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Attractors of dynamical systems in locally compact spaces

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Abstract: In this article the properties of attractors of dynamical systems in locally compact metric space are discussed. Existing conditions of attractors and related results are obtained by the near isolating block which we present.

Keywords: Flow, attraction neighborhood, near isolating block

MSC: 34C35 54H20

1 Introduction

The dynamical system theory studies the rules of changes in the state which depends on time. In the investigation of dynamical systems, one of very interesting topics is the study of attractors (see [1 – 4] and the references therein). In [1 – 3], the authors gave some definitions of attractors and also made further investigations about the properties of them. In [5], the limit set of a neighborhood was used in the definition of an attractor, and in [6] Hale and Waterman also emphasized the importance of the limit set of a set in the analysis of the limiting behavior of a dynamical system. In [7 – 9], the authors defined intertwining attractor of dynamical systems in metric space and obtained some existing conditions of intertwining attractor. In [10], the author studied the properties of limit sets of subsets and attractors in a compact metric space. In [11], the author studied a positively bounded dynamical system in the plane, and obtained the conditions of compactness of the set of all bounded solutions. In [12], the uniform attractor was defined as the minimal compact uniformly pullback attracting random set, several existence criteria for uniform attractors were given and the relationship between uniform and co-cycle attractors was carefully studied. In [13], the authors established conditions for the existence and stability of invariant sets for dynamical systems defined on metric space of fuzzy subsets of R^n. In [14], the authors studied the recurrence and the gradient-like structure of a flow and gave some properties of connecting orbit. The paper [15] defined an attractor of a dynamical system on a locally compact metric space and investigated topological properties of the attraction domain of dynamical systems. In [16], the authors obtained equivalent conditions for the existence of global attractors for transformation semigroups on principal bundles. The article [17] investigated the structure of a global attractor for an abstract evolutionary system and obtained weak and strong uniform tracking properties of omega-limits and global attractors.

Motivated by the above discussion, in this paper we study properties of attractors of dynamical systems. Main results are as follows. First of all, we define the near isolating block, then we present some basic

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properties about attractors, give the existing conditions of attractors of dynamical systems and obtain the relation between the attractor and the attraction neighborhood in a locally compact metric space. At the end of this article, as byproduct, we obtain some results about the compactness of the set formed by bounded orbits using near isolating block. To the best of our knowledge, results similar to these presented in our work have never been reported.

2 Preliminaries

Let X be a locally compact metric space with a metric ρ. In X, there is a flow $\pi : X \times R \to X$. Denote $x \cdot t = \pi(x, t)$, and $A \cdot B = \{x \cdot t | x \in A, t \in B\}$ where $A \subset X$ and $B \subset R$. $x \cdot R$, $x \cdot R^+$ and $x \cdot R^-$ denote the trajectory, the positive trajectory and the negative trajectory of A point $x \in X$, respectively. The ω-limit set of x(or the trajectory $x \cdot R^+$), $\omega(x, \pi) = \{y \in X|$ there exists a sequence $\{t_n\}_{n=1}^{+\infty} \subset R^+$ such that $t_n \to +\infty$ and $x \cdot t_n \to y$ as $n \to +\infty\}$. The α-limit set of x(or the trajectory $x \cdot R^-$), $\alpha(x, \pi) = \{y \in X|$ there exists a sequence $\{t_n\}_{n=1}^{+\infty} \subset R^-$ such that $t_n \to -\infty$ and $x \cdot t_n \to y$ as $n \to +\infty\}$. For $Y \subset X$, if $Y \cdot R = Y$, call Y invariant. For the set $Y \subset X$, $\mathcal{C}l\{Y\}$, ∂Y, $\mathcal{I}ntY$ and $\mathcal{E}xtY$ denote the closure, the boundary, the interior and the complement of Y, respectively. The ω-limit set of $Y \subset X$ defines $\omega(Y, \pi) = \cap_{t \geq 0} \mathcal{C}l\{Y \cdot [t, +\infty)\}$. In [5], the authors have proven that $\omega(Y, \pi)$ is the maximal invariant subset in $\mathcal{C}l\{Y \cdot [0, +\infty)\}$. For $r > 0$, $B(x, r) = \{y \mid \rho(x, y) < r\}$ denotes the ball with the center x and the radius r.

