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Pentavalent arc-transitive Cayley graphs on Frobenius groups with soluble vertex stabilizer

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Abstract: A Cayley graph Γ is said to be *arc-transitive* if its full automorphism group $\mathsf{Aut}\Gamma$ is transitive on the arc set of Γ. In this paper we give a characterization of pentavalent arc-transitive Cayley graphs on a class of Frobenius groups with soluble vertex stabilizer.

Keywords: Arc-transitive graph, Frobenius group, Cayley graph, Soluble vertex stabilizer

MSC: 20B25, 05C25

1 Introduction

Throughout the paper, graphs considered are simple, connected and undirected. For a graph Γ, we denote the vertex set, edge set, arc set, valency and full automorphism group of Γ by $V\Gamma$, $E\Gamma$, $A\Gamma$, $\mathsf{val}(\Gamma)$ and $\mathsf{Aut}\Gamma$, respectively. Γ is said to be *G-vertex-transitive*, *G-edge-transitive* or *G-arc-transitive* if $G \leq \mathsf{Aut}\Gamma$ is transitive on $V\Gamma$, $E\Gamma$ or $A\Gamma$; in particular, if $G = \mathsf{Aut}\Gamma$, then Γ is simply called vertex-transitive, edge-transitive or arc-transitive. An *s-arc* of Γ is a sequence of vertices $(u_0, u_1, \ldots, u_s)$ such that u_i is adjacent to u_{i+1} and $u_{i-1} \neq u_{i+1}$ for all possible i. For a subgroup $G \leqslant \mathsf{Aut}\Gamma$, Γ is said to be (G, s)-arc-transitive if G is transitive on the set of s-arcs in Γ. In particular, a 0-arc is called a vertex, and a 1-arc is called an arc for short. As we all know, a graph Γ is G-arc-transitive for some $G \leq \mathsf{Aut}\Gamma$ if and only if G is transitive on $V\Gamma$ and the vertex stabilizer G_v of $v \in V\Gamma$ in G is transitive on the neighborhood $\Gamma(v)$ of v.

A graph Γ is called a *Cayley graph* if there exist a group G and a subset $S \subset G \setminus \{1\}$ with $S = S^{-1} := \{g^{-1} \mid g \in S\}$ such that the vertices of Γ may be identified with the elements of G in such a way that x is adjacent to y if and only if $yx^{-1} \in S$. The Cayley graph Γ is denoted by $\mathsf{Cay}(G, S)$. Throughout this paper, we denote the vertex of $\mathsf{Cay}(G, S)$ corresponding to the identity of G by 1.

A graph Γ is a Cayley graph on G if and only if $\mathsf{Aut}\Gamma$ contains a subgroup which is regular on vertices and isomorphic to G. It is well-known that a Cayley graph is vertex-transitive. However, a Cayley graph is of course not necessarily arc-transitive. Thus much excellent work has dealt with arc-transitive Cayley graphs. In particular, there are many works about cubic and pentavalent Cayley graphs. For the cubic case, see [1–3] for cubic symmetric Cayley graphs on finite nonabelian simple groups, which are normal except for A_{47}, see [4] for a characterisation of connected cubic s-transitive Cayley graphs, see [5] for a classification of the connected arc-transitive cubic Cayley graphs on $\mathrm{PSL}(2, p)$ where $p \geqslant 5$ is a prime, and see [6] for a classification of cubic arc-transitive Cayley graphs on a class of Frobenius groups. For the pentavalent case, see [7] for a classification of arc-transitive pentavalent Cayley graphs on finite nonabelian simple groups, see

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[8] for a construction of 2-arc transitive pentavalent Cayley graph of A_{39}, and see [9] for a characterization of connected core-free pentavalent 1-transitive Cayley graphs.

The objective of this paper is to give a characterization of pentavalent arc-transitive Cayley graphs on a class of primitive Frobenius groups with soluble vertex stabilizer.

A group G is said to be a *Frobenius group* if G has the form $G = W{:}H$ such that $xy \neq yx$ for any $x \in W \setminus \{1\}$ and $y \in H \setminus \{1\}$. In particular, G is called a *primitive* Frobenius group if H acts irreducibly on W.

