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Sufficient and necessary conditions of convergence for $\widetilde{\rho}$-mixing random variables

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Abstract: In the present paper, the sufficient and necessary conditions of the complete convergence and complete moment convergence for $\widetilde{\rho}$-mixing random variables are established, which extend some well-known results.

Keywords: Complete convergence, complete moment convergence, $\widetilde{\rho}$-mixing random variables, weakly bounded.

MSC: 60F15

1 Introduction

1.1 $\widetilde{\rho}$-mixing sequence

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\{X_n, n \geq 1\}$ be a sequence of random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$, $S_n = \sum_{i=1}^{n} X_i$ for $n \geq 1$. For any $S \subset N = \{1, 2, \cdots\}$, define $\mathcal{F}_S = \sigma(X_i, i \in S)$. Let $\mathcal{A}$ and $\mathcal{B}$ be two sub σ-algebra on $\mathcal{F}$, put

$$\rho(\mathcal{A}, \mathcal{B}) = \sup\left\{ \frac{|\mathbb{E}XY - \mathbb{E}X\mathbb{E}Y|}{\sqrt{\mathbb{E}(X - \mathbb{E}X)^2\mathbb{E}(Y - \mathbb{E}Y)^2}} : X \in L_2(\mathcal{A}), Y \in L_2(\mathcal{B}) \right\}. \tag{1.1}$$

Define the $\widetilde{\rho}$-mixing coefficients by

$$\widetilde{\rho}_n = \sup\{\rho(\mathcal{F}_S, \mathcal{F}_T) : S, T \subset N \text{ with } dist(S, T) \geq n\}, \tag{1.2}$$

where $dist(S, T) = \inf\{|s-t| : s \in S, t \in T\}$. Obviously, $0 \leq \widetilde{\rho}_{n+1} \leq \widetilde{\rho}_n \leq \widetilde{\rho}_0 = 1$. Then the sequence $\{X_n, n \geq 1\}$ is called $\widetilde{\rho}$-mixing if there exists $k \in N$ such that $\widetilde{\rho}_k < 1$.

The notion of $\widetilde{\rho}$-mixing random variables was first introduced by Bradley [1], and a number of limits results for $\widetilde{\rho}$-mixing random variables have been established by many authors. One can refer to Bradley [1] for the central limit theorem; Sung [2, 3], An and Yuan [4], Lan [5], Guo and Zhu [6] for complete convergence; Zhang [7] for complete moment convergence; Peligrad and Gut [8], Utev and Peligrad [9] for the moment inequalities; Gan [10], Wu and Jiang [11], Kuczmaszewska [12] for strong law of large numbers.

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1.2 Some notations and known results

Let $\{X_n, n \geq 1\}$ be a sequence of random variables and if there exist a positive constant $C_1(C_2)$ and a random variable X, such that the left-hand side (right-hand side) of the following inequalities is satisfied for all $n \geq 1$ and $x \geq 0$,

$$C_1\mathbb{P}(|X| > x) \leq \frac{1}{n}\sum_{k=1}^{n}\mathbb{P}(|X_k| > x) \leq C_2\mathbb{P}(|X| > x), \tag{1.3}$$

then the sequence $\{X_n, n \geq 1\}$ is said to be weakly lower (upper) bounded by X. The sequence $\{X_n, n \geq 1\}$ is said to be weakly bounded by X if it is both weakly lower and upper bounded by X. A sequence of random variables $\{U_n, n \geq 1\}$ is said to converge completely to a constant C if

$$\sum_{n=1}^{\infty}\mathbb{P}\left(|U_n - C| > \varepsilon\right) < \infty, \quad \text{for all } \varepsilon > 0. \tag{1.4}$$

The concept of complete convergence was introduced firstly by Hsu and Robbins [13]. In view of the Borel-Cantelli lemma, complete convergence implies that $U_n \to C$ almost surely.

