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Random attractors for stochastic retarded strongly damped wave equations with additive noise on bounded domains

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Abstract: In this paper we study the asymptotic behavior for a class of stochastic retarded strongly damped wave equation with additive noise on a bounded smooth domain in \mathbb{R}^d . We get the existence of the random attractor for the random dynamical systems associated with the equation.

Keywords: stochastic wave function, time-delay, random attractor, random dynamical system, additive noise

MSC: 35B40, 35B41, 35L05, 35L20

1 Introduction

The main aim of this paper is to investigate the asymptotic behavior of the solution to the following stochastic strongly damped wave equation with time-delay and with additive noise on a bounded set $D \subset \mathbb{R}^d$:

$$u_{tt} - \alpha \Delta u_t - \Delta u + u_t + \lambda u = f(u^t) + g(x) + \sum_{j=1}^m h_j(x) \frac{dW_j}{dt}, \quad (1)$$

with the initial value conditions

$$u(t, x) = u_0(t, x), \quad u_t(t, x) = \frac{\partial}{\partial t} u_0(t, x), \quad \text{for } t \in [-h, 0], \quad x \in D,$$

and the boundary condition

$$u(t, x) = 0, \quad \text{for } t \in [-h, \infty), \quad x \in \partial D.$$

Here λ and α are positive constants, $W_j (j = 1, 2, \dots, m)$ is a real-valued two-sided Wiener process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which will be specified later, g is a given function defined on D , u is a real function, f is a nonlinear functional satisfying some conditions which will be specified later, $u^t(\cdot) = u(t + \cdot)$.

Wave equation is a kind of hyperbolic equation, which can be used to describe the wave phenomena in nature and engineering. Hence, wave equation is a very important research field. Some evolution systems in physics, chemistry and life science, depend not only on the current status, but also on the status in the past period. These systems can be described by time-delay partial differential equations.

In this paper we study the asymptotic behavior of the solution to the stochastic time-delay wave equation (1), when time tends to infinite. As we know, the asymptotic behavior of random system can be studied by

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its random attractor, which was first introduced by Crauel and Flandoli [9]. In recent years, the properties of random attractors have been studied by many authors, see [10, 14, 16–19, 21, 24, 25, 30–32] and references therein.

The asymptotic behavior of the solution to deterministic wave equation has been investigated by many authors, see [2, 22, 23, 29, 33] and reference therein. In [33], S. Zhou obtained the uniform boundedness of the global attractor, and the estimate of the upper bound of the Hausdorff dimension of the global attractor for strongly damped nonlinear wave equations. In [2], J.M. Ball proved the existence of a global attractor for the semi-linear wave equation on a bounded domain $\Omega \subset \mathbb{R}^n$ with Dirichlet boundary conditions. The existence of the attractor for a class of strongly damped semi-linear wave equation on a bounded domain $\Omega \subset \mathbb{R}^3$ had been obtained by V. Pata [22]. In [29], the authors got the existence of a global attractor for the wave equation with nonlinear damping on a bounded domain. In [23], D. Prazak proved the global attractor for a class of semi-linear damped wave equation on bounded domain in \mathbb{R}^n ($n = 2, 3$) has finite fractal dimension.

The random attractor for the stochastic wave equation has been studied also by many authors [12, 26–28, 34]. In [12] X. Fan got the existence and fractal dimension of random attractors for stochastic wave equations with multiplicative noise on a bounded domain $\Omega \subset \mathbb{R}^n$ ($n = 1, 2, 3$). Z. Wang and S. Zhou investigated the random attractor for stochastic damped wave equation with multiplicative noise [26] and additive noise [27] on unbounded domain.

However, the results for the stochastic retarded wave equation are very few. Retarded wave equations are widely used in engineering, biology, physics and chemistry. Therefore, it is important for us to study the asymptotic behavior of random attractor for stochastic retarded strongly damped wave equation. To prove the existence of random attractor, we need to get some kind of compactness. Hence, we need the higher regularity for the solution. To this end, by using the method in [15, 20], we decompose the solution into two parts. One part has exponential decay with time and the other part has higher regularity. It follows from the higher regularity of the solution and Ascoli theorem that the random absorbing set are compact.

This paper is organized as follows. In Section 2, we recall a theorem for the existence of random attractor, and show that Equation (1) generates a random dynamical system. In Section 3, we prove the dynamical system has a random absorbing set, and give the uniform estimates of the solution. In Section 4, we prove the existence of random attractor.

2 Preliminaries and Random Dynamical Systems

In this section, we first recall a result for the existence of random attractor for a continuous random dynamical system (RDS), and then introduce some notations which will be used in this paper. At last, we show that the equation (1) generates a random dynamical system.

Let $(X, \|\cdot\|_X)$ be a Banach space with Borel σ -algebra $\mathcal{B}(X)$. Suppose that $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ is a metric dynamical system on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Suppose that ϕ is a continuous RDS of X over $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$. And suppose \mathcal{D} be a collection of subsets of X . The reader can refer to [3] [7] for more basic knowledge about random dynamical systems.

Now, we refer to [3] [13] for the following result for the existence of random attractor for continuous RDS.

Proposition 2.1. *Let $\{K(\omega)\}_{\omega \in \Omega} \in \mathcal{D}$ be a random absorbing set for the continuous RDS ϕ in \mathcal{D} and ϕ is \mathcal{D} -pullback asymptotically compact in X . Then ϕ has a unique \mathcal{D} -random attractor $\{A(\omega)\}_{\omega \in \Omega}$ which is given by*

$$A(\omega) = \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} \phi(t, \theta_{-t}\omega, K(\theta_{-t}\omega))}. \quad (2)$$

This result will be used to prove the existence of random attractor for the RDS generating by stochastic strongly damped wave equation with time-delay (1).

The following notation will be used in the rest of the paper. We use $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ to denote the inner product and the norm in $L^2(D)$, and use the notation $\|\cdot\|_X$ to denote the norm of a general Banach space

X. For $h > 0$, let u^t be the function defined on $[-h, 0]$ by the relation $u^t(s) = u(t + s)$, $s \in [-h, 0]$. For any continuous function $\xi : [-h, 0] \rightarrow L^2(D)$, define

$$\|\xi\|_S = \sup_{s \in [-h, 0]} \|\xi(s)\|. \quad (3)$$

Let S be the collection of all continuous function $\xi : [-h, 0] \rightarrow L^2(D)$ with $\|\xi\|_S < \infty$. Then, it is easy to check that $(S, \|\cdot\|_S)$ is a Banach space.

In the following, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, with

$$\Omega = \{\omega = (\omega_1, \omega_2, \dots, \omega_m) \in C(\mathbb{R}, \mathbb{R}^m) : \omega(0) = 0\}. \quad (4)$$

The Borel σ -algebra \mathcal{F} on Ω is generated by the compact open topology [1] and \mathbb{P} is the corresponding Wiener measure on \mathcal{F} . $(\theta_t)_{t \in \mathbb{R}}$ on Ω is defined by

$$\theta_t \omega(s) = \omega(s + t) - \omega(t), \quad s, t \in \mathbb{R}. \quad (5)$$

Then, $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t \in \mathbb{R}})$ is an ergodic metric dynamical system.

In the rest of this section, we show that there is a RDS generated by the following stochastic strongly damped wave equation:

$$u_{tt} - \alpha \Delta u_t - \Delta u + u_t + \lambda u = f(u^t) + g(x) + \sum_{j=1}^m h_j(x) \frac{dW_j}{dt}, \quad (6)$$

with the initial value conditions

$$u(t, x) = u_0(t, x), \quad u_t(t, x) = \frac{\partial}{\partial t} u_0(t, x), \quad \text{for } t \in [-h, 0], \quad x \in D, \quad (7)$$

and the boundary condition

$$u(t, x) = 0, \quad \text{for } t \in [-h, \infty), \quad x \in \partial D. \quad (8)$$

Here α, λ and h are positive constants, g is a given function in $L^2(D)$, and h'_j ($j = 1, 2, \dots, m$) are given functions in $H_0^2(D)$, $W(t) \equiv (W_1(t), W_2(t), \dots, W_m(t))$ is a two-sided Wiener process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We identify $\omega(t)$ with $W(t)$, i.e. $W_j(t) = \omega_j(t)$ ($j = 1, 2, \dots, m$). $f : S \rightarrow L^2(D)$ is a continuous functional satisfying the following conditions:

(A1) $f(0) = 0$;

(A2) there exists a constant $L_f > 0$ such that, for all $\xi, \eta \in S$,

$$\|f(\xi) - f(\eta)\| \leq L_f \|\xi - \eta\|_S; \quad (9)$$

(A3) there exist positive constants $\beta_0 > 0$, $C_f > 0$ such that $\forall \beta \in (0, \beta_0)$, $t > 0$, $u \in C([-h, t]; L^2(D))$,

$$\int_0^t e^{\beta s} |f(u^s)|^2 ds \leq C_f^2 \int_{-h}^t e^{\beta s} |u(s)|^2 ds. \quad (10)$$

Let $\sigma > 0$ be a fixed constant such that

$$\lambda + \sigma^2 - \sigma > 0, \quad 1 - \sigma > 0, \quad 1 - \sigma\alpha > 0. \quad (11)$$

For convenience, we introduce the following notations. Set

$$\beta_1 = \lambda + \sigma^2 - \sigma, \quad \beta_2 = 1 - \sigma, \quad \beta_3 = 1 - \sigma\alpha. \quad (12)$$

Set $H = H^1(D) \times L^2(D)$, and the norm on H is the following

$$\|(u, v)\|_H^2 = \beta_1 \|u\|^2 + \beta_3 \|\nabla u\|^2 + \|v\|^2, \quad \text{for } (u, v) \in H. \quad (13)$$

Set

$$\mathcal{H} = \{(u, v) : u \in S, |\nabla u| \in S, v \in S\}, \quad (14)$$

with the norm $\|(u, v)\|_{\mathcal{H}}^2 = \beta_1 \|u\|_S^2 + \beta_3 \|\nabla u\|_S^2 + \|v\|_S^2$.

