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Research Article

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Statistical approximation properties of λ-Bernstein operators based on q-integers

Abstract: In this paper, we introduce a new generalization of λ-Bernstein operators based on q-integers, we obtain the moments and central moments of these operators, establish a statistical approximation theorem and give an example to show the convergence of these operators to $f(x)$. It can be seen that in some cases the absolute error bounds are smaller than the case of classical q-Bernstein operators to $f(x)$.

Keywords: q-integers, λ-Bernstein operators, Statistical convergence, basis function

MSC: 41A10; 41A25; 41A36

1 Introduction

A generalization of Bernstein polynomials based on q-integers was proposed by Lupaş in 1987 in [1]. However, the Lupaş q-Bernstein operators are rational functions rather than polynomials. The Phillips q-Bernstein polynomials were introduced by Phillips in 1997 in [2]. After that, there are many papers mentioning the approximation properties of positive linear operators in the area of q-calculus due to its applications in the field of approximation theory, such as generalized q-Bernstein polynomials [3–6], Durrmeyer type q-Bernstein operators [7–9], Kantorovich type q-Bernstein operators [10–12] and so on.

As we know, Phillips [2] defined the following q-Bernstein operators:

$$B_n(f; x) = \sum_{i=0}^{n} b_{n,i}(x; q) f\left(\frac{[i]_q}{[n]_q}\right),\tag{1}$$

where $b_{n,i}(x; q)$ $(i = 0, 1, ..., n)$ are q-Bernstein basis functions and defined as

$$b_{n,i}(x; q) = \begin{bmatrix} n \\ i \end{bmatrix}_q x^i \prod_{s=0}^{n-i-1} \left(1 - q^s x\right).\tag{2}$$

Motivated by above work, in this paper, we introduce λ-Bernstein operators based on q-integers as

$$\widetilde{B}_{n,q,\lambda}(f; x) = \sum_{i=0}^{n} \widetilde{b}_{n,i}(x; q, \lambda) f\left(\frac{[i]_q}{[n]_q}\right),\tag{3}$$

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where

$$
\begin{cases}
\widetilde{b}_{n,0}(x;q,\lambda) = b_{n,0}(x;q) - \frac{\lambda}{[n]_q+1} b_{n+1,1}(x;q), \\
\widetilde{b}_{n,i}(x;q,\lambda) = b_{n,i}(x;q) + \lambda\left(\frac{[n]_q-2[i]_q+1}{[n]_q^2-1} b_{n+1,i}(x;q) - \frac{[n]_q-2q[i]_q-1}{[n]_q^2-1} b_{n+1,i+1}(x;q)\right), \\
\qquad\qquad (i = 1, 2, ..., n-1) \\
\widetilde{b}_{n,n}(x;q,\lambda) = b_{n,n}(x;q) - \frac{\lambda}{[n]_q+1} b_{n+1,n}(x;q),
\end{cases}
\tag{4}
$$

$b_{n,i}(x;q)$ are defined in (2), $\lambda \in [-1, 1]$, $n \geq 2$, $x \in [0, 1]$ and $0 < q \leq 1$.

Obviously, when $\lambda = 0$, $\widetilde{B}_{n,q,0}(f;x)$ turn out to be q-Bernstein operators defined in (1); when $\lambda = 0$, $q = 1$, $\widetilde{B}_{n,1,0}(f;x)$ turn out to be classical Bernstein operators; when $q = 1$, $\widetilde{B}_{n,1,\lambda}(f;x)$ turn out to be the λ-Bernstein operators which are studied by Cai, et al. in [13].

Details of q-integers can be found in [14, 15]. For any fixed real number $0 < q \leq 1$ and each nonnegative integer k, we denote q-integers by $[k]_q$, where $[k]_q := \begin{cases} \frac{1-q^k}{1-q}, & q \neq 1, \\ k, & q = 1. \end{cases}$ Also q-factorial and q-binomial coefficients are defined as follows:

$$
[k]_q! := \begin{cases} [k]_q[k-1]_q...[1]_q, & k = 1, 2, ..., \\ 1, & k = 0, \end{cases},
$$

$$
\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!} \quad (n \geq k \geq 0).
$$

Now we recall the concepts of statistical convergence.

Let K be a subset of $\mathbb{N}$, the set of all natural numbers. The density of K is defined by

$$
\delta(K) := \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_K(k),
$$

provided the limit exists [16], where χ_K is the characteristic function of K.

A sequence $x := \{x_k\}$ is called statistically convergent to a number L, if for every $\epsilon > 0$, $\delta(k \in \mathbb{N} : |x_k - L| \geq \epsilon) = 0$ [17]. This convergence is denoted as $st - \lim_{n\to\infty} x_n = L$.

We also recall the following theorem given by Gadjiev and Orhan [18].

Theorem 1.1. *(See [18]) If the sequence of positive linear operators $A_n : C_M[a, b] \to C[a, b]$ satisfies the conditions $st - \lim_{n\to\infty} ||A_n(e_v; \cdot) - e_v||_{C[a,b]} = 0$, with $e_v(t) = t^v$ for $v = 0, 1, 2$, then for any function $f \in C_M[a, b]$, we have*

$$
st - \lim_{n\to\infty} ||A_n(f; \cdot) - f||_{C[a,b]} = 0,
$$

where $C_M[a, b]$ denotes the space of all functions f which are continuous in $[a, b]$ and bounded on the all positive axis.

The aims of this paper are to introduce a new generalization of λ-Bernstein operators based on q-integers and give the statistical approximation properties of these operators using the concept of statistical convergence.

The rest of this paper is organized as follows. In section 2, we give some lemmas which are needed to prove our main results. In section 3, we obtain the statistical approximation theorem and give an example to show that, in some cases, the absolute error bounds of these operators defined in (3) to $f(x)$ are smaller than the bounds of classical q-Bernstein operators to $f(x)$.