Theorem 1.1. *Let $G{=}W{:}H{\cong}\mathbb{Z}_p^d{:}\mathbb{Z}_n$ be a primitive Frobenius group, where p is a prime, and d, n are positive integers. Suppose that Γ is a connected pentavalent arc-transitive Cayley graph on G with soluble stabilizer. Let $A{=}\mathrm{Aut}\Gamma$. Then one of the following statements holds:*
(i) $G \cong \mathrm{D}_{2p}$, $\Gamma \cong G(2p, 5)$ *with* $p \equiv 1 (\mathrm{mod}\ 5)$ *and* $p > 11$, *and* $A = (\mathbb{Z}_p{:}\mathbb{Z}_5){:}\mathbb{Z}_2$*;*
(ii) G *is normal in* A, *and* $A_1 \cong \mathbb{Z}_5, \mathrm{D}_{10}$, *or* F_{20}*;*
(iii) $(\Gamma, G) = (\mathcal{C}_{12}, \mathbb{Z}_2^2{:}\mathbb{Z}_3)$, *where* $\mathcal{C}_{12}$ *is constructed in Example 3.1.*

Remarks on Theorem 1.1.
(a) The Cayley graph Γ in part (ii) is called a *normal* Cayley graph, introduced in [10].
(b) H acts irreducibly on W if and only if n does not divide $p^m - 1$ where $m < d$. We call such n a *primitive divisor* of $p^d - 1$, refer to [11, Proposition 2.3].

After this introductory section, some preliminary results are given in Section 2 and a few examples are given in Section 3. Then we complete the proof of Theorem 1.1 in Section 4.

2 Preliminary Results

In this section we give some preliminary results, which will be used in the subsequent sections.

Let Γ be a G-vertex-transitive graph. Then, for $\alpha \in V\Gamma$, the stabilizer G_α is a core-free subgroup in G, that is, $\cap_{g\in G} G_\alpha^g = 1$. Set $H = G_\alpha$ and $D = \{x \mid \alpha^x \in \Gamma(\alpha)\}$. Then D is a union of several double cosets HxH. Moreover, Γ is isomorphic to the coset graph $\mathsf{Cos}(G, H, D)$ defined over $\{Hx \mid x \in G\}$ with edge set $\{\{Hg_1, Hg_2\} \mid g_2g_1^{-1} \in D\}$.

The following statements for coset graphs are well known.
(a) Γ is undirected if and only if $D = D^{-1} := \{x^{-1} \mid x \in D\}$.
(b) Γ is connected if and only if $\langle H, D\rangle = G$.
(c) Γ is G-arc-transitive if and only if $D = HgH$ for $g \in G$ with $g^2 \in H$; moreover, g can be chosen as a 2-element such that $g \in N_G(H \cap H^g)$.

For a Cayley graph $\Gamma = \mathsf{Cay}(G, S)$. Let $\mathsf{Aut}(G, S) = \{\alpha \in \mathsf{Aut}(G) \mid S^\alpha = S\}$. Then we have the following basic result.

Lemma 2.1. ([12, Lemma 2.1]) *Let* $\Gamma = \mathsf{Cay}(G, S)$ *be a Cayley graph. Then the normalizer* $N_{\mathsf{Aut}\Gamma}(G) = G{:}\mathsf{Aut}(G, S)$.

The soluble vertex stabilizer for arc-transitive graphs of valency 5 is known.

Lemma 2.2. [14] *Let Γ be a pentavalent (G, s)-transitive graph for some $G \leq \mathsf{Aut}\Gamma$ and $s \geq 1$. Let $v \in V\Gamma$. If G_v is soluble, then $|G_v| \mid 80$ and $s \leq 3$. Furthermore, one of the following holds:*
(1) $s = 1$, $G_v \cong \mathbb{Z}_5, \mathrm{D}_{10}$ *or* D_{20}*;*
(2) $s = 2$, $G_v \cong \mathrm{F}_{20}$, *or* $\mathrm{F}_{20} \times \mathbb{Z}_2$*;*
(3) $s = 3$, $G_v \cong \mathrm{F}_{20} \times \mathbb{Z}_4$.