Set $\eta = u_t + \sigma u$, then (6)-(8) can be rewritten as the following form:

$$\begin{cases} \frac{du}{dt} = \eta - \sigma u, \\ \frac{d\eta}{dt} = -\beta_2 \eta + \alpha \Delta \eta - \beta_1 u + \beta_3 \Delta u + f(u^t) + g(x) + \sum_{j=1}^m h_j(x) \frac{dW_j}{dt}, \end{cases} \quad (15)$$

with the initial value conditions

$$u(t, x) = u_0(t, x), \quad \eta(t, x) = \eta_0(t, x) \equiv \frac{\partial}{\partial t} u_0(t, x) + \sigma u_0(t, x), \quad \text{for } t \in [-h, 0], \quad x \in D, \quad (16)$$

and the boundary condition

$$u(t, x) = 0, \quad \text{for } t \in [-h, \infty), \quad x \in \partial D. \quad (17)$$

To show that the equation (15) generates a continuous RDS, we first transform (15) into a deterministic equation with random parameters. We use famous Ornstein-Uhlenbeck process to do that. For $j = 1, 2, \dots, m$, set

$$z_j(\theta_t \omega_j) \equiv - \int_{-\infty}^0 e^{s(\theta_t \omega_j)}(s) ds, \quad t \in \mathbb{R}. \quad (18)$$

It is an Ornstein-Uhlenbeck process, and it is the solution of the following Itô equation:

$$dz_j + z_j dt = dW_j. \quad (19)$$

Moreover, the random variable $z_j(\theta_t \omega_j)$ is tempered, and $z_j(\theta_t \omega_j)$ is \mathbb{P} -a.e. continuous [3]. Set

$$z(\theta_t \omega) = \sum_{j=1}^m h_j(x) z_j(\theta_t \omega_j),$$

then (19) implies that

$$dz + z dt = \sum_{j=1}^m h_j dW_j. \quad (20)$$

Notice that $h_j \in H_0^2(D)$ ($j = 1, 2, \dots, m$). Thus, there exists a constant $c > 0$, such that,

$$\|z(\omega)\|^2 \leq c \sum_{j=1}^m |z_j(\omega_j)|^2; \quad \|\nabla z(\omega)\|^2 \leq c \sum_{j=1}^m |z_j(\omega_j)|^2; \quad \|\Delta z(\omega)\|^2 \leq c \sum_{j=1}^m |z_j(\omega_j)|^2. \quad (21)$$

It follows from Proposition 4.3.3 [1] that, there exists a tempered function $r(\omega) > 0$ such that

$$\sum_{j=1}^m |z_j(\omega_j)|^2 \leq r(\omega), \quad (22)$$

where $r(\omega)$ satisfies, for \mathbb{P} -a.e. $\omega \in \Omega$,

$$r(\theta_t \omega) \leq e^{\frac{\beta}{2}|t|} r(\omega), \quad t \in \mathbb{R}. \quad (23)$$

Here β is a positive constant which will be fixed later. Then, it follows that, for \mathbb{P} -a.e. $\omega \in \Omega$, $t \in \mathbb{R}$

$$\|z(\theta_t \omega)\|^2 + \|\nabla z(\theta_t \omega)\|^2 + \|\Delta z(\theta_t \omega)\|^2 \leq c e^{\frac{\beta}{2}|t|} r(\omega). \quad (24)$$

Set

$$v(t, x) = \eta(t, x) - z(\theta_t \omega). \quad (25)$$

Then the equation (15) can be rewritten as the following form

$$\begin{cases} \frac{du}{dt} = v - \sigma u + z(\theta_t \omega), \\ \frac{dv}{dt} = -\beta_2 v + \alpha \Delta v - \beta_1 u + \beta_3 \Delta u + f(u^t) + g(x) + \sigma z(\theta_t \omega) + \alpha \Delta z(\theta_t \omega), \end{cases} \quad (26)$$

with the initial conditions

$$u(t, x) = u_0(t, x), \quad v(t, x) = v_0(t, x) \equiv \frac{\partial}{\partial t} u_0(t, x) + \sigma u_0(t, x) - z(\theta_t \omega), \quad \text{for } t \in [-h, 0], \quad x \in D, \quad (27)$$

and the boundary condition

$$u(t, x) = 0, \quad \text{for } t \in [-h, \infty), \quad x \in \partial D. \quad (28)$$

The equation (26) is a deterministic equation with random parameters. By [6], under the conditions (A1)-(A3), for each $(u_0, v_0) \in \mathcal{H}$, (26) has a unique solution $(u(t, \omega, u_0), v(t, \omega, v_0))$ for a.e. $\omega \in \Omega$. For any $T > 0$, by the regularity of solutions for an analytic semigroup [4], we can get that

$$(u, v) \in C([-h, T]; H^1(D)) \times C([-h, T]; L^2(D)). \quad (29)$$

The global solution can be obtained by the boundedness of solution of (26), by Lemma 3.1 below. Hence, equation (26) generates a continuous random dynamical system ϕ with

$$\phi(t, \omega, \phi_0) = (u^t, v^t), \quad \text{for } t \geq 0, \quad \omega \in \Omega, \quad \text{and } \phi_0 \equiv (u_0, v_0) \in \mathcal{H}. \quad (30)$$

Notice $\eta(t, \omega, \eta_0) = v(t, \omega, \eta_0) + z(\theta_t \omega)$. Define ψ by

$$\psi(t, \omega, \psi_0) = (u^t, \eta^t), \quad \text{for } t \geq 0, \quad \omega \in \Omega, \quad \text{and } \psi_0 \equiv (u_0, \eta_0) \in \mathcal{H}. \quad (31)$$

Then, ψ also generates a continuous RDS associate with (15). It is easy to see that two random dynamical systems are equivalent. Thus, we need only to consider the random dynamical system ϕ .

3 Uniform estimates of solutions

In this section we prove some uniform estimates for the solution of the equation (26). To show that the RDS ϕ has an absorbing set, we need the following assumption:

$$C_f^2 < 4\sigma\beta_1\beta_2. \quad (32)$$

By condition (32), there exist two positive constants β_4, β_5 , such that

$$C_f^2 = 4\beta_4\beta_5, \quad \beta_4 < \sigma\beta_1, \quad \beta_5 < \beta_2. \quad (33)$$

Throughout the rest of this paper we assume that \mathcal{D} is the collection of all tempered random subsets of \mathcal{H} .