2 Some lemmas

In the sequel, let $e_v(t) = t^v$ for $v = 0, 1, 2$. In order to obtain the main results, we need the following lemmas:

Lemma 2.1. *Let $\lambda \in [-1, 1]$, $x \in [0, 1]$ and $q \in (0, 1]$, for the operators $\widetilde{B}_{n,q,\lambda}(f; x)$, we have*

$$\widetilde{B}_{n,q,\lambda}(e_0; x) = 1. \tag{5}$$

Proof. Using $[n]_q = 1 + q[n-1]_q$, we have

$$\sum_{i=0}^{n} \widetilde{b}_{n,i}(x; q, \lambda) = \sum_{i=0}^{n} b_{n,i}(x; q) - \frac{\lambda}{[n]_q + 1} b_{n+1,1}(x; q) + \lambda \frac{[n]_q - 2[1]_q + 1}{[n]_q^2 - 1} b_{n+1,1}(x; q)$$

$$- \lambda \frac{[n]_q - 2q[1]_q - 1}{[n]_q^2 - 1} b_{n+1,2}(x; q) + \lambda \frac{[n]_q - 2[2]_q + 1}{[n]_q^2 - 1} b_{n+1,2}(x; q)$$

$$- \lambda \frac{[n]_q - 2q[2]_q - 1}{[n]_q^2 - 1} b_{n+1,3}(x; q) + \dots + \lambda \frac{[n]_q - 2[n-1]_q + 1}{[n]_q^2 - 1} b_{n+1,n-1}(x; q)$$

$$- \lambda \frac{[n]_q - 2q[n-1]_q - 1}{[n]_q^2 - 1} b_{n+1,n}(x; q) - \frac{\lambda}{[n]_q + 1} b_{n+1,n}(x; q)$$

$$= \sum_{i=0}^{n} b_{n,i}(x; q) + \frac{2q[n-1]_q + 1 - [n]_q}{[n]_q^2 - 1} \lambda b_{n+1,n}(x; q) - \frac{\lambda}{[n]_q + 1} b_{n+1,n}(x; q)$$

$$= \sum_{i=0}^{n} b_{n,i}(x; q) = 1.$$

Then we can obtain (5) since $\widetilde{B}_{n,q,\lambda}(e_0; x) = \sum_{k=0}^{n} \widetilde{b}_{n,k}(x; q, \lambda)$. $\qquad\square$

Lemma 2.2. *Let $\lambda \in [-1, 1]$, $x \in [0, 1]$ and $q \in (0, 1]$, for the operators $\widetilde{B}_{n,q,\lambda}(f; x)$, we have*

$$\widetilde{B}_{n,q,\lambda}(e_1; x) = x + \frac{[n+1]_q x \left(1 - x^n\right) \lambda}{[n]_q \left([n]_q - 1\right)} - \frac{2[n+1]_q x \lambda}{[n]_q^2 - 1} \left[\frac{1 - x^n}{[n]_q} + qx \left(1 - x^{n-1}\right) \right]$$

$$+ \frac{\lambda}{q[n]_q \left([n]_q + 1\right)} \left[1 - \prod_{s=0}^{n} \left(1 - q^s x\right) - x^{n+1} - [n+1]_q x \left(1 - x^n\right) \right]$$

$$+ \frac{\lambda}{[n]_q^2 - 1} \left\{ 2[n+1]_q x^2 \left(1 - x^{n-1}\right) - \frac{2[n+1]_q x}{q[n]_q} \left(1 - x^n\right) \right.$$

$$\left. + \frac{2}{q[n]_q} \left[1 - \prod_{s=0}^{n} \left(1 - q^s x\right) - x^{n+1} \right] \right\}. \tag{6}$$

Proof. From (3), we have

$$\widetilde{B}_{n,q,\lambda}(e_1; x) = \sum_{i=0}^{n} \widetilde{b}_{n,i}(x; q, \lambda) \frac{[i]_q}{[n]_q}$$

$$= \sum_{i=1}^{n-1} \frac{[i]_q}{[n]_q} \widetilde{b}_{n,i}(x; q, \lambda) + \widetilde{b}_{n,n}(x; q, \lambda)$$

$$= \sum_{i=1}^{n-1} \frac{[i]_q}{[n]_q} \left[b_{n,i}(x; q) + \lambda \left(\frac{[n]_q - 2[i]_q + 1}{[n]_q^2 - 1} b_{n+1,i}(x; q) - \frac{[n]_q - 2q[i]_q - 1}{[n]_q^2 - 1} b_{n+1,i+1}(x; q) \right) \right]$$

$$+ b_{n,n}(x; q) - \frac{\lambda}{[n]_q + 1} b_{n+1,n}(x; q)$$

$$= \sum_{i=0}^{n} \frac{[i]_q}{[n]_q} b_{n,i}(x; q) + \lambda \sum_{i=0}^{n} \frac{[i]_q}{[n]_q} \frac{[n]_q - 2[i]_q + 1}{[n]_q^2 - 1} b_{n+1,i}(x; q)$$

$$- \lambda \sum_{i=1}^{n-1} \frac{[i]_q}{[n]_q} \frac{[n]_q - 2q[i]_q - 1}{[n]_q^2 - 1} b_{n+1,i+1}(x; q).$$

As we know, for the q-Bernstein operators [2], we have $\sum_{i=0}^{n} \frac{[i]_q}{[n]_q} b_{n,i}(x; q) = x$, then

$$\widetilde{B}_{n,q,\lambda}(e_1; x) = x + \frac{[n+1]_q \lambda}{[n]_q ([n]_q - 1)} \sum_{i=1}^{n} \frac{[i]_q}{[n+1]_q} \frac{[n+1]_q!}{[i]_q![n+1-i]_q!} x^i \prod_{s=0}^{n-i} (1 - q^s x)$$

$$- \frac{2[n+1]_q \lambda}{[n]_q^2 - 1} \sum_{i=1}^{n} \frac{[i]_q^2}{[n]_q[n+1]_q} \frac{[n+1]_q!}{[i]_q![n+1-i]_q!} x^i \prod_{s=0}^{n-i} (1 - q^s x)$$

$$- \frac{[n+1]_q \lambda}{[n]_q ([n]_q + 1)} \sum_{i=1}^{n-1} \frac{[i]_q}{[n+1]_q} \frac{[n+1]_q!}{[i+1]_q![n-i]_q!} x^{i+1} \prod_{s=0}^{n-i-1} (1 - q^s x)$$

$$+ \frac{2q[n+1]_q \lambda}{[n]_q^2 - 1} \sum_{i=1}^{n-1} \frac{[i]_q^2}{[n]_q[n+1]_q} \frac{[n+1]_q!}{[i+1]_q![n-i]_q!} x^{i+1} \prod_{s=0}^{n-i-1} (1 - q^s x)$$

$$:= x + \triangle_1 + \triangle_2 + \triangle_3 + \triangle_4. \tag{7}$$

Firstly, we have

$$\triangle_1 = \frac{[n+1]_q \lambda}{[n]_q ([n]_q - 1)} \sum_{i=1}^{n} \frac{[i]_q}{[n+1]_q} \frac{[n+1]_q!}{[i]_q![n+1-i]_q!} x^i \prod_{s=0}^{n-i} (1 - q^s x)$$

$$= \frac{[n+1]_q \lambda x}{[n]_q ([n]_q - 1)} \sum_{i=0}^{n-1} b_{n,i}(x; q) = \frac{[n+1]_q x (1 - x^n) \lambda}{[n]_q ([n]_q - 1)}. \tag{8}$$