We give two basic results for pentavalent graphs.

Lemma 2.3. *Let $\Gamma = \mathrm{Cay}(G, S)$ be a connected pentavalent graph with soluble stabilizer. Assume that* $\mathrm{Aut}\Gamma$ *contains a subgroup X such that Γ is X-arc-transitive and $G \trianglelefteq X$. Then $X_1 \cong \mathbb{Z}_5$, D_{10}, or F_{20}.*

Proof. Since Γ is connected, $G = \langle S\rangle$, and thus $\mathrm{Aut}(G, S)$ acts faithfully on S. So $\mathrm{Aut}(G, S) \lesssim \mathrm{S}_5$. By Lemma 2.1, $X \le N_{\mathrm{Aut}\Gamma}(G) = G{:}\mathrm{Aut}(G, S)$. Thus $X_1 \leqslant \mathrm{Aut}(G, S) \lesssim \mathrm{S}_5$. Note that X_1 is transitive on S, so $X_1 \cong \mathbb{Z}_5$, D_{10}, or F_{20}. □

We say a vertex-transitive graph Γ is a normal cover of its quotient graph Γ_N if Γ and Γ_N have the same valency, where $N \lhd \mathrm{Aut}\Gamma$ is not transitive on $V\Gamma$.

Lemma 2.4. *Let Γ be a connected pentavalent X-arc-transitive graph with soluble stabilizer, and let $N \lhd X$ such that X/N is insoluble, where $X \le \mathrm{Aut}\Gamma$. Then Γ is a normal cover of Γ_N.*

Proof. Let $u \in V\Gamma$, and let $B = u^N$ be an orbit of N acting on $V\Gamma$. Let K be the kernel of X acting on $V\Gamma_N$. Then K_u is soluble as $K_u \trianglelefteq X_u$. By the Frattini argument, we have that $K = NK_u$. Note that $K/N \cong NK_u/N \cong K_u/(N \cap K_u)$, so K/N is soluble. Since X/N is insoluble, $X/K \cong (X/N)/(K/N)$ is insoluble. Thus Γ is a normal cover of Γ_N. □

The next lemma gives a classification of locally primitive Cayley graphs on abelian groups.

Lemma 2.5. [15, Theorem 1.1] *Let Γ be a connected locally primitive Cayley graph of an abelian group of valency at least 3. Then one of the following holds:*

(1) $\Gamma = K_n, K_{n,n}, K_n \times \cdots \times K_n$;
(2) Γ is the standard double cover of K_n^l;
(3) Γ is a normal or bi-normal Cayley graph of an elementary abelian 2-group or a meta-abelian 2-group.

3 Examples

In this section we give some examples of pentavalent arc-transitive graphs.

Example 3.1. Let $T = \mathrm{PSL}(2, 4)$. Take a subgroup $H \cong \mathbb{Z}_5$ of T. Let $g \in N_T(L) \setminus L$ be an involution such that $\langle H, g\rangle = T$, where $L = H^g \cap H$. Let $G \cong \mathbb{Z}_2^2{:}\mathbb{Z}_3$ be a subgroup of T. Then G is transitive on $[T{:}H]$, and so the coset graph $\mathcal{C}_{12} = \mathrm{Cos}(T, H, HgH)$ is a pentavalent T-arc-transitive Cayely graph on G.

Example 3.2. Let $T = \mathrm{PSL}(2, 9)$. Take a subgroup $H \cong \mathrm{D}_{10}$ of T. Let $g \in N_T(L) \setminus L$ be an involution such that $\langle H, g\rangle = T$, where $L = H^g \cap H$. Let $G \cong \mathbb{Z}_3^2{:}\mathbb{Z}_4$ be a subgroup of T. Then G is intransitive on $[T{:}H]$, and so the coset graph $\mathcal{C}_{36} = \mathrm{Cos}(T, H, HgH)$ is a pentavalent T-arc-transitive graph but not a Cayely graph on G.