Lemma 3.1. *Assume that conditions (A1)-(A3) and (32) hold. Then, there exists a random absorbing set $\{K(\omega)\} \in \mathcal{D}$ for ϕ , that is, for any $\{B(\omega)\} \in \mathcal{D}$, and for \mathbb{P} -a.e. $\omega \in \Omega$, there exists $T_1(B, \omega) > 0$, such that*

$$\phi(t, \theta_{-t}\omega, B(\theta_{-t}\omega)) \subset K(\omega), \quad \text{for all } t \geq T_1(B, \omega). \quad (34)$$

Proof. Taking the inner product of the first equation in (26) with u , $-\Delta u$, and the second equation in (26) with v , respectively, we have that

$$\frac{1}{2} \frac{d}{dt} \|u\|^2 = \langle v, u \rangle - \sigma \|u\|^2 + \langle z(\theta_t \omega), u \rangle, \quad (35)$$

$$\frac{1}{2} \frac{d}{dt} \|\nabla u\|^2 = \langle \nabla u, \nabla v \rangle - \sigma \|\nabla u\|^2 + \langle \nabla z(\theta_t \omega), \nabla u \rangle, \quad (36)$$

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|v\|^2 = & -\beta_2 \|v\|^2 - \alpha \|\nabla v\|^2 - \beta_1 \langle u, v \rangle - \beta_3 \langle \nabla u, \nabla v \rangle + \langle f(u^t), v \rangle + \\ & \langle g, v \rangle + \sigma \langle z(\theta_t \omega), v \rangle + \alpha \langle \Delta z(\theta_t \omega), v \rangle. \end{aligned} \quad (37)$$

Then, summing up (35) $\times \beta_1$, (36) $\times \beta_3$ and (37), one has that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|(u, v)\|_H^2 = & -\sigma \beta_1 \|u\|^2 - \sigma \beta_3 \|\nabla u\|^2 - \beta_2 \|v\|^2 - \alpha \|\nabla v\|^2 + \\ & \langle f(u^t), v \rangle + \langle g, v \rangle + \beta_1 \langle z(\theta_t \omega), u \rangle + \beta_3 \langle \nabla z(\theta_t \omega), \nabla u \rangle + \sigma \langle z(\theta_t \omega), v \rangle + \alpha \langle \Delta z(\theta_t \omega), v \rangle. \end{aligned} \quad (38)$$

Now, we estimate the terms on the right hand side of (38). By (33), there exists a constant $\beta_6 > 0$ small enough, such that $\beta_6 < \beta_2 - \beta_5$. Using Young inequality, we obtain that

$$\langle f(u^t), v \rangle + \langle g, v \rangle \leq \|f(u^t)\| \|v\| + \|g\| \|v\| \leq \frac{1}{4\beta_5} \|f(u^t)\|^2 + \beta_5 \|v\|^2 + \frac{1}{4\beta_6} \|g\|^2 + \beta_6 \|v\|^2. \quad (39)$$

Using Young inequality again, one has that

$$\begin{aligned} & \beta_1 \langle z(\theta_t \omega), u \rangle + \beta_3 \langle \nabla z(\theta_t \omega), \nabla u \rangle + \sigma \langle z(\theta_t \omega), v \rangle + \alpha \langle \Delta z(\theta_t \omega), v \rangle \\ & \leq \beta_1 \|z(\theta_t \omega)\| \|u\| + \beta_3 \|\nabla z(\theta_t \omega)\| \|\nabla u\| + \sigma \|z(\theta_t \omega)\| \|v\| + \alpha \|\Delta z(\theta_t \omega)\| \|v\| \\ & \leq \frac{\beta_1^2}{4\beta_7} \|z(\theta_t \omega)\|^2 + \beta_7 \|u\|^2 + \frac{\beta_3^2}{4\beta_9} \|\nabla z(\theta_t \omega)\|^2 + \beta_9 \|\nabla u\|^2 + \frac{\sigma^2}{2\beta_8} \|z(\theta_t \omega)\|^2 + \frac{\beta_8}{2} \|v\|^2 + \frac{\alpha^2}{2\beta_8} \|\Delta z(\theta_t \omega)\|^2 + \frac{\beta_8}{2} \|v\|^2 \\ & \leq c_1 r(\theta_t \omega) + (\beta_7 \|u\|^2 + \beta_8 \|v\|^2 + \beta_9 \|\nabla u\|^2). \end{aligned} \quad (40)$$

In the third step, we use (21) and (22). Here $\beta_7, \beta_8, \beta_9$ are positive constants such that $\beta_7 < \sigma\beta_1 - \beta_4, \beta_8 < \beta_2 - \beta_5 - \beta_6, \beta_9 < \sigma\beta_3$, and $c_1 = c \left(\frac{\beta_1^2}{4\beta_7} + \frac{\sigma^2 + \alpha^2}{2\beta_8} + \frac{\beta_3^2}{4\beta_9} \right)$. Then, it follows from (38)-(40) that,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|(u, v)\|_H^2 \leq & -(\sigma\beta_1 - \beta_4 - \beta_7) \|u\|^2 - (\sigma\beta_3 - \beta_9) \|\nabla u\|^2 - (\beta_2 - \beta_5 - \beta_6 - \beta_8) \|v\|^2 - \alpha \|\nabla v\|^2 + \\ & \frac{1}{4\beta_5} \|f(u^t)\|^2 - \beta_4 \|u\|^2 + \frac{1}{4\beta_6} \|g\|^2 + c_1 r(\theta_t \omega). \end{aligned} \quad (41)$$

Set

$$\beta_{10} = \min \left\{ \frac{\sigma\beta_1 - \beta_4 - \beta_7}{\beta_1}, \frac{\sigma\beta_3 - \beta_9}{\beta_3}, \beta_2 - \beta_5 - \beta_6 - \beta_8 \right\}. \quad (42)$$

Then, one has that

$$\frac{d}{dt} \|(u, v)\|_H^2 \leq -2\beta_{10} \|(u, v)\|_H^2 - 2\alpha \|\nabla v\|^2 + \frac{1}{2\beta_5} \|f(u^t)\|^2 - 2\beta_4 \|u\|^2 + \frac{1}{2\beta_6} \|g\|^2 + 2c_1 r(\theta_t \omega). \quad (43)$$

Choose $\beta \in (0, \beta_{10})$ small enough, such that $\beta < \beta_{10}$. It follows from (43) that

$$\begin{aligned} \frac{d}{dt} \left(e^{\beta t} \|(u, v)\|_H^2 \right) \leq & -(2\beta_{10} - \beta) e^{\beta t} \|(u, v)\|_H^2 - 2\alpha e^{\beta t} \|\nabla v\|^2 + \frac{e^{\beta t}}{2\beta_5} \|f(u^t)\|^2 - \\ & 2\beta_4 e^{\beta t} \|u\|^2 + \frac{e^{\beta t}}{2\beta_6} \|g\|^2 + 2c_1 e^{\beta t} r(\theta_t \omega). \end{aligned} \quad (44)$$

Integrate over t in both side of the last inequality,

$$e^{\beta t} \|(u(t), v(t))\|_H^2 \leq \|(u(0), v(0))\|_H^2 - (2\beta_{10} - \beta) \int_0^t e^{\beta s} \|(u(s), v(s))\|_H^2 ds - 2\alpha \int_0^t e^{\beta s} \|\nabla v(s)\|^2 ds +$$

$$\frac{1}{2\beta_5} \int_0^t e^{\beta s} \|f(u^s)\|^2 ds - 2\beta_4 \int_0^t e^{\beta s} \|u(s)\|^2 ds + \int_0^t \frac{e^{\beta s}}{2\beta_6} \|g\|^2 ds + 2c_1 \int_0^t e^{\beta s} r(\theta_s \omega) ds. \quad (45)$$

By condition (A3) and (32), we find that

$$\begin{aligned} & \frac{1}{2\beta_5} \int_0^t e^{\beta s} \|f(u^s)\|^2 ds - 2\beta_4 \int_0^t e^{\beta s} \|u(s)\|^2 ds \leq \frac{C_f^2}{2\beta_5} \int_{-h}^t e^{\beta s} \|u(s)\|^2 ds - 2\beta_4 \int_0^t e^{\beta s} \|u(s)\|^2 ds \\ & = 2\beta_4 \int_{-h}^0 e^{\beta s} \|u(s)\|^2 ds \leq 2\beta_4 h \|u_0\|_S^2. \end{aligned} \quad (46)$$

Noting that $\beta < 2\beta_{10}$, (45) and (46) imply that

$$\begin{aligned} & e^{\beta t} \| (u(t), v(t)) \|_H^2 + 2\alpha \int_0^t e^{\beta s} \|\nabla v(s)\|^2 ds \\ & \leq \| (u(0), v(0)) \|_H^2 + 2\beta_4 h \|u_0\|_S^2 + \frac{e^{\beta t}}{2\beta\beta_6} \|g\|^2 + 2c_1 \int_0^t e^{\beta s} r(\theta_s \omega) ds. \end{aligned} \quad (47)$$

Since $\|u_0\|_S \leq \frac{1}{\beta_1} \|(u_0, v_0)\|_{\mathcal{H}}$, and $\| (u(0), v(0)) \|_H^2 \leq \| (u_0, v_0) \|_{\mathcal{H}}^2$, it follows from (47) that

$$\begin{aligned} & \| (u(t), v(t)) \|_H^2 + 2\alpha \int_0^t e^{\beta(s-t)} \|\nabla v(s)\|^2 ds \\ & \leq \left(1 + \frac{2\beta_4 h}{\beta_1}\right) e^{-\beta t} \| (u_0, v_0) \|_{\mathcal{H}}^2 + \frac{1}{2\beta\beta_6} \|g\|^2 + 2c_1 \int_0^t e^{\beta(s-t)} r(\theta_s \omega) ds. \end{aligned} \quad (48)$$