The complete moment convergence is a more general concept than the complete convergence, which was introduced by Chow [14]. Let $\{Z_n, n \geq 1\}$ be a sequence of random variables and $a_n > 0$, $b_n > 0$, $q > 0$, if

$$\sum_{n=1}^{\infty} a_n\mathbb{E}\{b_n^{-1}|Z_n| - \epsilon\}_+^q < \infty, \quad \text{for some or all } \epsilon > 0,$$

then the above result was called the complete moment convergence. The complete convergence and complete moment convergence have been studied by many authors. For instance, see Wang [15], Zhao [16], Zhang [17] and so on.

Baum and Katz [18] obtained the following equivalent conditions for the i.i.d. random variables.

Theorem A. (Baum and Katz [18]) *Let $0 < r < 2$, $r \leq p$. Suppose that $\{X_n,\ n \geq 1\}$ is a sequence of i.i.d. random variables with mean zero, then $\mathbb{E}|X_1|^p < \infty$ is equivalent to the condition that*

$$\sum_{n=1}^{\infty} n^{p/r-2}\mathbb{P}\left(|S_n| > \varepsilon n^{1/r}\right) < \infty, \quad \text{for all } \varepsilon > 0, \tag{1.5}$$

and also equivalent to the condition that

$$\sum_{n=1}^{\infty} n^{p/r-2}\mathbb{P}\left(\max_{1\leq k\leq n}|S_k| > \varepsilon n^{1/r}\right) < \infty, \quad \text{for all } \varepsilon > 0. \tag{1.6}$$

For the i.i.d. case, related results are fruitful and detailed. It is natural to extend them to dependent case, for examples, martingale difference, negatively associated, mixing random variables and so on. In the present paper, we are interested in the $\widetilde{\rho}$-mixing random variables.

For identically distributed $\widetilde{\rho}$-mixing random variables, Peligrad and Gut [8] extended the results of Baum and Katz to $\widetilde{\rho}$-mixing random variables (see Theorem B); subsequently, An and Yuan [4] extended the results of Peligrad and Gut [8] to weighted sums of $\widetilde{\rho}$-mixing random variables (see Theorem C); Gan [10] obtained a sufficient condition on complete convergence (see Theorem D).

Theorem B. (Peligrad and Gut [8]) *Let $\{X_n,\ n \geq 1\}$ be a sequence of identically distributed $\widetilde{\rho}$-mixing random variables, $\alpha p > 1$, $\alpha > 1/2$, and suppose that $\mathbb{E}X_1 = 0$ for $\alpha \leq 1$. Assume that $\lim_{n\to\infty}\widetilde{\rho}_n < 1$, then $\mathbb{E}|X_1|^p < \infty$ is equivalent to the condition that*

$$\sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{P}\left(\max_{1\leq j\leq n}|S_j| > \varepsilon n^{\alpha}\right) < \infty, \quad \text{for all } \varepsilon > 0. \tag{1.7}$$

Theorem C. (An and Yuan [4]) *Let $\{X_n,\ n \geq 1\}$ be a sequence of identically distributed $\widetilde{\rho}$-mixing random variables, $\alpha p > 1$, $\alpha > 1/2$ and suppose that $\mathbb{E}X_1 = 0$ for $\alpha \leq 1$. Assume that $\{a_{ni}, 1 \leq i \leq n\}$ is an array of real numbers satisfying*

$$\sum_{i=1}^{n} |a_{ni}|^p = O(n^\delta),\ 0 < \delta < 1,$$

and

$$\sharp A_{nk} = \sharp\{1 \leq i \leq n : |a_{ni}|^p > (k+1)^{-1}\} \geq ne^{-1/k}.$$

Then $\mathbb{E}|X_1|^p < \infty$ is equivalent to

$$\sum_{n=1}^{\infty} n^{\alpha p-2}\mathbb{P}\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > \varepsilon n^\alpha\right) < \infty,\ \ \textit{for all}\ \varepsilon > 0. \tag{1.8}$$