4 Proof of Theorem 1.1

In this section we will prove Theorem 1.1 by a series of lemmas.

Let $G = W{:}H \cong \mathbb{Z}_p^d{:}\mathbb{Z}_n$ be a primitive Frobenius group, where p is a prime, and d, n are integers. Assume that $\Gamma = \mathrm{Cay}(G, S)$ is a connected pentavalent arc-transitive Cayley graph on G with soluble vertex stabilizer. First of all, we study the case where the full automorphism group $A := \mathrm{Aut}\Gamma$ is soluble.

Suppose that $G \cong \mathrm{D}_{2p}$. Then we have the following lemma, see [16, Proposition 2.7].

Lemma 4.1. *Let G be a dihedral group of order $2p$, and let Γ be a connected pentavalent arc-transitive Cayley graph on G. Then $\Gamma \cong G(2p, 5)$ with $p \equiv 1 (\mathrm{mod}\ 5)$ and $p > 11$, $\mathrm{Aut}\Gamma \cong (\mathbb{Z}_p{:}\mathbb{Z}_5){:}\mathbb{Z}_2$.*

Let F be the Fitting subgroup of A, that is, F is the largest nilpotent normal subgroup of A. Then $F \neq 1$, and $\mathrm{C}_A(F) \leqslant F$ as A is soluble.

For a group H and a prime p, we denote the Sylow p-subgroup of H by H_p.

Lemma 4.2. *If $G \not\cong \mathrm{D}_{2p}$, then W is normal in A.*

Proof. We claim that $G \cap F \neq 1$. Suppose that $G \cap F = 1$. Since $F \leq A_u$, $|F| \mid |A_u|$. It follows that $|F| \mid 80$ by Lemma 2.2, where $u \in V\Gamma$.

Assume that F is transitive on $V\Gamma$. Then $|G| \mid |F|$. Since H acts irreducibly on W, $G \cong \mathbb{Z}_5{:}\mathbb{Z}_4$ or $\mathbb{Z}_2^4{:}\mathbb{Z}_5$. Since there exists no connected arc-transitive pentavalent graphs of order $4p$ for each prime $p \geq 5$ by [18, Theorem 1.1], the former case does not occur. By [17], there exists no connected arc-transitive pentavalent graphs of order 80, so the latter case is excluded. Similarly, we also exclude the case where $\Gamma_F \cong K_2$.

Thus Γ is a normal cover of Γ_F. Then $|F|$ divides $|G|$. Since $\mathrm{C}_G(F) \leqslant F$, $\mathrm{C}_G(F) = 1$. Therefore G acts faithfully on F. It follows that $G \lesssim \mathrm{Aut}(F)$. Thus $2 \mid |F|$.

Suppose that $p = 2$. By the previous paragraph, we have that $n \geqslant 7$. Assume that $\Phi(F) = 1$. Since $|F|$ divides both 80 and $|G|$, we obtain that $F_2 \cong \mathbb{Z}_2^k$ and $F_{2'} \cong \mathbb{Z}_5$ or 1, where $k \leqslant \min\{4, d\}$. Note that $\mathrm{C}_A(F) \leqslant F$, so $G \lesssim \mathrm{GL}(k, 2)$. By Atlas [19], there exists no k and G satisfying the above relation. Thus $\Phi(F) \neq 1$. Since $F_5 \lesssim \mathbb{Z}_5$, we conclude that $\Phi(F_2) = \Phi(F) \leqslant F_2$. Let $\overline{F}_2 = F_2/\Phi(F)$ and $\overline{H} = H\Phi(F)/\Phi(F)$. Then $\overline{H} \cong H$. If $\mathrm{C}_{\overline{H}}(\overline{F}_2) \neq 1$, then $\mathrm{C}_H(F_2) \neq 1$ by [20, p.174, Theorem 1.4], a contradiction. Thus $\mathrm{C}_{\overline{H}}(\overline{F}_2) = 1$, and so $\overline{H} \lesssim \mathrm{GL}(k, 2)$, where $k \leq 3$. By Atlas [19], $\overline{F}_2 \cong \mathbb{Z}_2^3$ and $\overline{H} \cong \mathbb{Z}_7$. It follows that $|F_2| = 16$ and $d = 3$, which is a contradiction.