For $\tau \in [-h, 0]$, by (48), one has that, for $t + \tau > 0$,

$$\begin{aligned} & \| (u(t + \tau), v(t + \tau)) \|_H^2 + 2\alpha \int_0^{t+\tau} e^{\beta(s-t-\tau)} \|\nabla v(s)\|^2 ds \\ & \leq \left(1 + \frac{2\beta_4 h}{\beta_1}\right) e^{-\beta(t+\tau)} \| (u_0, v_0) \|_{\mathcal{H}}^2 + \frac{1}{2\beta\beta_6} \|g\|^2 + 2c_1 \int_0^{t+\tau} e^{\beta(s-t-\tau)} r(\theta_s \omega) ds \\ & \leq \left(1 + \frac{2\beta_4 h}{\beta_1}\right) e^{\beta(h-t)} \| (u_0, v_0) \|_{\mathcal{H}}^2 + \frac{1}{2\beta\beta_6} \|g\|^2 + 2c_1 e^{\beta h} \int_0^t e^{\beta(s-t)} r(\theta_s \omega) ds, \end{aligned} \quad (49)$$

and for $t + \tau \leq 0$

$$\begin{aligned} & \| (u(t + \tau), v(t + \tau)) \|_H^2 + 2\alpha \int_0^{t+\tau} e^{\beta(s-t-\tau)} \|\nabla v(s)\|^2 ds \leq \| (u_0, v_0) \|_{\mathcal{H}}^2 \\ & \leq \left(1 + \frac{2\beta_4 h}{\beta_1}\right) e^{\beta(h-t)} \| (u_0, v_0) \|_{\mathcal{H}}^2 + \frac{1}{2\beta\beta_6} \|g\|^2 + 2c_1 e^{\beta h} \int_0^t e^{\beta(s-t)} r(\theta_s \omega) ds. \end{aligned} \quad (50)$$

By (49) and (50), we obtain that, for $t \geq 0$, $\tau \in [-h, 0]$

$$\left\| (u^t, v^t) \right\|_{\mathcal{H}}^2 + 2\alpha \int_0^{t+\tau} e^{\beta(s-t-\tau)} \|\nabla v(s)\|^2 ds$$

$$\leq \left(1 + \frac{2\beta_4 h}{\beta_1}\right) e^{\beta(h-t)} \| (u_0, v_0) \|_{\mathcal{H}}^2 + \frac{1}{2\beta\beta_6} \|g\|^2 + 2c_1 e^{\beta h} \int_0^t e^{\beta(s-t)} r(\theta_s \omega) ds. \quad (51)$$

Replacing ω by $\theta_{-t}\omega$ in (51), we obtain that,

$$\begin{aligned} & \left\| \left(u^t(\theta_{-t}\omega, u_0(\theta_{-t}\omega)), v^t(\theta_{-t}\omega, v_0(\theta_{-t}\omega)) \right) \right\|_{\mathcal{H}}^2 + 2\alpha \int_0^{t+\tau} e^{\beta(s-t-\tau)} \|\nabla v(s, \theta_{-t}\omega, v_0(\theta_{-t}\omega))\|^2 ds \\ & \leq \left(1 + \frac{2\beta_4 h}{\beta_1}\right) e^{\beta(h-t)} \| (u_0(\theta_{-t}\omega), v_0(\theta_{-t}\omega)) \|_{\mathcal{H}}^2 + \frac{1}{2\beta\beta_6} \|g\|^2 + 2c_1 e^{\beta h} \int_0^t e^{\beta(s-t)} r(\theta_{s-t}\omega) ds \\ & = \left(1 + \frac{2\beta_4 h}{\beta_1}\right) e^{\beta(h-t)} \| (u_0(\theta_{-t}\omega), v_0(\theta_{-t}\omega)) \|_{\mathcal{H}}^2 + \frac{1}{2\beta\beta_6} \|g\|^2 + \frac{4c_1}{\beta} e^{\beta h} r(\omega). \end{aligned} \quad (52)$$

In the last term, we use (23). Notice that $(u_0(\theta_{-t}\omega), v_0(\theta_{-t}\omega)) \in B(\theta_{-t}\omega)$, and $\{B(\omega)\}$ is tempered. Then,

$$\lim_{t \rightarrow +\infty} \left(1 + \frac{2\beta_4 h}{\beta_1}\right) e^{\beta(h-t)} \| (u_0(\theta_{-t}\omega), v_0(\theta_{-t}\omega)) \|_{\mathcal{H}}^2 = 0. \quad (53)$$

Hence, for any $B \in \mathcal{D}$, there exists a random variable $T_1(B, \omega) > 0$ such that, for all $t > T_1(B, \omega)$,

$$\begin{aligned} & \left\| \left(u^t(\theta_{-t}\omega, u_0(\theta_{-t}\omega)), v^t(\theta_{-t}\omega, v_0(\theta_{-t}\omega)) \right) \right\|_{\mathcal{H}}^2 + 2\alpha \int_0^{t+\tau} e^{\beta(s-t-\tau)} \|\nabla v(s, \theta_{-t}\omega, v_0(\theta_{-t}\omega))\|^2 ds \\ & \leq 1 + \frac{1}{2\beta\beta_6} \|g\|^2 + \frac{4c_1}{\beta} e^{\beta h} r(\omega) \equiv r_1(\omega). \end{aligned} \quad (54)$$

It is easy to see that $r_1(\omega)$ is tempered. This ends the proof. \square

To show the existence of random attractor, we need to get some estimates for Δu and Δv . To this end, we decompose the solution of (26) into two parts, and then obtain some priori estimates for the solutions. Set

$$u = \tilde{u} + \hat{u}; \quad v = \tilde{v} + \hat{v}. \quad (55)$$

Here (\tilde{u}, \tilde{v}) and (\hat{u}, \hat{v}) are the functions which satisfy the following equations respectively

$$\begin{cases} \frac{d\tilde{u}}{dt} = \tilde{v} - \sigma\tilde{u}, \\ \frac{d\tilde{v}}{dt} = -\beta_2\tilde{v} + \alpha\Delta\tilde{v} - \beta_1\tilde{u} + \beta_3\Delta\tilde{u}, \\ \tilde{u}(t, x) = u_0(t, x), \quad \tilde{v}(t, x) = v_0(t, x), \quad t \in [-h, 0], \quad x \in D, \\ \tilde{u}(t, x) = 0, \quad t \in [-h, +\infty), \quad x \in \partial D. \end{cases} \quad (56)$$

and

$$\begin{cases} \frac{d\hat{u}}{dt} = \hat{v} - \sigma\hat{u} + z(\theta_t\omega), \\ \frac{d\hat{v}}{dt} = -\beta_2\hat{v} + \alpha\Delta\hat{v} - \beta_1\hat{u} + \beta_3\Delta\hat{u} + f(u^t) + g(x) + \sigma z(\theta_t\omega) + \alpha\Delta z(\theta_t\omega), \\ \hat{u}(t, x) = 0, \quad \hat{v}(t, x) = 0, \quad t \in [-h, 0], \quad x \in D, \\ \hat{u}(t, x) = 0, \quad t \in [-h, +\infty), \quad x \in \partial D. \end{cases} \quad (57)$$

First, we estimate (\tilde{u}, \tilde{v}) . By the proof of Lemma 3.1, it follows from (52) that, for $t > 0$, $\tau \in [-h, 0]$

$$\begin{aligned} & \left\| \left(\tilde{u}^t(\theta_{-t}\omega, u_0(\theta_{-t}\omega)), \tilde{v}^t(\theta_{-t}\omega, v_0(\theta_{-t}\omega)) \right) \right\|_{\mathcal{H}}^2 + 2\alpha \int_0^{t+\tau} e^{\beta(s-t-\tau)} \|\nabla \tilde{v}(s, \theta_{-t}\omega, v_0(\theta_{-t}\omega))\|^2 ds \\ & \leq \left(1 + \frac{2\beta_4 h}{\beta_1}\right) e^{\beta(h-t)} \| (u_0(\theta_{-t}\omega), v_0(\theta_{-t}\omega)) \|_{\mathcal{H}}^2. \end{aligned} \quad (58)$$