Secondly, since $[i]_q^2 = [i]_q + q[i]_q[i-1]_q$, we get

$$\triangle_2 = - \frac{2[n+1]_q \lambda}{[n]_q^2 - 1} \sum_{i=1}^{n} \frac{[i]_q^2}{[n]_q[n+1]_q} \frac{[n+1]_q!}{[i]_q![n+1-i]_q!} x^i \prod_{s=0}^{n-i} (1 - q^s x)$$

$$= \frac{-2[n+1]_q \lambda}{[n]_q ([n]_q^2 - 1)} \sum_{i=1}^{n} \frac{[i]_q}{[n+1]_q} \frac{[n+1]_q!}{[i]_q![n+1-i]_q!} x^i \prod_{s=0}^{n-i} (1 - q^s x)$$

$$+ \frac{-2q[n+1]_q \lambda}{[n]_q^2 - 1} \sum_{i=1}^{n} \frac{[i]_q[i-1]_q}{[n+1]_q[n]_q} \frac{[n+1]_q!}{[i]_q![n+1-i]_q!} x^i \prod_{s=0}^{n-i} (1 - q^s x)$$

$$= \frac{-2[n+1]_q \lambda x}{[n]_q ([n]_q^2 - 1)} \sum_{i=0}^{n-1} b_{n,i}(x; q) + \frac{-2q[n+1]_q x^2 \lambda}{[n]_q^2 - 1} \sum_{i=0}^{n-2} b_{n-1,i}(x; q)$$

$$= - \frac{2[n+1]_q x \lambda}{[n]_q^2 - 1} \left[\frac{1 - x^n}{[n]_q} + qx \left(1 - x^{n-1}\right) \right]. \tag{9}$$

Thirdly, since $[i]_q = [i+1]_q/q - 1/q$, we have

$$\triangle_3 = - \frac{[n+1]_q \lambda}{[n]_q ([n]_q + 1)} \sum_{i=1}^{n-1} \frac{[i]_q}{[n+1]_q} \frac{[n+1]_q!}{[i+1]_q![n-i]_q!} x^{i+1} \prod_{s=0}^{n-i-1} (1 - q^s x)$$

$$= - \frac{[n+1]_q \lambda}{q[n]_q ([n]_q + 1)} \sum_{i=1}^{n-1} \frac{[i+1]_q}{[n+1]_q} \frac{[n+1]_q!}{[i+1]_q![n-i]_q!} x^{i+1} \prod_{s=0}^{n-i-1} (1 - q^s x)$$

$$+ \frac{\lambda}{q ([n]_q + 1) [n]_q} \sum_{i=1}^{n-1} b_{n+1,i+1}(x; q)$$

$$= - \frac{[n+1]_q x \lambda}{q[n]_q ([n]_q + 1)} \sum_{i=1}^{n-1} b_{n,i}(x; q) + \frac{\lambda}{q ([n]_q + 1) [n]_q} \sum_{i=1}^{n-1} b_{n+1,i+1}(x; q)$$

$$= - \frac{[n+1]_q x \lambda}{q[n]_q ([n]_q + 1)} \left[1 - \prod_{s=0}^{n-1} (1 - q^s x) - x^n \right]$$

$$+\frac{\lambda}{q[n]_q\left([n]_q+1\right)}\left[1-\prod_{s=0}^{n}\left(1-q^sx\right)-[n+1]x\prod_{s=0}^{n-1}\left(1-q^sx\right)-x^{n+1}\right]$$

$$=\frac{\lambda}{q[n]_q\left([n]_q+1\right)}\left[1-\prod_{s=0}^{n}\left(1-q^sx\right)-x^{n+1}-[n+1]_qx\left(1-x^n\right)\right].\tag{10}$$

Finally, since $[i]_q^2=[i]_q[i+1]_q/q-[i]_q/q$, we have

$$\triangle_4=\frac{2q[n+1]_q\lambda}{[n]_q^2-1}\sum_{i=1}^{n-1}\frac{[i]_q^2}{[n]_q[n+1]_q}\frac{[n+1]_q!}{[i+1]_q![n-i]_q!}x^{i+1}\prod_{s=0}^{n-i-1}\left(1-q^sx\right)$$

$$=\frac{2[n+1]_q\lambda}{[n]_2-1}\sum_{i=0}^{n-2}b_{n-1,i}(x;q)-\frac{2[n+1]_qx\lambda}{q[n]_q\left([n]_q^2-1\right)}\sum_{i=1}^{n-1}b_{n,i}(x;q)$$

$$+\frac{2\lambda}{q[n]_q\left([n]_q^2-1\right)}\sum_{i=1}^{n-1}b_{n+1,i+1}(x;q)$$

$$=\frac{\lambda}{[n]_q^2-1}\left\{2[n+1]_qx^2\left(1-x^{n-1}\right)-\frac{2[n+1]_qx}{q[n]_q}\left(1-x^n\right)\right.$$

$$\left.+\frac{2}{q[n]_q}\left[1-\prod_{s=0}^{n}\left(1-q^sx\right)-x^{n+1}\right]\right\}.\tag{11}$$