Suppose that p is odd. Assume that $\Phi(F) = 1$. Then $F_2 \cong \mathbb{Z}_2^k$ and $2^k \mid n$, where $k \leqslant 4$. Note that $G \lesssim \mathrm{GL}(k, 2)$, it follows that $W \lesssim \mathrm{GL}(k, 2)$, and $\mathbb{Z}_{2^k} \lesssim \mathrm{GL}(k, 2)$. By Atlas [19], there exists no k and W satisfying the above relation. Thus $\Phi(F) \neq 1$. Since $F_5 \lesssim \mathbb{Z}_5$, we conclude that $\Phi(F_2) = \Phi(F) \leqslant F_2$. Let $\overline{W} = W\Phi(F_2)/\Phi(F_2)$ and $\overline{F}_2 = F_2/\Phi(F_2)$. If $\mathrm{C}_{\overline{W}}(\overline{F}_2) \neq 1$, then $\mathrm{C}_W(F_2) \neq 1$, refer to [20, p.174, Theorem 1.4], a contradiction. Thus $\mathrm{C}_{\overline{W}}(\overline{F}_2) = 1$, and so $\overline{W} \lesssim \mathrm{GL}(k, 2)$, where $k \leq 3$. By Atlas [19], this is impossible.

To sum up, $F \cap G \neq 1$. Since W is minimal in G, $W \leqslant F$. Note that $G \not\cong \mathrm{D}_{2p}$, so Γ is a cover of Γ_{F_p}, and thus $W = F_p$. Therefore, W is normal in A. This completes the proof. □

Lemma 4.3. *With the hypothesis of Lemma 4.2, then G is normal in A, and $A_u \cong \mathbb{Z}_5, \mathrm{D}_{10}$, or F_{20}, where $u \in V\Gamma$.*

Proof. By Lemma 4.2, W is normal in A. Since $G \not\cong \mathrm{D}_{2p}$, $n > 2$. Then Γ is a normal cover of Γ_W. By Lemma 2.5, either $G/W \trianglelefteq A/W$ or $\Gamma_W \cong K_6$ or $K_{5,5}$. In the former case, $G \trianglelefteq A$. By Lemma 2.3, $A_u \cong \mathbb{Z}_5, \mathrm{D}_{10}$, or F_{20}, where $u \in V\Gamma$. If $\Gamma_W \cong K_6$, then $A/W \leq \mathrm{Aut}\Gamma_W \cong \mathrm{S}_6$. Note that $5 \cdot 6 \mid |A/W|$, so A_u is insoluble, which is a contradiction. Similarly, we can exclude the case where $\Gamma_W \cong K_{5,5}$. This completes the proof of Lemma 4.3. □

In the remaining section, we study the case where the full automorphism group A is insoluble. Denote by R the radical of A, that is, R is the largest soluble normal subgroup of A.

Suppose that $R = 1$. Then we have the following lemma.

Lemma 4.4. *If $R = 1$, then $(G, \Gamma) = (\mathbb{Z}_2^2{:}\mathbb{Z}_3, \mathcal{C}_{12})$.*

Proof. Let N be a minimal normal subgroup of A. Since $R = 1$, $N = T_1 \times \cdots \times T_\ell \cong T^\ell$, where $T_i \cong T$ is non-abelian simple. By [21], T is one of the following:

$$\mathrm{PSL}(4,2),\ \mathrm{PSU}(3,8),\ \mathrm{M}_{11},\ \mathrm{PSp}(4,3),\ \mathrm{PSL}(3,q)(q<9),\ \mathrm{PSL}(2,q)(q>3).$$

Let $W_i = T_i \cap W$, where $1 \leqslant i \leqslant \ell$, and let $L = N \cap H$. Since G is a Frobenius group, L is a diagonal subgroup of N. Write $L = \langle h_1 h_2 \cdots h_\ell \rangle$, where $\langle h_i \rangle \cong L$. Let $H_i = \langle h_i \rangle$ and $G_i = W_i{:}H_i$, where $1 \leqslant i \leqslant \ell$. Then G_i is a Frobenius group. Let $m = \frac{|T_1|}{|G_1|} \prod_{i=2}^{\ell} \frac{|T_i|}{|W_i|}$. Since $N \cap G = W{:}L$, it follows that $\frac{|N|}{|N \cap G|} \mid |A_u|$, and $m = \frac{|N|}{|N \cap G|}$. Thus $m \mid 80$.