It follows from (52) and (58) that

$$\begin{aligned}
 & \left\| \left(\hat{u}^t(\theta_{-t}\omega, 0), \hat{v}^t(\theta_{-t}\omega, 0) \right) \right\|_{\mathcal{H}}^2 + 2\alpha \int_0^{t+\tau} e^{\beta(s-t-\tau)} \|\nabla \hat{v}(s, \theta_{-t}\omega, 0)\|^2 ds \\
 & \leq 2 \left\| \left(u^t(\theta_{-t}\omega, u_0(\theta_{-t}\omega)), v^t(\theta_{-t}\omega, v_0(\theta_{-t}\omega)) \right) \right\|_{\mathcal{H}}^2 + 2 \left\| \left(\tilde{u}^t(\theta_{-t}\omega, u_0(\theta_{-t}\omega)), \tilde{v}^t(\theta_{-t}\omega, v_0(\theta_{-t}\omega)) \right) \right\|_{\mathcal{H}}^2 + \\
 & \quad + 4\alpha \int_0^{t+\tau} e^{\beta(s-t-\tau)} \left(\|\nabla v(s, \theta_{-t}\omega, v_0(\theta_{-t}\omega))\|^2 + \|\nabla \hat{v}(s, \theta_{-t}\omega, v_0(\theta_{-t}\omega))\|^2 \right) ds \\
 & \leq 4 \left(1 + \frac{2\beta_4 h}{\beta_1} \right) e^{\beta(h-t)} \| (u_0(\theta_{-t}\omega), v_0(\theta_{-t}\omega)) \|_{\mathcal{H}}^2 + \frac{1}{\beta\beta_6} \|g\|^2 + \frac{8c_1}{\beta} e^{\beta h} r(\omega). \tag{59}
 \end{aligned}$$

Thus, (54) and (59) imply that, for $t > T_1(B, \omega)$, $\tau \in [-h, 0]$,

$$\left\| \left(\hat{u}(t, \theta_{-t}\omega, 0), \hat{v}(t, \theta_{-t}\omega, 0) \right) \right\|_{\mathcal{H}}^2 + 2\alpha \int_0^{t+\tau} e^{\beta(s-t-\tau)} \|\nabla \hat{v}(s, \theta_{-t}\omega, 0)\|^2 ds \leq 2r_1(\omega) + 2. \tag{60}$$

Here $T_1(B, \omega)$ and $r_1(\omega)$ are random variables defined in Lemma 3.1. Next, we give a priori estimate for $(\Delta \hat{u}, \Delta \hat{v})$.

Lemma 3.2. Assume that conditions (A1)-(A3) and (32) hold. Let $B \in \mathcal{D}$ and $(u_0(\omega), v_0(\omega)) \in B(\omega)$. Then there exist two random variables $r_2(\omega)$ and $T_2(B, \omega) > 0$, such that, the solution $(\hat{u}(t, \omega, 0), \hat{v}(t, \omega, 0))$ of (57) satisfies, for $t > T_2(B, \omega)$,

$$(\beta_3 \|\Delta \hat{u}(t, \theta_{-t}\omega, 0)\|^2 + \|\nabla \hat{v}(t, \theta_{-t}\omega, 0)\|^2) + \alpha \int_0^t e^{\beta(s-t)} \|\Delta \hat{v}(s, \theta_{-t}\omega, 0)\|^2 ds \leq r_2(\omega). \tag{61}$$

Proof. Using the first equation in (57), one has that

$$\frac{d}{dt} \Delta \hat{u} = \Delta \hat{v} - \sigma \Delta \hat{u} + \Delta z(\theta_t \omega). \tag{62}$$

Taking the inner product of (62) and the second equation in (57) with $\Delta \hat{u}$ and $-\Delta \hat{v}$ respectively, we obtain

$$\frac{1}{2} \frac{d}{dt} \|\Delta \hat{u}\|^2 = \langle \Delta \hat{v}, \Delta \hat{u} \rangle - \sigma \|\Delta \hat{u}\|^2 + \langle \Delta z(\theta_t \omega), \Delta \hat{u} \rangle, \tag{63}$$

$$\frac{1}{2} \frac{d}{dt} \|\nabla \hat{v}\|^2 = -\beta_2 \|\nabla \hat{v}\|^2 - \alpha \|\Delta \hat{v}\|^2 - \beta_3 \langle \Delta \hat{u}, \Delta \hat{v} \rangle + \beta_1 \langle \hat{u}, \Delta \hat{v} \rangle - \langle f(u^t) + g + \sigma z(\theta_t \omega) + \alpha \Delta z(\theta_t \omega), \Delta \hat{v} \rangle. \tag{64}$$

Adding up (63) $\times \beta_3$ and (64), we get

$$\begin{aligned}
 \frac{1}{2} \frac{d}{dt} \left(\beta_3 \|\Delta \hat{u}\|^2 + \|\nabla \hat{v}\|^2 \right) &= -\sigma \beta_3 \|\Delta \hat{u}\|^2 - \beta_2 \|\nabla \hat{v}\|^2 - \alpha \|\Delta \hat{v}\|^2 + \beta_1 \langle \hat{u}, \Delta \hat{v} \rangle + \\
 & \quad \beta_3 \langle \Delta z(\theta_t \omega), \Delta \hat{u} \rangle - \langle f(u^t) + g + \sigma z(\theta_t \omega) + \alpha \Delta z(\theta_t \omega), \Delta \hat{v} \rangle. \tag{65}
 \end{aligned}$$

It follows from (21), (22) and Young inequality, that

$$|\beta_3 \langle \Delta z(\theta_t \omega), \Delta \hat{u} \rangle| \leq \frac{\sigma \beta_3}{2} \|\Delta \hat{u}\|^2 + \frac{\beta_3}{2\sigma} \|\Delta z\|^2 \leq \frac{\sigma \beta_3}{2} \|\Delta \hat{u}\|^2 + \frac{c \beta_3}{2\sigma} r(\theta_t \omega), \tag{66}$$

and

$$\begin{aligned}
 & \left| \beta_1 \langle \hat{u}, \Delta \hat{v} \rangle - \langle f(u^t) + g + \sigma z(\theta_t \omega) + \alpha \Delta z(\theta_t \omega), \Delta \hat{v} \rangle \right| \\
 & \leq \frac{\alpha}{2} \|\Delta \hat{v}\|^2 + \frac{5\beta_1^2}{2\alpha} \|\hat{u}\|^2 + \frac{5}{2\alpha} \|f(u^t)\|^2 + \frac{5}{2\alpha} \|g\|^2 + \frac{5c(\sigma^2 + \alpha^2)}{2\alpha} r(\theta_t \omega). \tag{67}
 \end{aligned}$$

It follows from (65)-(67) that

$$\frac{d}{dt} \left(\beta_3 \|\Delta \hat{u}\|^2 + \|\nabla \hat{v}\|^2 \right) \leq -\beta_{11} \left(\beta_3 \|\Delta \hat{u}\|^2 + \|\nabla \hat{v}\|^2 \right) - \alpha \|\Delta \hat{v}\|^2 +$$

$$\frac{5\beta_1^2}{\alpha} \|\hat{u}\|^2 + \frac{5}{\alpha} \|f(u^t)\|^2 + \frac{5}{\alpha} \|g\|^2 + \beta_{12} r(\theta_t \omega), \quad (68)$$

with $\beta_{11} = \min\{\sigma, 2\beta_2\}$ and $\beta_{12} = \frac{5c(\sigma^2 + \alpha^2)}{\alpha} + \frac{c\beta_3}{\sigma}$. Let $\beta > 0$ be the constant defined in Lemma 3.1. Noting that $\beta_{10} < \beta_{11}$, we have $\beta < \beta_{11}$. Thus, multiplying $e^{\beta t}$ in both side of (68), we have

$$\begin{aligned} \frac{d}{dt} \left(\beta_3 e^{\beta t} \|\Delta \hat{u}\|^2 + e^{\beta t} \|\nabla \hat{v}\|^2 \right) &\leq -(\beta_{11} - \beta) \left(\beta_3 e^{\beta t} \|\Delta \hat{u}\|^2 + e^{\beta t} \|\nabla \hat{v}\|^2 \right) - \alpha e^{\beta t} \|\Delta \hat{v}\|^2 + \\ &\frac{5\beta_1^2}{\alpha} e^{\beta t} \|\hat{u}\|^2 + \frac{5}{\alpha} e^{\beta t} \|f(u^t)\|^2 + \frac{5}{\alpha} e^{\beta t} \|g\|^2 + \beta_{12} e^{\beta t} r(\theta_t \omega), \end{aligned} \quad (69)$$

Noticing $(\hat{u}(0), \hat{v}(0)) = (0, 0)$ and $\beta < \beta_{11}$, integrate with respect to t in both side of (69),

$$\begin{aligned} &\left(\beta_3 \|\Delta \hat{u}(t)\|^2 + \|\nabla \hat{v}(t)\|^2 \right) + \alpha \int_0^t e^{\beta(s-t)} \|\Delta \hat{v}(s)\|^2 ds \\ &\leq \frac{5\beta_1^2}{\alpha} \int_0^t e^{\beta(s-t)} \|\hat{u}(s)\|^2 ds + \frac{5}{\alpha} \int_0^t e^{\beta(s-t)} \|f(u^s)\|^2 ds + \frac{5}{\alpha\beta} \|g\|^2 + \beta_{12} \int_0^t e^{\beta(s-t)} r(\theta_s \omega) ds. \end{aligned} \quad (70)$$