Then, (6) can be obtained by (7)-(11). $\qquad\square$

Lemma 2.3. *Let* $\lambda\in[-1,1]$, $x\in[0,1]$, $n>1$ *and* $q\in(0,1]$, *for the operators* $\widetilde{B}_{n,q,\lambda}(f;x)$, *we have*

$$\widetilde{B}_{n,q,\lambda}(e_2;x)$$

$$=x^2+\frac{x(1-x)}{[n]_q}+\frac{[n+1]_qx\lambda}{[n]_q\left([n]_q-1\right)}\left[qx\left(1-x^{n-1}\right)+\frac{1-x^n}{[n]_q}\right]-\frac{2[n+1]_q\lambda}{[n]_q\left([n]_q^2-1\right)}$$

$$\times\left[\frac{x\left(1-x^n\right)}{[n]_q}+q(2+q)x^2\left(1-x^{n-1}\right)+q^3[n-1]_qx^3\left(1-x^{n-2}\right)\right]$$

$$-\frac{\lambda}{q[n]_q\left([n]_q+1\right)}\left\{[n+1]_qx^2\left(1-x^{n-1}\right)-\frac{[n+1]_qx\left(1-x^n\right)}{q[n]_q}\right.$$

$$+\frac{1-\prod_{s=0}^{n}\left(1-q^sx\right)-x^{n+1}}{q[n]_q}\right\}+\frac{2\lambda}{[n]_q\left([n]_q^2-1\right)}$$

$$\times\left\{q[n-1]_q[n+1]_qx^3\left(1-x^{n-2}\right)-\frac{(1-q)[n+1]_qx^2\left(1-x^{n-1}\right)}{q}\right.$$

$$\left.+\frac{[n+1]_qx\left(1-x^n\right)}{q^2[n]_q}-\frac{1-\prod_{s=0}^{n}\left(1-q^sx\right)-x^{n+1}}{q^2[n]_q}\right\}.\tag{12}$$

Proof. From (3), we have

$$\widetilde{B}_{n,q,\lambda}(e_2;x)=\sum_{i=0}^{n}\frac{[i]_q^2}{[n]_q^2}\widetilde{b}_{n,i}(x;q,\lambda)$$

$$=\sum_{i=0}^{n}\frac{[i]_q^2}{[n]_q^2}b_{n,i}(x;q)+\lambda\sum_{i=0}^{n}\frac{[i]_q^2}{[n]_q^2}\frac{[n]_q-2[i]_q+1}{[n]_q^2-1}b_{n+1,i}(x;q)$$

$$-\lambda\sum_{i=1}^{n-1}\frac{[i]_q^2}{[n]_q^2}\frac{[n]_q-2q[i]_q-1}{[n]_q^2-1}b_{n+1,i+1}(x;q).$$

For the q-Bernstein polynomials [2], we have $\sum_{i=0}^{n} \frac{[i]_q^2}{[n]_q^2} b_{n,i}(x;q) = x^2 + \frac{x(1-x)}{[n]_q}$, thus

$$\widetilde{B}_{n,q,\lambda}(e_2;x) = x^2 + \frac{x(1-x)}{[n]_q} + \frac{\lambda}{[n]_q - 1} \sum_{i=0}^{n} \frac{[i]_q^2}{[n]_q^2} b_{n+1,i}(x;q) - \frac{2\lambda}{[n]_q^2 - 1} \sum_{i=0}^{n} \frac{[i]_q^3}{[n]_q^2} b_{n+1,i}(x;q)$$

$$- \frac{\lambda}{[n]_q + 1} \sum_{i=1}^{n-1} \frac{[i]_q^2}{[n]_q^2} b_{n+1,i+1}(x;q) + \frac{2q\lambda}{[n]_q^2 - 1} \sum_{i=1}^{n-1} \frac{[i]_q^3}{[n]_q^2} b_{n+1,i+1}(x;q)$$

$$:= x^2 + \frac{x(1-x)}{[n]_q} + \triangle_5 + \triangle_6 + \triangle_7 + \triangle_8. \tag{13}$$

We have

$$\triangle_5 = \frac{\lambda}{[n]_q - 1} \sum_{i=0}^{n} \frac{[i]_q^2}{[n]_q^2} b_{n+1,i}(x;q)$$

$$= \frac{[n+1]_q q x^2 \lambda}{[n]_q ([n]_q - 1)} \sum_{i=0}^{n-2} b_{n-1,i}(x;q) + \frac{[n+1]_q x \lambda}{[n]_q^2 ([n]_q - 1)} \sum_{i=0}^{n-1} b_{n,i}(x;q)$$

$$= \frac{[n+1]_q x \lambda}{[n]_q ([n]_q - 1)} \left[qx \left(1 - x^{n-1}\right) + \frac{1 - x^n}{[n]_q} \right]. \tag{14}$$

Since $[i]_q^3 = [i]_q + q(2+q)[i]_q[i-1]_q + q^3[i]_q[i-1]_q[i-2]_q$, we have

$$\triangle_6 = -\frac{2\lambda}{[n]_q^2 - 1} \sum_{i=0}^{n} \frac{[i]_q^3}{[n]_q^2} b_{n+1,i}(x;q)$$

$$= -\frac{2[n+1]_q x \lambda}{[n]_q^2 ([n]_q^2 - 1)} \sum_{i=0}^{n-1} b_{n,i}(x;q) - \frac{2q(2+q)[n+1]_q x^2 \lambda}{[n]_q ([n]_q^2 - 1)} \sum_{i=0}^{n-2} b_{n-1,i}(x;q)$$

$$- \frac{2q^3[n-1]_q[n+1]_q x^3 \lambda}{[n]_q ([n]_q^2 - 1)} \sum_{i=0}^{n-3} b_{n-2,i}(x;q)$$

$$= \frac{-2[n+1]_q \lambda}{[n]_q ([n]_q^2 - 1)} \left[\frac{x \left(1 - x^n\right)}{[n]_q} + q(2+q)x^2 \left(1 - x^{n-1}\right) + q^3[n-1]_q x^3 \left(1 - x^{n-2}\right) \right]. \tag{15}$$