Suppose that $T = \mathrm{PSL}(4,2)$. By Atlas [19], $G_i \lesssim \mathrm{A}_7$ and A_7 has no subgroup with index 2, 5 or 10. So m does not divide $|A_u|$, which is a contradiction. Similarly, we can exclude the cases where $T = \mathrm{PSU}(3,8)$, M_{11}, $\mathrm{PSp}(4,3)$ and $\mathrm{PSL}(3,q)(q<9)$.

Suppose that $T = \mathrm{PSL}(2, q)$ where $q = r^e$ with r a prime. According to [22, Theorem 6.25], G_i is isomorphic to a subgroup of one of the following groups:

$$\mathbb{Z}_r^e{:}\mathbb{Z}_{\frac{(r^e-1)}{d}},\ \mathrm{D}_{\frac{2(r^e\pm1)}{d}},\ \mathrm{A}_5 \text{ and } \mathrm{PGL}(2, r^f),$$

where $d = (2, r-1)$, and $f \mid e$. In the following, we process our analysis by several cases.

Case 1: $G_i \lesssim \mathbb{Z}_r^e{:}\mathbb{Z}_{\frac{(r^e-1)}{d}}$.

If $\ell > 1$, then $(q+1)^2$ divides 80 since m divides $|A_u|$. It follows that $(q+1)^2$ divides 16, and thus $q = 3$, which is a contradiction. Thus $\ell = 1$, namely, $N \cong T$. For this case, $q+1$ divides 80. It implies that $q = 4, 7, 9, 19$ or 79. Since $W \cong \mathbb{Z}_r^{e_1}$ with $e_1 \le e$, we conclude that $W \cong \mathbb{Z}_2^2$, $\mathbb{Z}_7$, $\mathbb{Z}_3^2$, $\mathbb{Z}_{19}$ or $\mathbb{Z}_{79}$.

Assume that $W \cong \mathbb{Z}_2^2$. Then $G \cong \mathbb{Z}_2^2{:}\mathbb{Z}_3$. By Example 3.1, $\Gamma \cong \mathcal{C}_{12}$. Assume that $W \cong \mathbb{Z}_3^2$. Then $G \cong \mathbb{Z}_3^2{:}\mathbb{Z}_8$ or $\mathbb{Z}_3^2{:}\mathbb{Z}_4$. Since PSL(2, 9) has no subgroup with order 72, we can exclude the former case. For the latter case, by Example 3.2, we can also exclude this case.

Assume that $W \cong \mathbb{Z}_7$, $\mathbb{Z}_{19}$ or $\mathbb{Z}_{79}$. Then G is isomorphic to one of the following

$$\mathrm{D}_{14},\ \mathbb{Z}_7{:}\mathbb{Z}_6,\ \mathrm{D}_{38},\ \mathbb{Z}_{19}{:}\mathbb{Z}_6,\ \mathbb{Z}_{19}{:}\mathbb{Z}_{18},\ \mathrm{D}_{158},\ \mathbb{Z}_{79}{:}\mathbb{Z}_6,\ \mathbb{Z}_{79}{:}\mathbb{Z}_{26},\ \mathbb{Z}_{79}{:}\mathbb{Z}_{78}.$$

Since PSL(2, 19) has no subgroup with order 114 and 342 by MAGMA [23], we can exclude the cases where $G \cong \mathbb{Z}_{19}{:}\mathbb{Z}_6$, and $\mathbb{Z}_{19}{:}\mathbb{Z}_{18}$. Besides, we can also exclude the cases where $G \cong \mathrm{D}_{14}, \mathrm{D}_{38}$ and D_{158} by [16, Proposition 2.7], and where $G \cong \mathbb{Z}_7{:}\mathbb{Z}_6$, $\mathbb{Z}_{79}{:}\mathbb{Z}_6$, $\mathbb{Z}_{79}{:}\mathbb{Z}_{26}$ and $\mathbb{Z}_{79}{:}\mathbb{Z}_{78}$ by [16, Theprem 4.2].