To avoid confusion, we first fix some notations and definitions.

Definition 2.1. For a set $A \subset X$, A is called an attractor for the flow π if A admits a neighborhood N such that $A = \omega(N, \pi)$.

In [18], the author considered the differential system defined in the plane

$$\begin{cases} \frac{dx}{dt} = X(x, y), \\ \frac{dy}{dt} = Y(x, y). \end{cases} \tag{2.1}$$

Suppose $X, Y \in C^1$. Let the vector field $V = (X, Y)$ define a flow π. Let $B \subset R^n$ be the closure of a bounded and connected open set with the boundary ∂B; Let $\mathcal{L}_1 \ldots \mathcal{L}_n$ denote its boundary components, where $\mathcal{L}_i \cap \mathcal{L}_j =$ for $i \neq j$, and $\mathcal{L}_1$ denotes the external boundary. Each of them is a smoothly simply closed curve. We define three subsets b^+, b^- and τ as follows: $b^+ = \{p \in \partial B | \exists \varepsilon > 0$ with $\pi(p, (-\varepsilon, 0)) \cap B = \}$; $b^- = \{p \in \partial B | \exists \varepsilon > 0$ with $\pi(p, (0, \varepsilon)) \cap B = \}$; $\tau = \{p \in \partial B | V$ is tangent to ∂B at $p\}$.

Definition 2.2. If $b^+ \cap b^- = \tau$ and $b^+ \cup b^- = \partial B$, then B is called an isolating block of the flow defined by (2.1).

In [5],[11], [14] and [18,19], the properties of attractors and connecting orbits of dynamical systems were discussed by an isolating block. Similar to Definition 2.2, we give the following definition which is very useful to our main results.

Definition 2.3. For $N \subset X$, the set N is called a near isolating block of a dynamical system if $x \cdot R^- \cap \mathcal{E}xtN \neq$ for any point $x \in \partial N$.

Remark 2.1. The condition of an isolating block is different to that of a near isolating block. Let $E = \{B \mid B$ is an isolating block $\}$ and $F = \{N \mid N$ satisfies $x \cdot R^- \cap \mathcal{E}xtN \neq$ for $x \in \partial N\}$. In general, $E \not\supset F$, $F \not\supset E$, but $E \cap F \neq$.

Example 2.1. Consider the dynamical system defined by the differential equations

$$\begin{cases} \frac{dx}{dt} = \lambda x, \\ \frac{dy}{dt} = \lambda y. \end{cases} \tag{2.2}$$

Denote $O = (0, 0)$. When $\lambda < 0$, the ball $B(O, 1)$ is an isolating block, of cause, and it is also a near isolating block. But the triangle region with the vertexes $(0, 1)$, $(-1, -1)$ and $(1, -1)$ is a near isolating block and not an isolating block.

Definition 2.4. An attractor neighborhood means a closed subset $N \subset X$ such that for any $x \in \partial N$, $\alpha(x, \pi) \subset \mathcal{E}xtN$.

Remark 2.2. From Definition 2.4 above, we can see that for any given attractor neighborhood N, $\omega(N, \pi) \subset N$ may not hold.

Example 2.2. Consider the dynamical system defined by the differential equations

$$\begin{cases} \frac{dx}{dt} = -x, \\ \frac{dy}{dt} = -2y. \end{cases} \tag{2.3}$$

The point $O = (0, 0)$ is only an attractor of the dynamical system. Any subset $N \subset R^2$, which does not contain $O = (0, 0)$, is an attractor neighborhood, but $\omega(N, \pi) \subset N$ does not hold.

At the end of this paper, we also use the two following definitions.