Next, since $[i]_q^2 = [i+1]_q[i]_q/q - [i+1]_q/q^2 + 1/q^2$, we get

$$\triangle_7 = -\frac{\lambda}{[n]_q + 1} \sum_{i=1}^{n-1} \frac{[i]_q^2}{[n]_q^2} b_{n+1,i+1}(x;q)$$

$$= -\frac{[n+1]_q x^2 \lambda}{q[n]_q ([n]_q + 1)} \sum_{i=0}^{n-2} b_{n-1,i}(x;q) + \frac{[n+1]_q x \lambda}{q^2[n]_q^2 ([n]_q + 1)} \sum_{i=1}^{n-1} b_{n,i}(x;q)$$

$$- \frac{\lambda}{q^2[n]_q^2 ([n]_q + 1)} \sum_{i=1}^{n-1} b_{n+1,i+1}(x;q)$$

$$= \frac{-\lambda}{q[n]_q ([n]_q + 1)} \left\{ [n+1]_q x^2 \left(1 - x^{n-1}\right) - \frac{[n+1]_q x \left(1 - x^n\right)}{q[n]_q} \right.$$

$$\left. + \frac{1 - \prod_{s=0}^{n} \left(1 - q^s x\right) - x^{n+1}}{q[n]_q} \right\}. \tag{16}$$

Finally, by some computations, we have

$$[i]_q^3 = [i+1]_q[i]_q[i-1]_q - \frac{1-q}{q^2}[i+1]_q[i]_q + \frac{[i+1]_q}{q^3} - \frac{1}{q^3}.$$

Then, we obtain

$$\triangle_8 = \frac{2q\lambda}{[n]_q^2 - 1} \sum_{i=1}^{n-1} \frac{[i]_q^3}{[n]_q^2} b_{n+1,i+1}(x;q)$$

$$= \frac{2q[n-1]_q[n+1]_qx^3\lambda}{[n]_q\left([n]_q^2-1\right)}\sum_{i=0}^{n-3}b_{n-2,i}(x;q) - \frac{2(1-q)[n+1]_qx^2\lambda}{q[n]_q\left([n]_q^2-1\right)}\sum_{i=0}^{n-2}b_{n-1,i}(x;q)$$

$$+ \frac{2[n+1]_qx\lambda}{q^2[n]_q^2\left([n]_q^2-1\right)}\sum_{i=1}^{n-1}b_{n,i}(x;q) - \frac{2\lambda}{q^2[n]_q^2\left([n]_q^2-1\right)}\sum_{i=1}^{n-1}b_{n+1,i+1}(x;q)$$

$$= \frac{2\lambda}{[n]_q\left([n]_q^2-1\right)}\left\{q[n-1]_q[n+1]_qx^3\left(1-x^{n-2}\right) - \frac{(1-q)[n+1]_qx^2\left(1-x^{n-1}\right)}{q}\right.$$

$$\left. + \frac{[n+1]_qx\left(1-x^n\right)}{q^2[n]_q} - \frac{1-\prod_{s=0}^n\left(1-q^sx\right)-x^{n+1}}{q^2[n]_q}\right\}. \tag{17}$$

Using (13)-(17), we can obtain (12), Lemma 2.1 is proved. $\qquad\square$

Remark 2.4. *When $\lambda = 0$, we get the moments of q-Bernstein operators (see [2]).*

Lemma 2.5. *For $\lambda \in [-1, 1]$, $x \in [0, 1]$ and $q \in (0, 1]$, the operators $\widetilde{B}_{n,q,\lambda}(f; x)$ are positive linear operators.*

Proof. Indeed, we only need to prove $\widetilde{b}_{n,i}(t; q, \lambda) \geq 0$ for $0 \leq i \leq n$. For $i = 0$, we have

$$\widetilde{b}_{n,0}(x;q,\lambda) = b_{n,0}(x;q) - \frac{\lambda}{[n]_q+1}b_{n+1,1}(x;q)$$

$$= \prod_{s=0}^{n-1}\left(1-q^sx\right) - \frac{\lambda}{[n]_q+1}[n+1]_qx\prod_{s=0}^{n-1}\left(1-q^sx\right)$$

$$= \prod_{s=0}^{n-1}\left(1-q^sx\right)\left(1-\frac{[n+1]_q}{[n]_q+1}\lambda x\right) \geq 0,$$

since $[n]_q + 1 \geq [n+1]_q$. Similarly, we can obtain $\widetilde{b}_{n,n}(x; q, \lambda) \geq 0$. Then we will prove $\widetilde{b}_{n,i}(x; q, \lambda) \geq 0$, where $1 \leq i \leq n - 1$. Actually, by (4), we have

$$\widetilde{b}_{n,i}(x;q,\lambda) = b_{n,i}(x;q) + \lambda\left[\frac{[n]_q-2[i]_q+1}{[n]_q^2-1}b_{n+1,i}(x;q) - \frac{[n]_q-2q[i]_q-1}{[n]_q^2-1}b_{n+1,i+1}(x;q)\right]$$

$$= b_{n,i}(x;q)\left\{1+\lambda\left[\frac{[n]_q-2[i]_q+1}{[n]_q^2-1}\frac{[n+1]_q}{[n+1-i]_q}\left(1-q^{n-i}x\right) - \frac{[n]_q-2q[i]_q-1}{[n]_q^2-1}\frac{[n+1]_q}{[i+1]_q}x\right]\right\}.$$

We need to prove

$$-1 \leq \frac{[n]_q-2[i]_q+1}{[n]_q^2-1}\frac{[n+1]_q}{[n+1-i]_q}\left(1-q^{n-i}x\right) - \frac{[n]_q-2q[i]_q-1}{[n]_q^2-1}\frac{[n+1]_q}{[i+1]_q}x \leq 1. \tag{18}$$

We discuss the above inequality in three cases as follows:

Case 1: For $1 \leq [i]_q \leq \min\left\{\frac{[n]_q-1}{2q}, \frac{[n]_q+1}{2}\right\}$, we have

$$0 \leq \frac{[n+1]_q}{[n+1-i]_q}\frac{[n]_q-2[i]_q+1}{[n]_q^2-1}, \frac{[n+1]_q}{[i+1]_q}\frac{[n]_q-2q[i]_q-1}{[n]_q^2-1} \leq 1.$$

Then we can obtain (18) by the fact that $0 \leq 1 - q^{n-i}x$, $0 \leq x \leq 1$.

Case 2: For $\max\left\{\frac{[n]_q+1}{2}, \frac{[n]_q-1}{2q}\right\} \leq [i]_q \leq [n-1]_q$, we have

$$-1 \leq \frac{[n+1]_q}{[n+1-i]_q}\frac{[n]_q-2[i]_q+1}{[n]_q^2-1}, \frac{[n+1]_q}{[i+1]_q}\frac{[n]_q-2q[i]_q-1}{[n]_q^2-1} \leq 0.$$

Then we can get (18) for $0 \leq 1 - q^{n-i}x$, $0 \leq x \leq 1$.