Case 2: $G_i \lesssim \mathrm{D}_{\frac{2(r^e\pm1)}{d}}$.

By the discussion as above, $N \cong T$. If $G_i \lesssim \mathrm{D}_{\frac{2(r^e-1)}{d}}$, then $q(q+1)$ divides 80. It follows that $q = 4$. Thus $G \cong \mathrm{D}_6$. By [16, Proposition 2.7], $\Gamma \cong K_6$, $\mathrm{Aut}\Gamma \cong \mathrm{S}_6$, and so $|A_u| \nmid 80$, which is a contradiction. Thus $G_i \lesssim \mathrm{D}_{\frac{2(r^e+1)}{d}}$. For this case, $q(q-1)$ divides 80. It follows that $q = 5$, and then $G \cong \mathrm{D}_6$. By above discussion, this case is also excluded.

Case 3: $G_i \lesssim \mathrm{A}_5$.

Then $G_i \cong \mathrm{D}_6$ or D_{10}, and $N \cong T$. Since $\frac{|\mathrm{A}_5|}{|\mathrm{D}_{10}|} \nmid 80$, we can exclude the case where $G \cong \mathrm{D}_{10}$. Arguing as Case 2, we can exclude the case where $G \cong \mathrm{D}_6$.

Case 4: $G_i \lesssim \mathrm{PGL}(2, r^f)$.

Let $\hat{r} = \frac{r^{e-f}(r^{2e}-1)}{r^{2f}-1}$. Then $\hat{r}$ divides 80. If $e - f = 1$, then $e = 2, f = 1$ and $r(r^2+1) \mid 80$. It follows that $r = 2$ and $N \cong T$. Hence $G \cong \mathrm{D}_6$ or D_{10}. However, the case does not occur by Case 3. Thus $e > f + 1$, and so $r = 2$. Then $\frac{r^{2e}-1}{r^{2f}-1} = 5$, which is impossible. This completes the proof of Lemma 4.4. □

In what follows, we will prove that $R = 1$.

Lemma 4.5. *The radical* $R = 1$.

Proof. Suppose that $R \not= 1$. Set $L = R \cap G$. Let $L \neq 1$. Since W is minimal and normal in G, $W \le L$. Note that A/R is insoluble, so Γ is a normal cover of Γ_R by Lemma 2.4.

Let $\overline{G} = GR/R$. Then $\overline{G} \cong G/(G \cap R)$ is cyclic. Since Γ is a Cayley graph of G, Γ_R is a Cayley graph of $\overline{G}$. By Lemma 2.5, we obtain that $\Gamma_R \cong K_6$, or $K_{5,5}$. Thus $\mathrm{Aut}\Gamma_R \cong \mathrm{S}_6$, or $\mathrm{S}_5 \wr \mathbb{Z}_2$. It follows that $|A_u| \nmid 80$, a contradiction.

Suppose that $L = 1$. Let $\overline{G} = GR/R$. Then $\overline{G} \cong G$. Let $\overline{A} = A/R$. Then $\overline{A} = \overline{G}\,\overline{A}_{\overline{u}}$, where $\overline{u} \in V\Gamma_R$. Arguing as the proof of Lemma 4.4 with $\overline{A} = \overline{G}\,\overline{A}_{\overline{u}}$ in place of $A = GA_u$, $\overline{A}$ is almost simple, and $\overline{G} \cong \mathbb{Z}_2^2{:}\mathbb{Z}_3$. Therefore, A is almost simple, which is a contradiction.

□

The assertion of Theorem 1.1 follows from Lemmas 4.1 and 4.3-4.5.

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