Definition 2.5. A simple closed curve is called a singular closed orbit if it is the union of alternating nonclosed whole orbits and equilibrium points, and is contained in the ω-(or α-)limit set of an orbit.

Definition 2.6. For an equilibrium point p, an orbit $\gamma(x) = x \cdot R(x \neq p)$ is called a homoclinic orbit with respect to p provided that $\lim_{t\to-\infty} x \cdot t = \lim_{t\to+\infty} x \cdot t = p$.

Definition 2.7. If there exists a point $p \in R^n$ such that $\lim_{t\to-\infty} p \cdot t = a$ and $\lim_{t\to+\infty} p \cdot t = b$, then the set $\pi(p, R) = \{p \cdot t \mid t \in R\}$ is called a connecting orbit from a to b.

To give the existing conditions of attractors of dynamical systems, we need first give the properties of the limit set, which have been proven in [7], [10] and will be used in our main theorems.

Theorem 2.1. The following facts about $\omega(Y, \pi)$ hold.

(1) $\omega(Y, \pi) = \cap_{n\geq 0}\mathcal{C}l\{Y \cdot [n, +\infty)\}$;

(2) Let $Y_i \subset X(i = 1, 2)$, then $\omega(Y_1 \cup Y_2) = \omega(Y_1) \cup \omega(Y_2)$; In particular, if $Y \subset Z$, then $\omega(Y) \subset \omega(Z)$;

(3) $z \in \omega(Y)$ if and only if there exist sequences $\{y_n\}$, $\{t_n\}$, $y_n \in Y$, $t_n \in R$, $t_n \to +\infty$ and $n \to +\infty$, such that $lim_{n\to+\infty} y_n \cdot t_n = z$.

Theorem 2.2. For any $Y \subset X$, $\omega(Y, \pi) = \omega(\mathcal{C}l\{Y\}, \pi)$ holds.

Theorem 2.3. For any $Y \subset X$, $\omega(Y, \pi)$ is the limit point in 2^X of the sequence $\{\mathcal{C}l\{Y \cdot [n, +\infty)\}\}_{n=1}^{\infty}$.

In the article, for simpler, we mainly discuss properties of the attractor and related results of the dynamical system in the space $X = R^n$ in the next section. Of course, the results given still hold in the general locally compact metric space. At the end of this section, we give an example which explains a fact that $\cup_{x\in M}\omega(x) = \omega(M, \pi)$ does not hold in general.

Example 2.3. Consider the dynamical system defined by differential equations as follows

$$\begin{cases} \frac{dR}{dt} = R(R-2), \\ \frac{d\theta}{dt} = -2. \end{cases} \tag{2.4}$$

Let $N = \{(R, \theta)|R \leq 2\}$ and $O = (0, 0)$. Obviously, $\cup_{x\in N}\omega(x) = \{(R, \theta)|R = 2\} \cup \{O\}$, but $\omega(N) = N$. Hence, $\omega(N) = \cup_{x\in N}\omega(x)$ does not hold.

3 Main results

In this section we give main results in the article. First of all, based on the properties of a near isolating block, we shall give the existing condition of an attractor.

Theorem 3.1. Assume that N is a bounded near isolating block of R^n. Then A is an attractor if $A = \{x \mid x \cdot R \subset N\} \neq$.