Case 3: For $\min\left\{\frac{[n]_q+1}{2}, \frac{[n]_q-1}{2q}\right\} < [i]_q < \max\left\{\frac{[n]_q+1}{2}, \frac{[n]_q-1}{2q}\right\}$.

If $\frac{[n]_q-1}{2q} < [i]_q < \frac{[n]_q+1}{2}$, we have

$$0 < \frac{[n]_q-2[i]_q+1}{[n]_q^2-1}\frac{[n+1]_q}{[n+1-i]_q}, -\frac{[n]_q-2q[i]_q-1}{[n]_q^2-1}\frac{[n+1]_q}{[i+1]_q} \leq \frac{1}{2};$$

If $\frac{[n]_q+1}{2} < [i]_q < \frac{[n]_q-1}{2q}$, we have

$$-\frac{1}{2} \leq \frac{[n]_q - 2[i]_q + 1}{[n]_q^2 - 1} \frac{[n+1]_q}{[n+1-i]_q}, \quad -\frac{[n]_q - 2q[i]_q - 1}{[n]_q^2 - 1} \frac{[n+1]_q}{[i+1]_q} < 0.$$

Then we can get(18) in both two subcases. Therefore, we get the proof of Lemma 2.5. □

3 Main results

Let us give the following statistical approximation theorem for the operators (3).

Theorem 3.1. *Let $q = \{q_n\}$ be a sequence satisfying $st-\lim_{n\to\infty} q_n = 1$ for $0 < q_n < 1$. Then, for all $f \in C[0, 1]$, $\lambda \in [-1, 1]$, $n > 1$ and the operators $\widetilde{B}_{n,q_n,\lambda}(f; x)$, we have*

$$st - \lim_{n\to\infty} \left\| \widetilde{B}_{n,q_n,\lambda}(f; \cdot) - f \right\| = 0. \tag{19}$$

Proof. By Lemma 2.1, it is clear that

$$st - \lim_{n\to\infty} \left\| \widetilde{B}_{n,q_n,\lambda}(e_0; \cdot) - e_0 \right\|_{C[0,1]} = 0. \tag{20}$$

In order to obtain the desired result, using Lemma 2.5 and Theorem 1.1, we only need to prove the following equalities

$$st - \lim_{n\to\infty} \left\| \widetilde{B}_{n,q_n,\lambda}(e_v; \cdot) - e_v \right\|_{C[0,1]} = 0, \quad v = 1, 2. \tag{21}$$

For $v = 1$, using Lemma 2.2, we have

$$\left| \widetilde{B}_{n,q_n,\lambda}(e_1; x) - e_1(x) \right| \leq \frac{[n+1]_{q_n}}{[n]_{q_n}([n]_{q_n} - 1)} + \frac{2[n+1]_{q_n}}{[n]_{q_n}^2 - 1}\left(\frac{1}{[n]_{q_n}} + 1\right) + \frac{1}{q_n[n]_{q_n}([n]_{q_n} + 1)}$$

$$+ \frac{1 + [n+1]_{q_n}}{q_n[n]_{q_n}([n]_{q_n} + 1)} + \frac{2}{[n]_{q_n}^2 - 1}\left([n+1]_{q_n} + \frac{[n+1]_{q_n}}{q_n[n]_{q_n}} + \frac{1}{q_n[n]_{q_n}}\right)$$

$$\leq \frac{q_n[n+1]_{q_n} + 2q_n + 4}{q_n[n]_{q_n}([n]_{q_n} - 1)} + \frac{5}{[n]_{q_n} - 1} + \frac{2}{q_n[n]_{q_n}([n]_{q_n}^2 - 1)},$$

since $[n+1]_{q_n} \leq [n]_{q_n} + 1$, we obtain

$$\left| \widetilde{B}_{n,q_n,\lambda}(e_1; x) - e_1(x) \right| \leq \frac{6}{[n]_{q_n} - 1} + \frac{3q_n + 4}{q_n[n]_{q_n}([n]_{q_n} - 1)} + \frac{2}{q_n[n]_{q_n}([n]_{q_n}^2 - 1)}.$$

For a given $\epsilon > 0$, let us define the following sets

$$U = \left\{ k : \left\| \widetilde{B}_{k,q_k,\lambda}(e_1; \cdot) - e_1 \right\|_{C[0,1]} \geq \epsilon \right\}; \quad U_1 = \left\{ k : \frac{6}{[k]_{q_k} - 1} \geq \frac{\epsilon}{3} \right\};$$

$$U_2 = \left\{ k : \frac{3q_k + 4}{q_k[k]_{q_k}([k]_{q_k} - 1)} \geq \frac{\epsilon}{3} \right\}; \quad U_3 = \left\{ k : \frac{2}{q_k[k]_{q_k}([k]_{q_k}^2 - 1)} \geq \frac{\epsilon}{3} \right\}.$$

It is obvious that $U \subset U_1 \cup U_2 \cup U_3$, which implies that

$$\delta \left\{ k \leq n : \left\| \widetilde{B}_{k,q_k,\lambda}(e_1; \cdot) - e_1 \right\|_{C[0,1]} \geq \epsilon \right\}$$

$$\leq \delta \left\{ k \leq n : \frac{6}{[k]_{q_k} - 1} \geq \frac{\epsilon}{3} \right\} + \delta \left\{ k \leq n : \frac{3q_k + 4}{q_k[k]_q([k]_{q_k} - 1)} \geq \frac{\epsilon}{3} \right\}$$

$$+ \delta \left\{ k \leq n : \frac{2}{q_k[k]_{q_k}([k]_{q_k}^2 - 1)} \geq \frac{\epsilon}{3} \right\}. \tag{22}$$

Since $st - \lim_{n\to\infty} q_n = 1$, we have

$$st - \lim_{n\to\infty} \frac{6}{[n]_{q_n} - 1} = 0; \quad st - \lim_{n\to\infty} \frac{3q_n + 4}{q_n[n]_{q_n}\left([n]_{q_n} - 1\right)} = 0;$$

$$st - \lim_{n\to\infty} \frac{2}{q_n[n]_{q_n}\left([n]_{\hat{q}}^2 - 1\right)} = 0,$$

that is to say, the right hand side of (22) is zero. Hence, (21) is proved for $\nu = 1$.