Theorem D. (Gan [10]) *Let $\{X_n,\ n \geq 1\}$ be a sequence of identically distributed $\widetilde{\rho}$-mixing random variables with $\widetilde{\rho}(1) < 1$ and $1 < p \leq 2$, $\delta > 0$, $\alpha \geq \max\{(1+\delta)/p, 1\}$. If $\mathbb{E}X_1 = 0$ and $\mathbb{E}|X_1|^p < \infty$, then*

$$\sum_{n=1}^{\infty} n^{\alpha p-2-\delta}\mathbb{P}\left(|S_n| > \varepsilon n^\alpha\right) < \infty,\ \ \textit{for all}\ \ \varepsilon > 0. \tag{1.9}$$

In this paper, the purpose is to study and establish the equivalent conditions on complete convergence and complete moment convergence for $\widetilde{\rho}$-mixing random variables. Our main results are stated in Section 2 and all proofs are given in Section 3. Throughout the paper, C denotes a positive constant not depending on n, which may be different in various places. Let $I(A)$ be the indicator function of the set A, $a_n = O(b_n)$ represent $a_n \leq Cb_n$ for all $n \geq 1$.

2 Main results

In the section, we state our main results and some remarks. Recall that a real-valued function $l(x)$, positive and measurable on $(0, \infty)$, is said to be slowly varying at infinity if $\lim_{x\to\infty} l(\lambda x)/l(x) = 1$ for each $\lambda > 0$. Let

$$\mathcal{L} = \Big\{f : f(x) \text{ is slowly varying function, and such that } \int_1^k x^s f(x)dx \geq Ck^{s+1}f(k),\quad \text{for all } k > 1, s > -1\Big\}.$$

Firstly, we state the complete convergence for the weighted sums of $\{X_n, n \geq 1\}$.

Theorem 2.1. *Let $\{X_n, n \geq 1\}$ be a sequence of $\widetilde{\rho}$-mixing random variables which is weakly bounded by X. Let $\alpha > 1/2$, $\alpha p \geq 1$ and $\mathbb{E}X_n = 0$ if $p > 1$. Assume that $l(x) \in \mathcal{L}$ and $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ is an array of real numbers satisfying $\max_{1\leq i\leq n} |a_{ni}| = O(n^{-\alpha})$. Then $\mathbb{E}|X|^p l(|X|^{1/\alpha}) < \infty$ is equivalent to*

$$\sum_{n=1}^{\infty} n^{\alpha p-2} l(n)\mathbb{P}\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni}X_i\right| > \varepsilon\right) < \infty,\ \ \textit{for all}\ \ \varepsilon > 0. \tag{2.1}$$

Remark 2.1. *When right-hand side of inequality (1.3) is satisfied and $l(x)$ is slowly varying function, the moment condition $\mathbb{E}|X|^p l(|X|^{1/\alpha}) < \infty$ still implied (2.1). Conversely, for the sufficient condition of Theorem 2.1, we need left-hand side of inequality (1.3) and $l \in \mathcal{L}$.*

Since $\log x \in \mathcal{L}$ and $1 \in \mathcal{L}$, we can obtain the following corollaries.

Corollary 2.1. *Let $\{X_n, n \ge 1\}$ be a sequence of $\widetilde{\rho}$-mixing random variables which is weakly bounded by X. Let $\alpha > 1/2$, $\alpha p \ge 1$ and $\mathbb{E}X_n = 0$ if $p > 1$. Assume that $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ is an array of real numbers satisfying $\max_{1\le i\le n} |a_{ni}| = O(n^{-\alpha})$. Then $\mathbb{E}|X|^p \log(|X|) < \infty$ is equivalent to*

$$\sum_{n=1}^{\infty} n^{\alpha p-2} \log n \mathbb{P}\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}X_i\right| > \varepsilon\right) < \infty, \quad \text{for all } \varepsilon > 0. \tag{2.2}$$