Proof. Let $M = \{x \mid x \cdot R^+ \subset N\}$, then $A \subset M \subset N$. Since M is a positive invariant set, $M \cdot R^+ \subset N$ and $\mathcal{C}l\{M \cdot R^+\} \subset \mathcal{C}l\{N\}$. N is a bounded near isolating block of R^n, *i.e.*, $x \cdot R^- \cap \mathcal{E}xtN \neq$ for $x \in \partial N$, hence $\omega(M, \pi) \subset N$. Because $\omega(M, \pi)$ is the maximal invariant subset in $\mathcal{C}l\{M \cdot [0, +\infty)\}$, we know $\omega(M, \pi) = A$. By Definition 2.1, we need only prove that M is a neighborhood of A. Now we prove it by contradiction. If M is not a neighborhood of A, then $A \cap \partial M \neq$. Choose $x_0 \in A \cap \partial M$. For one thing, since $x_0 \in A$, $x_0 \cdot R \subset \mathcal{I}ntN$; For other, for $x_0 \in \partial M$, there exists a sequence $\{x_n\}_{n=1}^{+\infty}$ such that $x_n \in N \setminus M$, $\lim_{n\to+\infty} x_n = x_0$. Let $t_n = max\{t \geq 0 : x_n \cdot [0, t] \subset N\}$, and then $\lim_{n\to+\infty} t_n = +\infty$ because $x_0 \cdot R^+ \subset \mathcal{I}ntN$ by the continuous dependence on initial values. By the definition of t_n, $x_n \cdot t_n \in \partial N$ and t_n is finite for $x_n \in N \setminus M$. Since N is a bounded subset of

R^n, $\mathcal{C}l\{N\}$ and ∂N are compact. The compactness of ∂N implies that there exists a convergent subsequence of $\{x_n \cdot t_n\}_{n=1}^{+\infty}$. Without loss of generalization, $\{x_n \cdot t_n\}_{n=1}^{+\infty}$ also denotes its convergent subsequence, and then $\lim_{n\to+\infty} x_n \cdot t_n = y \in \partial N$. For $y \in \partial N$, there exists a $T_y > 0$ such that $y \cdot (-T_y) \in \mathcal{E}xtN$; An enough large number $n > 0$ is chosen such that $x_n \cdot t_n \cdot (-T_y) \in \mathcal{E}xtN$ and $t_n > T_y$. Hence $x_n \cdot (t_n - T_y) \in \mathcal{E}xtN$ and $0 < t_n - T_y < t_n$, which contradict with the definition of t_n. So $A \cap \partial M = $, $A \subset \mathcal{I}ntM$ and M is a neighborhood of A. Hence A is an attractor by Definition 2.1. The proof is completed.

Remark 3.1. From Theorem 3.1, we can know that N is a bounded near isolating block of R^n which plays an important role in the proof of Theorem 3.1. Of cause, N may not be a closed set.

Theorem 3.2. Assume that $A \subset R^n$ is a bounded subset, then A is an attractor if and only if there exists an open neighborhood N of A such that $\cap_{t\geq 0} N \cdot t = A$.

Proof. The necessity. If the bounded subset A is an attractor of a dynamical system, by Definition 2.1, there exists an open neighborhood N such that $\omega(N, \pi) = A$, and $\omega(\mathcal{C}l\{N\}, \pi) = A$ by Theorem 2.2. Since A is an invariant subset of N, $A = A \cdot t \subset N \cdot t$ and $A \subset \cap_{t\geq 0} N \cdot t$ for each $t > 0$. Since $N \cdot t \subset N \cdot [t, +\infty)$, $\cap_{t\geq 0} N \cdot t \subset \cap_{t\geq 0} N \cdot [t, +\infty) \subset \cap_{t\geq 0} \mathcal{C}l\{N \cdot [t, +\infty)\} = \omega(N, \pi) = A$ by Theorem 2.1. Hence $\cap_{t\geq 0} N \cdot t = A$.

The sufficiency. If there exists an open neighborhood N of A such that $\cap_{t\geq 0} N \cdot t = A$, choose a relatively compact neighborhood M(*i.e.*, $\mathcal{C}l\{M\}$ is compact and $A \cap \partial M = $) of A satisfying $A \subset M \subset N$. For each $x \in \partial M$, $x \notin A = \cap_{t\geq 0} N \cdot t$, so there exists a $T_x > 0$ such that $x \notin N \cdot T_x$, which implies $x \cdot (-T_x) \notin N$. Then $x \cdot (-T_x) \notin M$, *i.e.*, M is a bounded near isolating block. From Theorem 3.1 we know that A is an attractor. The proof is completed.