For $\nu = 2$, by Lemma 2.3, we have

$$\left|\widetilde{B}_{n,q_n,\lambda}(e_2; x) - e_2(x)\right|$$

$$\leq \frac{1}{4[n]_{q_n}} + \frac{[n+1]_{q_n}}{[n]_{q_n}\left([n]_{q_n} - 1\right)}\left(1 + \frac{1}{[n]_{q_n}}\right) + \frac{2[n+1]_{q_n}}{[n]_{q_n}\left([n]_{q_n}^2 - 1\right)}\left(\frac{1}{[n]_{q_n}} + 3 + [n-1]_{q_n}\right)$$

$$+ \frac{1}{q_n[n]_{q_n}\left([n]_{q_n} + 1\right)}\left([n+1]_{q_n} + \frac{[n+1]_{q_n}}{q_n[n]_{q_n}} + \frac{1}{q_n[n]_{q_n}}\right) + \frac{2}{[n]_{q_n}\left([n]_{q_n}^2 - 1\right)}$$

$$\left([n-1]_{q_n}[n+1]_{q_n} + \frac{[n+1]_{q_n}}{q_n} + \frac{[n+1]_{q_n}}{q_n^2[n]_{q_n}} + \frac{1}{q_n^2[n]_{q_n}}\right)$$

$$\leq \frac{1}{4[n]_{q_n}} + \frac{5}{[n]_{q_n} - 1} + \frac{1}{q_n[n]_{q_n}} + \frac{2}{[n]_{q_n}\left([n]_{q_n} - 1\right)} + \frac{8}{[n]_{q_n}^2 - 1} + \frac{1}{q_n^2[n]_{q_n}^2}$$

$$+ \frac{2}{q_n[n]_{q_n}\left([n]_{q_n} - 1\right)} + \frac{1}{[n]_{\hat{q}}^2\left([n]_{q_n} - 1\right)} + \frac{8}{[n]_{q_n}\left([n]_{\hat{q}}^2 - 1\right)} + \frac{1}{q_n^2[n]_{\hat{q}}^2\left([n]_{q_n} + 1\right)}$$

$$+ \frac{2}{q_n^2[n]_{\hat{q}}^2\left([n]_{q_n} - 1\right)} + \frac{2}{[n]_{\hat{q}}^2\left([n]_{\hat{q}}^2 - 1\right)} + \frac{2}{q_n^2[n]_{\hat{q}}^2\left([n]_{\hat{q}}^2 - 1\right)}.$$

For a given $\epsilon > 0$, we define the following sets

$$V = \left\{k : \left\|\widetilde{B}_{k,q_k,\lambda}(e_2; \cdot) - e_2\right\|_{C[0,1]} \geq \epsilon\right\};$$

$$V_1 = \left\{k : \frac{1}{4[k]_{q_k}} + \frac{5}{[k]_{q_k} - 1} + \frac{1}{q_k[k]_{q_k}} \geq \frac{\epsilon}{4}\right\};$$

$$V_2 = \left\{k : \frac{2}{[k]_{q_k}\left([k]_{q_k} - 1\right)} + \frac{8}{[k]_{\hat{q}_k}^2 - 1} + \frac{1}{q_k^2[k]_{\hat{q}_k}^2} + \frac{2}{q_k[k]_{q_k}\left([k]_{q_k} - 1\right)} \geq \frac{\epsilon}{4}\right\};$$

$$V_3 = \left\{k : \frac{1}{[k]_{\hat{q}_k}^2\left([k]_{q_k} - 1\right)} + \frac{8}{[k]_{q_k}\left([k]_{\hat{q}_k}^2 - 1\right)} + \frac{1}{q_k^2[k]_{\hat{q}_k}^2\left([k]_{q_k} + 1\right)} + \frac{2}{q_k^2[k]_{\hat{q}_k}^2\left([k]_{q_k} - 1\right)} \geq \frac{\epsilon}{4}\right\};$$

$$V_4 = \left\{k : \frac{2}{[k]_{\hat{q}_k}^2\left([k]_{\hat{q}_k}^2 - 1\right)} + \frac{2}{q_k^2[k]_{\hat{q}_k}^2\left([k]_{\hat{q}_k}^2 - 1\right)} \geq \frac{\epsilon}{4}\right\}.$$

Obviously, $V \subset V_1 \cup V_2 \cup V_3 \cup V_4$, then one have

$$\delta\left\{k \leq n : \left\|\widetilde{B}_{k,q_k,\lambda}(e_2; \cdot) - e_2\right\|_{C[0,1]} \geq \epsilon\right\}$$

$$\leq \delta\left\{k \leq n : \frac{1}{4[k]_{q_k}} + \frac{5}{[k]_{q_k} - 1} + \frac{1}{q_k[k]_{q_k}} \geq \frac{\epsilon}{4}\right\}$$

$$+ \delta\left\{k \leq n : \frac{2}{[k]_{q_k}\left([k]_{q_k} - 1\right)} + \frac{8}{[k]_{\hat{q}_k}^2 - 1} + \frac{1}{q_k^2[k]_{\hat{q}_k}^2} + \frac{2}{q_k[k]_{q_k}\left([k]_{q_k} - 1\right)} \geq \frac{\epsilon}{4}\right\}$$

$$+ \delta\left\{k \leq n : \frac{1}{[k]_{\hat{q}_k}^2\left([k]_{q_k} - 1\right)} + \frac{8}{[k]_{q_k}\left([k]_{\hat{q}_k}^2 - 1\right)} + \frac{1}{q_k^2[k]_{\hat{q}_k}^2\left([k]_{q_k} + 1\right)} + \frac{2}{q_k^2[k]_{\hat{q}_k}^2\left([k]_{q_k} - 1\right)} \geq \frac{\epsilon}{4}\right\}$$

$$+ \delta\left\{k \leq n : \frac{2}{[k]_{\hat{q}_k}^2\left([k]_{\hat{q}_k}^2 - 1\right)} + \frac{2}{q_k^2[k]_{\hat{q}_k}^2\left([k]_{\hat{q}_k}^2 - 1\right)} \geq \frac{\epsilon}{4}\right\}. \tag{23}$$