Replace ω by $\theta_{-t}\omega$ in last inequality,

$$\begin{aligned} &\left(\beta_3 \|\Delta \hat{u}(t, \theta_{-t}\omega, 0)\|^2 + \|\nabla \hat{v}(t, \theta_{-t}\omega, 0)\|^2 \right) + \alpha \int_0^t e^{\beta(s-t)} \|\Delta \hat{v}(s, \theta_{-t}\omega, 0)\|^2 ds \\ &\leq \frac{5\beta_1^2}{\alpha} \int_0^t e^{\beta(s-t)} \|\hat{u}(s, \theta_{-t}\omega, 0)\|^2 ds + \frac{5}{\alpha} \int_0^t e^{\beta(s-t)} \|f(u^s(\theta_{-t}\omega, u_0(\theta_{-t}\omega)))\|^2 ds + \frac{5}{\alpha\beta} \|g\|^2 + \beta_{12} \int_0^t e^{\beta(s-t)} r(\theta_{s-t}\omega) ds \\ &\leq \frac{5\beta_1^2}{\alpha} \int_0^t e^{\beta(s-t)} \|\hat{u}(s, \theta_{-t}\omega, 0)\|^2 ds + \frac{5}{\alpha} \int_0^t e^{\beta(s-t)} \|f(u^s(\theta_{-t}\omega, u_0(\theta_{-t}\omega)))\|^2 ds + \frac{5}{\alpha\beta} \|g\|^2 + \frac{2\beta_{12}}{\beta} r(\omega). \end{aligned} \quad (71)$$

In the second step, we use (23). Next, we estimate the first two terms on the right hand side of (71). For the first term, using (23) and (59), we have

$$\begin{aligned} &\frac{5\beta_1^2}{\alpha} \int_0^t e^{\beta(s-t)} \|\hat{u}(s, \theta_{-t}\omega, 0)\|^2 ds \\ &\leq \frac{20\beta_1(\beta_1 + 2\beta_4 h)}{\alpha} \int_0^t e^{\beta(h-t)} \| (u_0(\theta_{-t}\omega), v_0(\theta_{-t}\omega)) \|_{\mathcal{H}}^2 ds + \frac{5\beta_1^2}{\alpha\beta^2\beta_6} \|g\|^2 + \frac{40c_1\beta_1^2}{\alpha\beta} e^{\beta h} \int_0^t e^{\beta(s-t)} r(\theta_{s-t}\omega) ds \\ &\leq \frac{20\beta_1(\beta_1 + 2\beta_4 h)}{\alpha} t e^{\beta(h-t)} \| (u_0(\theta_{-t}\omega), v_0(\theta_{-t}\omega)) \|_{\mathcal{H}}^2 + \frac{5\beta_1^2}{\alpha\beta^2\beta_6} \|g\|^2 + \frac{80c_1\beta_1^2}{\alpha\beta^2} e^{\beta h} r(\omega). \end{aligned} \quad (72)$$

By Condition (A3), one has that

$$\begin{aligned} &\frac{5}{\alpha} \int_0^t e^{\beta(s-t)} \|f(u^s(\theta_{-t}\omega, u_0(\theta_{-t}\omega)))\|^2 ds \leq \frac{5C_f^2}{\alpha} \int_{-h}^t e^{\beta(s-t)} \|u(s, \theta_{-t}\omega, u_0(\theta_{-t}\omega))\|^2 ds \\ &= \frac{5C_f^2}{\alpha} \int_{-h}^0 e^{\beta(s-t)} \|u(s, \theta_{-t}\omega, u_0(\theta_{-t}\omega))\|^2 ds + \frac{5C_f^2}{\alpha} \int_0^t e^{\beta(s-t)} \|u(s, \theta_{-t}\omega, u_0(\theta_{-t}\omega))\|^2 ds \\ &\leq \frac{5C_f^2 h}{\alpha} e^{-\beta t} \|u_0(\theta_{-t}\omega)\|_{\mathcal{H}}^2 + \frac{5C_f^2}{\alpha} \int_0^t e^{\beta(s-t)} \|u(s, \theta_{-t}\omega, u_0(\theta_{-t}\omega))\|^2 ds. \end{aligned} \quad (73)$$

Similarly as in (72), by (52), one has that

$$\begin{aligned} & \frac{5C_f^2}{\alpha} \int_0^t e^{\beta(s-t)} \|u(s, \theta_{-t}\omega, u_0(\theta_{-t}\omega))\|^2 ds \\ & \leq \frac{5C_f^2(\beta_1 + 2\beta_4 h)}{\alpha\beta_1} t e^{\beta(h-t)} \| (u_0(\theta_{-t}\omega), v_0(\theta_{-t}\omega)) \|_{\mathcal{H}}^2 + \frac{5C_f^2}{2\alpha\beta^2\beta_6} \|g\|^2 + \frac{20c_1 C_f^2}{\alpha\beta^2} e^{\beta h} r(\omega). \end{aligned} \quad (74)$$

Notice $(u_0(\omega), v_0(\omega)) \in B(\omega)$. It follows that

$$\lim_{t \rightarrow +\infty} t e^{\beta(h-t)} \| (u_0(\theta_{-t}\omega), v_0(\theta_{-t}\omega)) \|_{\mathcal{H}}^2 + e^{-\beta t} \|u_0(\theta_{-t}\omega)\|_S^2 = 0. \quad (75)$$

Therefore, it follows from (71)-(75) that, for any $(u_0(\omega), v_0(\omega)) \in B(\omega)$, there exists $T_2(B, \omega) > 0$, such that, for all $t > T_2(B, \omega)$,

$$\begin{aligned} & (\beta_3 \|\Delta \hat{u}(t, \theta_{-t}\omega, 0)\|^2 + \|\nabla \hat{v}(t, \theta_{-t}\omega, 0)\|^2) + \alpha \int_0^t e^{\beta(s-t)} \|\Delta \hat{v}(s, \theta_{-t}\omega, 0)\|^2 ds \\ & \leq 1 + \left(\frac{5}{\alpha\beta} + \frac{5\beta_1^2}{\alpha\beta^2\beta_6} + \frac{5C_f^2}{2\alpha\beta^2\beta_6} \right) \|g\|^2 + \left(\frac{2\beta_{12}}{\beta} + \frac{80c_1\beta_1^2}{\alpha\beta^2} e^{\beta h} + \frac{20c_1 C_f^2}{\alpha\beta^2} e^{\beta h} \right) r(\omega) \\ & \equiv r_2(\omega). \end{aligned} \quad (76)$$

This completes the proof. \square

Lemma 3.3. Assume that conditions (A1)-(A3) and (32) hold. Let $B(\omega) \in \mathcal{D}$ and $(u_0(\omega), v_0(\omega)) \in B(\omega)$. Then there exist two random variables $r_3(\omega)$ and $T_3(B, \omega)$, such that, the solution $(\hat{u}(t, \omega, 0), \hat{v}(t, \omega, 0))$ of (57) satisfies, for $t > T_3(B, \omega)$ and $-h \leq \sigma_1 < \sigma_2 \leq 0$,

$$\left\| \left(\hat{u}^t(\sigma_1, \theta_{-t}\omega, 0), \hat{v}^t(\sigma_1, \theta_{-t}\omega, 0) \right) - \left(\hat{u}^t(\sigma_2, \theta_{-t}\omega, 0), \hat{v}^t(\sigma_2, \theta_{-t}\omega, 0) \right) \right\|_H \leq r_3(\omega) |\sigma_1 - \sigma_2|^{1/2}. \quad (77)$$

Proof. By the definition of $\|\cdot\|_H$, one has that

$$\begin{aligned} & \left\| \left(\hat{u}^t(\sigma_1, \theta_{-t}\omega, 0), \hat{v}^t(\sigma_1, \theta_{-t}\omega, 0) \right) - \left(\hat{u}^t(\sigma_2, \theta_{-t}\omega, 0), \hat{v}^t(\sigma_2, \theta_{-t}\omega, 0) \right) \right\|_H \\ & \leq \beta_1 \left\| \hat{u}(t + \sigma_1, \theta_{-t}\omega, 0) - \hat{u}(t + \sigma_2, \theta_{-t}\omega, 0) \right\| + \beta_3 \left\| \nabla \hat{u}(t + \sigma_1, \theta_{-t}\omega, 0) - \nabla \hat{u}(t + \sigma_2, \theta_{-t}\omega, 0) \right\| \\ & \quad + \left\| \hat{v}(t + \sigma_1, \theta_{-t}\omega, 0) - \hat{v}(t + \sigma_2, \theta_{-t}\omega, 0) \right\| \\ & \leq \beta_1 \int_{t+\sigma_1}^{t+\sigma_2} \left\| \hat{u}'(s, \theta_{-t}\omega, 0) \right\| ds + \beta_3 \int_{t+\sigma_1}^{t+\sigma_2} \left\| \nabla \hat{u}'(s, \theta_{-t}\omega, 0) \right\| ds + \int_{t+\sigma_1}^{t+\sigma_2} \left\| \hat{v}'(s, \theta_{-t}\omega, 0) \right\| ds. \end{aligned} \quad (78)$$