Remark 3.2. If A is a bounded subset of the locally compact space X, then the result of Theorem 3.2 still holds. In fact, choosing a relatively compact neighborhood M of A satisfying $A \subset M \subset N$ is important.

Next we shall give an equivalent condition of existence of a near isolating block of a dynamical system.

Theorem 3.3. Assume that the set N is a bounded subset of R^n, then for each $x \in \partial N$, $\alpha(x, \pi) \cap \mathcal{E}xtN \neq$ or $x \cdot R^- \cap \mathcal{E}xtN \neq$ hold if and only if N is a bounded near isolating block of R^n.

Proof. Now we prove the necessity. If $x \cdot R^- \cap \mathcal{E}xtN \neq$ for $x \in \partial N$, it is obvious to satisfy the condition of a near isolating block. If $\alpha(x, \pi) \neq$ and $\alpha(x, \pi) \cap \mathcal{E}xtN \neq$ hold for $x \in \partial N$, there is a $\delta > 0$ such that $B(y, \delta) \subset \mathcal{E}xtN$ for $y \in \alpha(x, \pi) \cap \mathcal{E}xtN$. From the definition of $\alpha(x, \pi)$, it follows that $x \cdot R^- \cap \mathcal{E}xtN \neq$.

The sufficiency. Assume that the set N is a bounded near isolating block of R^n, *i.e.*, for each $x \in \partial N$, $x \cdot R^- \cap \mathcal{E}xtN \neq$, so we only prove if $\alpha(x, \pi) \neq$ for $x \in \partial N$, then $\alpha(x, \pi) \cap \mathcal{E}xtN \neq$. Since N is a bounded near isolating block, there exists a $t_1 < 0$ such that $x \cdot t_1 \in \mathcal{E}xtN$. If $x \cdot (-\infty, t_1] \cap N = $, it follows that $\alpha(x, \pi) \subset \mathcal{C}l\{\mathcal{E}xtN\}$; Otherwise, by the connectedness of $x \cdot (-\infty, t_1]$, there exists a $t_2 (< t_1)$ such that $x \cdot t_2 \in \partial N$. Thus there is a negative number $t_3 (< t_2)$ such that $x \cdot t_3 \in \mathcal{E}xtN$ since N is a bounded near isolating block of R^n. Now we consider two cases as follows:

Case I: If there exists a $T_x < 0$ such that $x \cdot (-\infty, T_x] \cap N = $ for $x \in \partial N$, then $\alpha(x, \pi) \subset \mathcal{C}l\{\mathcal{E}xtN\}$. So $\alpha(x, \pi) \cap \mathcal{E}xtN \neq$. Otherwise, since the set $\alpha(x, \pi)$ is invariant, it follows that $\alpha(x, \pi) \subset \partial N$, which contradicts with $x \cdot R^- \cap \mathcal{E}xtN \neq$ for each $x \in \partial N$.

Case II: If there does not exist a $T_x < 0$ such that $x \cdot (-\infty, T_x] \cap N = $, for any fixed T_x, we can find a sequence $\{t_{2n+1}\}_{n=1}^{+\infty}$ such that $x \cdot t_{2n+1} \in \mathcal{E}xtN$ and $t_{2n+1} \to -\infty$ as $n \to -\infty$. Choose a sequence $\{\theta_{2n+1}\}_{n=1}^{+\infty}$ such that $x \cdot \theta_{2n+1} \in \partial N$ with $t_{2n-1} < \theta_{2n-1} < t_{2n+1} < 0$. By the compactness of ∂N, it follows that $\{x \cdot \theta_{2n+1}\}_{n=1}^{+\infty}$ is a convergent subsequence, that is, $\alpha(x, \pi) \subset \mathcal{C}l\{\mathcal{E}xtN\}$. Similar to the first case, we can get $\alpha(x, \pi) \cap \mathcal{E}xtN \neq$. The proof is completed.

Theorem 3.4. Assume that N is a bounded near isolating block of R^n. If $A = \{x \mid x \cdot R \subset N\} \neq$, there is a subset M of N such that $A \subset M$ and $\alpha(x, \pi) \subset \mathcal{E}xtM$ for each $x \in \partial M$ when $\alpha(x, \pi) \neq$.