Since $st - \lim_{n\to\infty} q_n = 1$, we obtain the following equalities

$$st - \lim_{n\to\infty}\left(\frac{1}{4[n]_{q_n}} + \frac{5}{[n]_{q_n} - 1} + \frac{1}{q_n[n]_{q_n}}\right) = 0;$$

$$st - \lim_{n\to\infty} \left[\frac{2}{[n]_{q_n} \left([n]_{q_n} - 1\right)} + \frac{8}{[n]_{q_n}^2 - 1} + \frac{1}{q_n^2 [n]_{q_n}^2} + \frac{2}{q_n [n]_{q_n} \left([n]_{q_n} - 1\right)} \right] = 0;$$

$$st - \lim_{n\to\infty} \left[\frac{1}{[n]_{q_n}^2 \left([n]_{q_n} - 1\right)} + \frac{8}{[n]_{q_n} \left([n]_{q_n}^2 - 1\right)} + \frac{1}{q_n^2 [n]_{q_n}^2 \left([n]_{q_n} + 1\right)} + \frac{2}{q_n^2 [n]_{q_n}^2 \left([n]_{q_n} - 1\right)} \right] = 0;$$

$$st - \lim_{n\to\infty} \left[\frac{2}{[n]_{q_n}^2 \left([n]_{q_n}^2 - 1\right)} + \frac{2}{q_n^2 [n]_{q_n}^2 \left([n]_{q_n}^2 - 1\right)} \right] = 0,$$

that is, the right hand side of (23) is zero. Therefore, (21) is proved for $v = 2$. Theorem 3.1 is proved. □

Finally, we give an example to show the convergence of $\widetilde{B}_{n,q,\lambda}(f; x)$ to $f(x)$ with different values of parameters, we also obtain the absolute error bounds in some cases.

Example 3.2. *Let $f(x) = 1 - cos(4e^x)$, the graphs of $\widetilde{B}_{n,q,-1}(f; x)$ and $\widetilde{B}_{n,q,1}(f; x)$ with different cases of n and q are shown in Figure 1. Figure 2 shows the graphs of $\widetilde{B}_{n,q,1}(f; x)$ and $\widetilde{B}_{n,q,0}(f; x)$ with n = 10 and q = 0.9. In Table 1, we give the absolute error bounds of $\widetilde{B}_{n,q,\lambda}(f; x)$ to $f(x)$.*
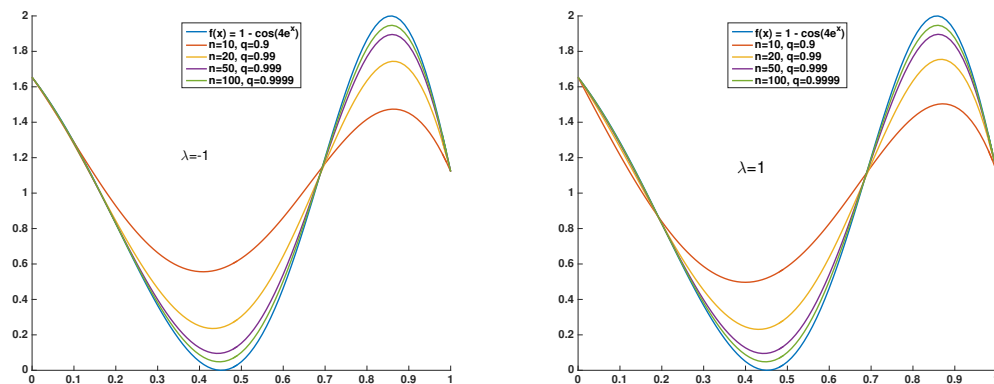

Figure 1: Convergence of $\widetilde{B}_{n,q,\lambda}(f; x)$ for $\lambda = -1$, $\lambda = 1$ and different values of q and n.
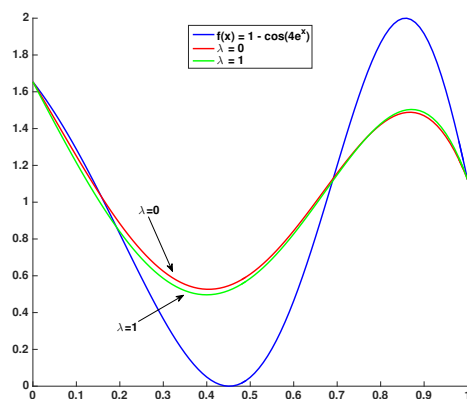

Figure 2: Convergence of $\widetilde{B}_{10,0.9,\lambda}(f; x)$ for $\lambda = 1$ and $\lambda = 0$.

Table 1: The errors of the approximation of $\widetilde{B}_{n,q,\lambda}(f;x)$ to $f(x)$.

		$\|f - \widetilde{B}_{n,q,\lambda}(f)\|_\infty$			
		$n = 10$	$n = 20$	$n = 50$	$n = 100$
$\lambda = -1$	$q = 0.9$	0.586703	0.469244	0.425725	0.423995
	$q = 0.99$	0.447241	0.259814	0.128908	0.082675
	$q = 0.999$	0.438761	0.244595	0.107170	0.056663
	$q = 0.9999$	0.437909	0.243088	0.105111	0.054359
$\lambda = 0$	$q = 0.9$	0.562861	0.459109	0.418998	0.417386
	$q = 0.99$	0.432402	0.256685	0.128892	0.082737
	$q = 0.999$	0.423813	0.241516	0.107168	0.056722
	$q = 0.9999$	0.422953	0.240017	0.105112	0.054418
$\lambda = 1$	$q = 0.9$	0.539161	0.449016	0.412299	0.410807
	$q = 0.99$	0.418391	0.253900	0.128901	0.082802
	$q = 0.999$	0.409770	0.238817	0.107201	0.056785
	$q = 0.9999$	0.408905	0.237324	0.105145	0.054481

One can see from Table 1 that in some cases, such as $n = 10$, 20 and $\lambda = 1$, the absolute error bounds of $\|f - \widetilde{B}_{n,q,\lambda}(f)\|_\infty$ are smaller than $\|f - \widetilde{B}_{n,q,0}(f)\|_\infty$, where $\widetilde{B}_{n,q,0}(f;x)$ are the classical q-Bernstein operators. So it is meaningful to consider the operators in the form (3).

Data availability statement

The data used to support the findings of this study are included within the article.

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