Now, we estimate the terms on the right hand side of (78). For the first term, using (57), (60) and (24), we obtain that, for $t > T_1(B, \omega)$,

$$\begin{aligned} & \int_{t+\sigma_1}^{t+\sigma_2} \left\| \hat{u}'(s, \theta_{-t}\omega, 0) \right\| ds \\ & \leq \int_{t+\sigma_1}^{t+\sigma_2} \left\| \hat{v}(s, \theta_{-t}\omega, 0) \right\| ds + \sigma \int_{t+\sigma_1}^{t+\sigma_2} \left\| \hat{u}(s, \theta_{-t}\omega, 0) \right\| ds + \int_{t+\sigma_1}^{t+\sigma_2} \left\| z(\theta_{s-t}\omega) \right\| ds \\ & \leq \left\{ \left(\int_{t+\sigma_1}^{t+\sigma_2} \left\| \hat{v}(s, \theta_{-t}\omega, 0) \right\|^2 ds \right)^{1/2} + \sigma \left(\int_{t+\sigma_1}^{t+\sigma_2} \left\| \hat{u}(s, \theta_{-t}\omega, 0) \right\|^2 ds \right)^{1/2} + \left(\int_{t+\sigma_1}^{t+\sigma_2} \left\| z(\theta_{s-t}\omega) \right\|^2 ds \right)^{1/2} \right\} |\sigma_1 - \sigma_2|^{1/2} \\ & \leq \left\{ \left(1 + \frac{\sigma}{\sqrt{\beta_1}} \right) \left(\int_{t+\sigma_1}^{t+\sigma_2} 2r_1(\theta_{s-t}\omega) + 2 ds \right)^{1/2} + \left(\int_{t+\sigma_1}^{t+\sigma_2} \left\| z(\theta_{s-t}\omega) \right\|^2 ds \right)^{1/2} \right\} |\sigma_1 - \sigma_2|^{1/2} \end{aligned}$$

$$\leq \left\{ \left(1 + \frac{\sigma}{\sqrt{\beta_1}} \right) h^{1/2} \left(2 \sup_{\tau \in [-h, 0]} r_1(\theta_\tau \omega) + 2 \right)^{1/2} + c^{1/2} h^{1/2} e^{\beta h/4} r^{1/2}(\omega) \right\} |\sigma_1 - \sigma_2|^{1/2}. \quad (79)$$

In the third step, we use (60), and in the last step, we use (24). Notice that

$$\int_0^{t+\sigma_2} e^{\beta(s-t-\sigma_2)} \|\nabla \hat{v}(s, \theta_{-t}\omega, 0)\|^2 ds \geq \int_{t+\sigma_1}^{t+\sigma_2} e^{\beta(s-t-\sigma_2)} \|\nabla \hat{v}(s, \theta_{-t}\omega, 0)\|^2 ds \geq e^{-\beta h} \int_{t+\sigma_1}^{t+\sigma_2} \|\nabla \hat{v}(s, \theta_{-t}\omega, 0)\|^2 ds.$$

Hence, it follows from (60) that, for $t > T_1(B, \omega)$,

$$\int_{t+\sigma_1}^{t+\sigma_2} \|\nabla \hat{v}(s, \theta_{-t}\omega, 0)\|^2 ds \leq \frac{e^{\beta h}}{\alpha} (r_1(\omega) + 1). \quad (80)$$

Using similar computation as (79), we can get the following estimate: for $t > T_1(B, \omega)$,

$$\begin{aligned} & \int_{t+\sigma_1}^{t+\sigma_2} \|\nabla \hat{u}'(s, \theta_{-t}\omega, 0)\| ds \\ & \leq \left\{ \left(\int_{t+\sigma_1}^{t+\sigma_2} \|\nabla \hat{v}(s, \theta_{-t}\omega, 0)\|^2 ds \right)^{1/2} + \sigma \left(\int_{t+\sigma_1}^{t+\sigma_2} \|\nabla \hat{u}(s, \theta_{-t}\omega, 0)\|^2 ds \right)^{1/2} \right. \\ & \quad \left. + \left(\int_{t+\sigma_1}^{t+\sigma_2} \|\nabla z(\theta_{s-t}\omega)\|^2 ds \right)^{1/2} \right\} |\sigma_1 - \sigma_2|^{1/2} \\ & \leq \left\{ \frac{e^{\beta h/2}}{\sqrt{\alpha}} (r_1(\omega) + 1)^{1/2} + \frac{\sigma}{\sqrt{\beta_3}} h^{1/2} \left(2 \sup_{\tau \in [-h, 0]} r_1(\theta_\tau \omega) + 2 \right)^{1/2} + c^{1/2} h^{1/2} e^{\beta h/4} r^{1/2}(\omega) \right\} |\sigma_1 - \sigma_2|^{1/2}. \end{aligned} \quad (81)$$

Next, we estimate the third term on the right hand side of (78). By the second equation in (58), we get that

$$\begin{aligned} & \int_{t+\sigma_1}^{t+\sigma_2} \|\hat{v}'(s, \theta_{-t}\omega, 0)\| ds \leq \int_{t+\sigma_1}^{t+\sigma_2} \beta_2 \|\hat{v}(s, \theta_{-t}\omega, 0)\| + \alpha \|\Delta \hat{v}(s, \theta_{-t}\omega, 0)\| + \beta_1 \|\hat{u}(s, \theta_{-t}\omega, 0)\| + \\ & \quad \beta_3 \|\Delta \hat{u}(s, \theta_{-t}\omega, 0)\| + \|f(u^s(\theta_{-t}\omega, u_0(\theta_{-t}\omega)))\| + \|g\| + \sigma \|z(\theta_{s-t}\omega)\| + \alpha \|\Delta z(\theta_{s-t}\omega)\| ds. \end{aligned} \quad (82)$$

By (60), we obtain that, for $t > T_1(B, \omega)$,

$$\begin{aligned} & \int_{t+\sigma_1}^{t+\sigma_2} \beta_2 \|\hat{v}(s, \theta_{-t}\omega, 0)\| + \beta_1 \|\hat{u}(s, \theta_{-t}\omega, 0)\| ds \\ & \leq (1 + \beta_2) \left(\int_{t+\sigma_1}^{t+\sigma_2} \|(\hat{u}(s, \theta_{-t}\omega, 0), \hat{v}(s, \theta_{-t}\omega, 0))\|_{\mathcal{H}}^2 ds \right)^{1/2} |\sigma_1 - \sigma_2|^{1/2} \\ & \leq (1 + \beta_2) h^{1/2} \sup_{\tau \in [-h, 0]} (2r_1(\theta_\tau \omega) + 2)^{1/2} |\sigma_1 - \sigma_2|^{1/2}. \end{aligned} \quad (83)$$

By Lemma 3.2, for $t > T_2(B, \omega)$,

$$\int_{t+\sigma_1}^{t+\sigma_2} \beta_3 \|\Delta \hat{u}(s, \theta_{-t}\omega, 0)\| + \alpha \|\Delta \hat{v}(s, \theta_{-t}\omega, 0)\| ds$$

$$\begin{aligned}
&\leq \left[\beta_3 \left(\int_{t+\sigma_1}^{t+\sigma_2} \|\Delta \hat{u}(s, \theta_{-t}\omega, 0)\|^2 ds \right)^{1/2} + \alpha \left(\int_{t+\sigma_1}^{t+\sigma_2} \|\Delta \hat{v}(s, \theta_{-t}\omega, 0)\|^2 ds \right)^{1/2} \right] |\sigma_1 - \sigma_2|^{1/2} \\
&\leq \left[\beta_3 \left(h \sup_{\tau \in [-h, 0]} r_2(\theta_\tau \omega) \right)^{1/2} + \alpha r_2^{1/2}(\omega) \right] |\sigma_1 - \sigma_2|^{1/2}.
\end{aligned} \quad (84)$$

By Condition (A1), (A2) and Lemma 3.1, for $t > T_1(B, \omega)$

$$\int_{t+\sigma_1}^{t+\sigma_2} \|f(u^s(\theta_{-t}\omega, u_0(\theta_{-t}\omega)))\| ds \leq L_f \int_{t+\sigma_1}^{t+\sigma_2} \|u^s(\theta_{-t}\omega, u_0(\theta_{-t}\omega))\|_{\mathcal{H}} ds \leq L_f h^{1/2} \sup_{\tau \in [-h, 0]} r_1^{1/2}(\theta_\tau \omega) |\sigma_1 - \sigma_2|^{1/2}. \quad (85)$$