Proof. Let $N_1 = \{x \mid x \cdot R^+ \subset N\}$, then by the proof of Theorem 3.1, N_1 is a neighborhood of A and $\omega(N_1, \pi) = A$. By $\omega(N_1, \pi) = A$, for any neighborhood U of A there exists a $t_U > 0$ such that $N_1 \cdot [t_U, +\infty) \subset U$. Let $U = \mathcal{I}ntN_1$, then there exists a $t_{N_1} > 0$ such that $N_1 \cdot [t_{N_1}, +\infty) \subset \mathcal{I}ntN_1$. Now we show that there exists a $t_x > 0$ such that $x \cdot (-\infty, -t_x] \cap N_1 = $ for any $x \in \partial N_1$. Otherwise, there is a $P \in \partial N_1$, for any $t_P > 0$, $P \cdot (-\infty, -t_P] \cap N_1 \neq$. Choose $t_P > t_{N_1}$ and $Q \in P \cdot (-\infty, -t_P] \cap N_1$, and then there exists a $t_Q > t_P > t_{N_1}$ such that $Q = P \cdot (-t_Q) \in N_1$. Hence, $Q \cdot [t_{N_1}, +\infty) = P \cdot (-t_Q) \cdot [t_{N_1}, +\infty) = P \cdot [t_{N_1} - t_Q, +\infty) \subset \mathcal{I}ntN_1$. By $t_{N_1} - t_Q < 0$, it implies $P \in P \cdot [t_{N_1} - t_Q, +\infty) \subset \mathcal{I}ntN_1$, which contradicts with $P \in \partial N_1$. Let $M = N_1 \cdot t_{N_1}$. Since A is an invariant subset of $\mathcal{I}ntN_1$, it follows that $A = A \cdot t_{N_1} \subset N_1 \cdot t_{N_1} = M \subset \mathcal{I}ntN_1$ and the map

$\pi(\cdot, -t_{N_1}) : \partial M \to \partial N_1$ is a homeomorphism. Thus for any $x \in \partial M$, $x \cdot (-\infty, -t_x - t_{N_1}] \cap N_1 =$ holds. Hence $\alpha(x,\pi) \subset \mathcal{C}l\{\mathcal{E}xtN_1\}$ and $\alpha(x,\pi) \cap M =$ when $\alpha(x,\pi) \neq$. The proof is completed.

Remark 3.3. According to the argument above, M is a neighborhood of A satisfying $\omega(M,\pi) \subset M$. However, $\omega(M,\pi) \subset M$ holds for an attractor neighborhood N of A, and $\omega(N,\pi) \subset N$ may not hold yet.

Example 3.1 Consider the following planar dynamical system defined by the differential equations:

$$\frac{dR}{dt} = R \cdot (1 - R), \quad \frac{d\theta}{dt} = \begin{cases} -(sin^2\theta + 1), & 0 < R < 1 \\ -sin^2\theta, & R = 1 \\ -(sin^2\theta + 1), & R > 1 \end{cases} \tag{3.1}$$

There exist three equilibrium points: the origin $A = (0,0)$ and two points $B = (1,0)$, $C = (1,\pi)$ on the unit circle. There have two trajectories $\gamma_1 = \{(1,\theta) : 0 < \theta < \pi\}$, $\gamma_2 = \{(1,\theta) : \pi < \theta < 2\pi\}$ on the unit circle and spiralling trajectories through points $P = (R,\theta)$, $R \neq 0, 1$. For any point $P = (R,\theta)$ with $0 < R < 1$, $\alpha(P,\pi)$ is the unit circle and $\omega(P,\pi) = \{0\}$. For any point $P = (R,\theta)$ with $R > 1$, $\alpha(P,\pi)$ is the unit circle, and $\omega(P,\pi) =$; For any point $P \in \gamma_1$, $\alpha(P,\pi) = \{C\}$ and $\omega(P,\pi) = \{B\}$. For any point $P \in \gamma_2$, $\alpha(P,\pi) = \{B\}$ and $\omega(P,\pi) = \{C\}$. A is the only attractor of the system, and let $N = \{(R,\theta) | 0 < R < 1/2\} \cup \{(1, \frac{\pi}{2})\}$, and then $\omega(N,\pi) \subset N$ do not hold. Let $M = \{(R,\theta) | 0 < R < 1/2\}$, and then $\omega(M) \subset M$. That is, even if N is a neighborhood of an attractor, but $\omega(N,\pi) \subset N$ may not hold.

