi(m)\tau\left(\overline{\psi}\right). \tag{12}$$

If $p \equiv 5 \bmod 8$ and $3 \mid (p-1)$, then note that $\overline{\tau(\psi)} = -\tau\left(\overline{\psi}\right)$, $\chi_2(3) = 1$ and $\chi_2(6)\psi(-1) = 1$, from (12) we have

$$\overline{C(m)} = \chi_2(m)\sqrt{p} - \overline{\psi}(m)\tau(\psi) - \psi(m)\tau\left(\overline{\psi}\right). \tag{13}$$

In this case, from (13) we have

$$|C(m)|^2 = p - \left(\overline{\psi}(m)\tau(\psi) + \psi(m)\tau\left(\overline{\psi}\right)\right)^2 = 3p - \chi_2(m)\left(\tau^2(\psi) + \tau^2\left(\overline{\psi}\right)\right). \tag{14}$$

From (14), Lemma 2, Lemma 3 and Lemma 4 we have

$$\sum_{m=1}^{p-1}|C(m)|^2\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right|^2 = \sum_{m=1}^{p-1}\left(3p - \chi_2(m)\left(\tau^2(\psi) + \tau^2\left(\overline{\psi}\right)\right)\right)\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right|^2$$
$$= 3p^2(p-2) - \left(\tau^2(\psi) + \tau^2\left(\overline{\psi}\right)\right)\sum_{m=1}^{p-1}\chi_2(m)\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right|^2$$
$$= 3p^2(p-2) + 2p^{\frac{3}{2}}\alpha. \tag{15}$$

If $p \equiv 5 \bmod 8$ and $3 \nmid (p-1)$, then $\chi_2(3) = -1$. From the method of proving (15) we have

$$\sum_{m=1}^{p-1}|C(m)|^2\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right|^2 = \sum_{m=1}^{p-1}\left(3p - \chi_2(m)\left(\tau^2(\psi) + \tau^2\left(\overline{\psi}\right)\right)\right)\left|\sum_{a=0}^{p-1} e\left(\frac{ma^3+a}{p}\right)\right|^2$$

$$= 3p^3 - \left(\tau^2(\psi) + \tau^2\left(\overline{\psi}\right)\right) \sum_{m=1}^{p-1} \chi_2(m) \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right|^2$$
$$= 3p^3 - 2p^{\frac{3}{2}}\alpha. \tag{16}$$

Now Theorem 1 follows from (15) and (16).

If $p \equiv 1 \bmod 8$, then $\psi(-1) = 1$. So for any integer m with $(m, p) = 1$, from (12) we know that $\overline{C(m)} = C(m)$. That is, $C(m)$ is a real number. Then from (12) we have

$$\sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right) \right|^2 \cdot \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right|^2$$
$$= \sum_{m=1}^{p-1} \left(\chi_2(m)\sqrt{p} + \overline{\psi}(m)\tau(\psi) + \psi(m)\tau\left(\overline{\psi}\right)\right)^2 \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right|^2$$
$$= 3p \sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right|^2 + \left(\tau^2(\psi) + \tau^2\left(\overline{\psi}\right)\right) \sum_{m=1}^{p-1} \chi_2(m) \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right|^2$$
$$+ 2\sqrt{p} \sum_{m=1}^{p-1} \left(\psi(m)\tau(\psi) + \overline{\psi}(m)\tau\left(\overline{\psi}\right)\right) \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right|^2. \tag{17}$$

If $3 \mid (p-1)$, then from (17), Lemma 1, Lemma 2, Lemma 3 and Lemma 4

$$\sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right) \right|^2 \cdot \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right|^2 = 3p^2(p-2) - 2p^{\frac{3}{2}}\alpha + 2p^2\left(\tau(\psi) + \tau\left(\overline{\psi}\right)\right). \tag{18}$$

If $3 \nmid (p-1)$, then from the method of proving (18) we have

$$\sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right) \right|^2 \cdot \left| \sum_{a=0}^{p-1} e\left(\frac{ma^3 + a}{p}\right) \right|^2 = 3p^3 + 2p^{\frac{3}{2}}\alpha - 2p^2\left(\tau(\psi) + \tau\left(\overline{\psi}\right)\right). \tag{19}$$

Now Theorem 2 follows from (18) and (19).

This completes the proofs of our all results.

Authors' contributions

All authors have equally contributed to this work. All authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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References

[1] Shen S.M., Zhang W.P., On the quartic Gauss sums and their recurrence property, Advances in Difference Equations, 2017, 2017: 43

[2] Li X.X., Hu J.Y., The hybrid power mean quartic Gauss sums and Kloosterman sums, Open Mathematics, 2017, 15, 151-156

[3] Chen Z.Y., Zhang W.P., On the fourth-order linear recurrence formula related to classical Gauss sums, Open Mathematics, 2017, 15, 1251–1255

[4] Zhang W.P., Liu H.N., On the general Gauss sums and their fourth power mean, Osaka Journal of Mathematics, 2005, 42, 189-199

[5] Han D., A Hybrid mean value involving two-term exponential sums and polynomial character sums, Czechoslovak Mathematical Journal, 2014, 64, 53-62

[6] Zhang W.P., Han D., On the sixth power mean of the two-term exponential sums, Journal of Number Theory, 2014, 136, 403-413

[7] Zhang H., Zhang W.P., The fourth power mean of two-term exponential sums and its application, Mathematical Reports, 2017, 19, 75–83

[8] Zhang W.P., Hu J.Y., The number of solutions of the diagonal cubic congruence equation $\bmod p$, Mathematical Reports, 2018, 20, 60–66

[9] Zhang W.P., Li H.L., Elementary Number Theory, 2008, Shaanxi Normal University Press, Xi'an, Shaanxi.

[10] Hua L.K., Introduction to Number Theory, 1979, Science Press, Beijing.

[11] Apostol T.M., Introduction to Analytic Number Theory, 1976, Springer-Verlag, New York.