By (24), one has that

$$\int_{t+\sigma_1}^{t+\sigma_2} \|g\| + \sigma \|z(\theta_{s-t}\omega)\| + \alpha \|\Delta z(\theta_{s-t}\omega)\| ds \leq \left(\|g\| + (\sigma + \alpha) c^{1/2} e^{\beta h/4} r^{1/2}(\omega) \right) h^{1/2} |\sigma_1 - \sigma_2|^{1/2}. \quad (86)$$

Set

$$\begin{aligned}
T_3(B, \omega) &= \max \{ T_1(B, \omega), T_2(B, \omega) \}; \\
r_3(\omega) &\equiv \beta_1 \left\{ \left(1 + \frac{\sigma}{\sqrt{\beta_1}} \right) h^{1/2} \left(2 \sup_{\tau \in [-h, 0]} r_1(\theta_\tau \omega) + 2 \right)^{1/2} + c^{1/2} h^{1/2} e^{\beta h/4} r^{1/2}(\omega) \right\} + \\
&\quad \beta_3 \left\{ \frac{e^{\beta h/2}}{\sqrt{\alpha}} (r_1(\omega) + 1)^{1/2} + \frac{\sigma}{\sqrt{\beta_3}} h^{1/2} \left(2 \sup_{\tau \in [-h, 0]} r_1(\theta_\tau \omega) + 2 \right)^{1/2} + c^{1/2} h^{1/2} e^{\beta h/4} r^{1/2}(\omega) \right\} + \\
&\quad (1 + \beta_2) h^{1/2} \sup_{\tau \in [-h, 0]} (2r_1(\theta_\tau \omega) + 2)^{1/2} + \left[\beta_3 \left(h \sup_{\tau \in [-h, 0]} r_2(\theta_\tau \omega) \right)^{1/2} + \alpha r_2^{1/2}(\omega) \right] + \\
&\quad L_f h^{1/2} \sup_{\tau \in [-h, 0]} r_1^{1/2}(\theta_\tau \omega) + \left(\|g\| + (\sigma + \alpha) c^{1/2} e^{\beta h/4} r^{1/2}(\omega) \right) h^{1/2}.
\end{aligned} \quad (87)$$

Then, it follows from (78)-(86) that, for $t > T_3(B, \omega)$,

$$\left\| (\hat{u}^t(\sigma_1, \theta_{-t}\omega, 0), \hat{v}^t(\sigma_1, \theta_{-t}\omega, 0)) - (\hat{u}^t(\sigma_2, \theta_{-t}\omega, 0), \hat{v}^t(\sigma_2, \theta_{-t}\omega, 0)) \right\|_H^2 \leq r_3(\omega) |\sigma_1 - \sigma_2|^{1/2}. \quad (88)$$

This ends the proof. \square

4 Random attractor

In this section we prove the existence of Random attractor for the RDS generated by (26). First, we will use the prior estimates in Section 3 to show that RDS ϕ is asymptotically compact.

Lemma 4.1. Assume that conditions (A1)-(A3) and (32) hold. The RDS ϕ is \mathcal{D} -pullback asymptotically compact in \mathcal{H} , that is, for \mathbb{P} -a.e. $\omega \in \Omega$, the sequence $\{\phi(t_n, \theta_{-t_n}\omega, \phi_{0,n}(\theta_{-t_n}\omega))\}_{n=1}^\infty$ has convergent subsequence in \mathcal{H} , as $t_n \rightarrow \infty$, for any $B \in \mathcal{D}$ and $\phi_{0,n}(\theta_{-t_n}\omega) \in B(\theta_{-t_n}\omega)$.

Proof. Since $t_n \rightarrow +\infty$, there exists $N_1 = N_1(B, \omega)$ large enough, such that, for $n \geq N_1$, $t_n \geq T_3(B, \omega) + h$. Thus, it follows from (60) and Lemma 3.2 that, for $n \geq N_1$ and $\sigma \in [-h, 0]$,

$$\|\hat{u}(\sigma, \theta_{-t_n}\omega, 0)\|_{H^2(D)}^2 \leq \left(\frac{1}{\beta_1} + \frac{1}{\beta_3} \right) (2r_1(\omega) + 2) + \frac{1}{\beta_3} r_2(\omega), \quad \|\hat{v}(\sigma, \theta_{-t_n}\omega, 0)\|_{H^1(D)}^2 \leq 2r_1(\omega) + 2 + r_2(\omega). \quad (89)$$

By the compact embedding theorem, we obtain that $H^1(D) \hookrightarrow L^2(D)$ and $H^2(D) \hookrightarrow H^1(D)$ are compact. Hence, for $\sigma \in [-h, 0]$ and $n \geq N_1$, $\{(\hat{u}(\sigma, \theta_{-t_n}\omega, 0), \hat{v}(\sigma, \theta_{-t_n}\omega, 0))\}$ is relatively compact in H . By Lemma 3.3, for $n \geq N_1$, $\{(\hat{u}(\cdot, \theta_{-t_n}\omega, 0), \hat{v}(\cdot, \theta_{-t_n}\omega, 0))\}$ is equi-continuous in $C([-h, 0], H)$. Hence, it follows from Ascoli theorem that, $\{(\hat{u}(\sigma, \theta_{-t_n}\omega, 0), \hat{v}(\sigma, \theta_{-t_n}\omega, 0))\}$ is relatively compact in $C([-h, 0], H)$. Therefore, we can find a subsequence $\{t_{n_i}\}$ such that,

$$(\hat{u}^{t_n}(\cdot, \theta_{-t_n}\omega, 0), \hat{v}^{t_n}(\cdot, \theta_{-t_n}\omega, 0)) \rightarrow (\zeta(\cdot, \omega), \xi(\cdot, \omega)) \text{ in } C([-h, 0], H). \quad (90)$$

For any $(u_0, v_0) \in B$, $B \in \mathcal{D}$ and $\epsilon > 0$, there exists $N_2 = N_2(B, \omega, \epsilon)$, such that for $n > N_2(B, \omega, \epsilon)$, and $\sigma \in [-h, 0]$,

$$\|(\hat{u}^{t_n}(\cdot, \theta_{-t_n}\omega, 0) - \zeta(\cdot, \omega), \hat{v}^{t_n}(\cdot, \theta_{-t_n}\omega, 0) - \xi(\cdot, \omega))\|_{\mathcal{H}} < \epsilon. \quad (91)$$

By (58), there exists there exists $N_3 = N_3(B, \omega, \epsilon)$, such that for $n > N_3(B, \omega, \epsilon)$,

$$\|(\tilde{u}^{t_n}(\cdot, \theta_{-t_n}\omega, u_0(\theta_{-t_n}\omega)), \tilde{v}^{t_n}(\cdot, \theta_{-t_n}\omega, v_0(\theta_{-t_n}\omega)))\|_{\mathcal{H}} < \epsilon. \quad (92)$$

By (91) and (92), we have that, for $n > \max\{N_1, N_2, N_3\}$

$$\begin{aligned} & \|(\hat{u}^{t_n}(\cdot, \theta_{-t_n}\omega, u_0(\theta_{-t_n}\omega)) - \zeta(\cdot, \omega), \hat{v}^{t_n}(\cdot, \theta_{-t_n}\omega, v_0(\theta_{-t_n}\omega)) - \xi(\cdot, \omega))\|_{\mathcal{H}} \\ & \leq 2 \|(\hat{u}^{t_n}(\cdot, \theta_{-t_n}\omega, 0) - \zeta(\cdot, \omega), \hat{v}^{t_n}(\cdot, \theta_{-t_n}\omega, 0) - \xi(\cdot, \omega))\|_{\mathcal{H}} \\ & \quad + 2 \|(\tilde{u}^{t_n}(\cdot, \theta_{-t_n}\omega, u_0(\theta_{-t_n}\omega)), \tilde{v}^{t_n}(\cdot, \theta_{-t_n}\omega, v_0(\theta_{-t_n}\omega)))\|_{\mathcal{H}} \\ & < 4\epsilon. \end{aligned} \quad (93)$$

This completes the proof. \square

Theorem 4.2. Assume that conditions (A1)-(A3) and (32) hold. Then, the random dynamical system ϕ has a unique \mathcal{D} -random attractor in \mathcal{H} .

Proof. By Lemma 3.1, ϕ has a closed absorbing set $K(\omega)$ in \mathcal{D} , and by Lemma 4.1, ϕ is \mathcal{D} -pullback asymptotically compact in \mathcal{H} . Hence, by Proposition 2.1, ϕ has a unique \mathcal{D} -random attractor. This ends the proof. \square

Competing interests

The authors declare that there is no conflict of interest regarding the publication of this paper.

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