Now we consider the relations between $N \subset \mathcal{A}(M,\pi) = \{x \in X \mid \neq \omega(x,\pi) \subset M\}$ and $\omega(N,\pi) \subset N$, and give some conditions which assure that $\omega(N,\pi) \subset N$ holds.

Theorem 3.5. If N is a bounded near isolating block of R^n, and let $M = \{x \mid x \cdot R \subset N\} \neq$. Then $\omega(N,\pi) \subset N$ if and only if $N \subset \mathcal{A}(M,\pi) = \{x \in X \mid \neq \omega(x,\pi) \subset M\}$.

Proof. Now we consider the necessity. Suppose $\omega(N,\pi) \subset N$. For any $x \in N$, $\omega(x,\pi) = \omega(\{x\},\pi) \subset \omega(N,\pi) \subset N$ holds. Since M is the maximal invariant set in N, it follows $\omega(x,\pi) \subset \omega(N,\pi) \subset M$. Hence by the definition of $\mathcal{A}(M,\pi)$, $N \subset \mathcal{A}(M,\pi) = \{x \in X \mid \neq \omega(x,\pi) \subset M\}$.

The sufficiency. Assume that $N \subset \mathcal{A}(M,\pi) = \{x \in X \mid \neq \omega(x,\pi) \subset M\}$. By Theorem 3.1, M is an attractor. That is, there is a set N_1 such that $\omega(N_1,\pi) = A$; Since $\mathcal{C}l\{N\}$ is compact and $N \subset \mathcal{A}(M,\pi) = \{x \in X \mid \neq \omega(x,\pi) \subset M\}$, there exists a $T > 0$ such that $N \cdot T \subset N_1$ and $N \cdot (T + t) \subset N_1$ for any $t > 0$. Hence $\omega(N,\pi) \subset N$. The proof is completed.

Remark 3.4. Under the conditions above, if N is a closed set and $N \subset \mathcal{A}(M,\pi)$, we get $\alpha(x,\pi) \subset \mathcal{E}xtN$ for any $x \in \partial N$. Otherwise, there is a $x \in \partial N$ such that $\alpha(x,\pi) \cap N \neq$. We can choose a $y \in \alpha(x,\pi) \cap N$ and $\omega(y) \subset \alpha(x,\pi)$. Hence $\omega(y) \cap M =$, which is a contradiction with $x \in N \subset A(M,\pi)$.

Remark 3.5. In some literature, $\mathcal{A}(M,\pi) = \{x \in X \mid \neq \omega(x,\pi) \subset M\}$ also denotes the attraction region of the attractor M. So Theorem 3.5 explains the relations between two kinds of attraction regions. In fact, from the proofs above, we easily see that the results of Theorems 3.1-3.5 still hold in generally locally compact metric spaces.

Next, we can get the condition of the compactness of the set formed by bounded orbits by a near isolating block. For simpler, we only consider the dynamical systems defined in the space R^2.

Theorem 3.6. The set of the bounded orbits of the dynamical system is compact if there exists a bounded near isolating block $N \subset R^2$ such that all equilibrium points and closed orbits of the dynamical system are contained in the region N.

Proof. Assume that $p \cdot R^+$ is bounded for $p \in R^2$, so the positive limit set $\omega(p)$ contains an equilibrium point, or a closed orbit, or a connected set composed of some equilibrium points and some orbits whose positive semi-orbit and negative semi-orbit tend to a singular point respectively by Poincaré-Bendixson Theorem(see [20]). Similarly, if $p \cdot R^-$ is bounded for $p \in R^2$, so the negative limit set $\alpha(p)$ contains an equilibrium point, or a closed orbit, or a connected set composed of some singular points and some orbits whose positive semi-orbit and negative semi-orbit tend to a singular point, respectively. Since the set $N \subset R^2$ is a near isolating block and contains all equilibrium points and closed orbits of the dynamical system, it contains all bounded orbits of the dynamical system. From Theorem 3.1, we know that in N, the set of the bounded orbits is an attractor. That is, the attractor is formed by all bounded orbits of the dynamical system, and it is only one attractor. Hence, the set of all bounded orbits is compact by the boundedness of N. The proof is completed.

Theorem 3.7. Assume that $\omega(x, \pi) \neq$ for $x \in R^2$. Then the set of bounded orbits of the dynamical system is compact if and only if there exists a bounded near isolating block $N \subset R^2$ such that all equilibrium points and closed orbits of the dynamical system are contained in the region N.

Proof. From the proof of Theorem 3.6, we know that the sufficiency holds.

The necessity. Let M denote the set of bounded orbits of the dynamical system. If M is compact, M is bounded. There exists a bounded subset $N \subset R^2$ such that $M \subset N$. Since $\omega(x, \pi) \neq$ for $x \in R^2$, any bounded neighborhood of M is a near isolating block containing all equilibrium points and closed orbits of the dynamical system. In fact, $\omega(x, \pi) \subset M$ for $x \in N$, and N contains all bounded orbits. The proof is completed.

Theorem 3.8. If the dynamical system has a bounded near isolating block N which has only two equilibrium points and no homoclinic orbit or closed orbit, then there exist one or uncountable connecting orbits in N.

Proof. From Theorem 3.6, the set of bounded orbits of a dynamical system in N is compact. By Theorem 3.1, it is the attractor A which has exactly two equilibrium points, and there exist no closed orbit and homoclinic orbit. Then the set of all bounded orbits is composed of the equilibrium points and connecting orbits by the connectedness of the limit set and the Poincaré-Bendixson Theorem, and the possible numbers of the connecting orbits are one or uncountable. The proof is completed.

Corollary 3.1. If the dynamical system has a bounded near isolating block N which has at least two equilibrium points but no homoclinic orbit or closed orbit, there exist one or uncountable connecting orbits in N.

Example 3.2. Consider the dissipative dynamical system defined by the deferential equations in the plane:

$$\begin{cases} \frac{dx}{dt} = y, \\ \frac{dy}{dt} = x - x^3 - y. \end{cases} \tag{3.2}$$

Obviously, the dynamical system (3.2) has a saddle point $O = (0, 0)$ and two stable foci $P = (-1, 0)$, $Q = (1, 0)$. There exist two connecting orbits γ_1, γ_2 which connect O,P and O, Q, respectively. Each orbit through a point x in the plane goes to P or Q provided that x does not lie on the stable or unstable manifolds of O. That is, $\{O, P, Q\} \cup \gamma_1 \cup \gamma_2$ is an attractor, and we can choose the set N such that $\{O, P, Q\} \cup \gamma_1 \cup \gamma_2 \subset N$, which is an attractor neighborhood of the attractor $\{O, P, Q\} \cup \gamma_1 \cup \gamma_2$ and a near isolating block.

4 Conclusion

In this article, we investigate the properties of attractors of the dynamical system. The dynamical system considered here may not be the positively bounded system in the locally compact space, where near isolating block defined may or may not be a closed set. The results include mainly two aspects: (1) The existing conditions of attractors of the dynamical system are given by near isolating block; (2) The equivalent condition of compactness of the set which is formed by the bounded orbits of the planar dynamical system is presented. This paper shows that near isolating block plays an important role in investigating the properties of dynamical systems. Thus, the applications of near isolating block require further studies.

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