Corollary 2.2. *Let $\{X_n, n \ge 1\}$ be a sequence of $\widetilde{\rho}$-mixing random variables which is weakly bounded by X. Let $\alpha > 1/2$, $\alpha p \ge 1$ and $\mathbb{E}X_n = 0$ if $p > 1$. Assume that $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ is an array of real numbers satisfying $\max_{1\le i\le n} |a_{ni}| = O(n^{-\alpha})$. Then $\mathbb{E}|X|^p < \infty$ is equivalent to*

$$\sum_{n=1}^{\infty} n^{\alpha p-2} \mathbb{P}\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}X_i\right| > \varepsilon\right) < \infty, \quad \text{for all } \varepsilon > 0. \tag{2.3}$$

Remark 2.2. *Let $\{X_n, n \ge 1\}$ be a sequence of identically distributed $\widetilde{\rho}$-mixing random variables, then $\{X_n, n \ge 1\}$ is weakly bounded by X_1. By taking $a_{ni} = n^{-\alpha}$ for all $1 \le i \le n, n \ge 1$, then Theorem B can be obtained by Corollary 2.2. Moreover, we not only consider the case $\alpha p > 1$, but also consider the case $\alpha p = 1$, so Theorem 2.1 extended and improved well-known results.*

Remark 2.3. *For independent random variable sequence, we have $\widetilde{\rho}_n = 0$ for all $n \ge 1$. So our results extend the Baum-Katz theorem from i.i.d. case to non-identically distributed $\widetilde{\rho}$-mixing random variables.*

Next, we give the complete moment convergence for the weighted sums of $\{X_n, n \ge 1\}$.

Theorem 2.2. *Let $\{X_n, n \ge 1\}$ be a sequence of $\widetilde{\rho}$-mixing random variables which is weakly bounded by X. Let $p > 1, \alpha > 1/2, \alpha p \ge 1$ and $\mathbb{E}X_n = 0$. Assume that $l(x) \in \mathcal{L}$ and $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ is an array of real numbers satisfying $\max_{1\le i\le n} |a_{ni}| = O(n^{-\alpha})$. Then $\mathbb{E}|X|^p l(|X|^{1/\alpha}) < \infty$ is equivalent to*

$$\sum_{n=1}^{\infty} n^{\alpha p-2} l(n) \mathbb{E}\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}X_i - \epsilon\right|\right)^{+} < \infty, \quad \text{for all } \varepsilon > 0. \tag{2.4}$$

Remark 2.4. *Similar to Theorem 2.1, the necessary condition of Theorem 2.2 only need that the right-hand side of inequality (1.3) and $l(x)$ is slowly varying function, the sufficient condition of Theorem 2.2 need the left-hand side of inequality (1.3) and $l \in \mathcal{L}$.*

Corollary 2.3. *Let $\{X_n, n \ge 1\}$ be a sequence of $\widetilde{\rho}$-mixing random variables which is weakly bounded by X. Let $p > 1, \alpha > 1/2, \alpha p \ge 1$ and $\mathbb{E}X_n = 0$. Assume $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ is an array of real numbers satisfying $\max_{1\le i\le n} |a_{ni}| = O(n^{-\alpha})$. Then $\mathbb{E}|X|^p \log(|X|) < \infty$ is equivalent to*

$$\sum_{n=1}^{\infty} n^{\alpha p-2} \log(n) \mathbb{E}\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}X_i - \epsilon\right|\right)^{+} < \infty, \quad \text{for all } \varepsilon > 0. \tag{2.5}$$

Corollary 2.4. *Let $\{X_n, n \ge 1\}$ be a sequence of $\widetilde{\rho}$-mixing random variables which is weakly bounded by X. Let $p > 1, \alpha > 1/2, \alpha p \ge 1$ and $\mathbb{E}X_n = 0$. Assume $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ is an array of real numbers satisfying $\max_{1\le i\le n} |a_{ni}| = O(n^{-\alpha})$. Then $\mathbb{E}|X|^p < \infty$ is equivalent to*

$$\sum_{n=1}^{\infty} n^{\alpha p-2} \mathbb{E}\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}X_i - \epsilon\right|\right)^{+} < \infty, \quad \text{for all } \varepsilon > 0. \tag{2.6}$$

3 Proofs of Main results

3.1 Some lemmas

To prove our results, we first give some lemmas as follows.

Lemma 3.1. *(Kuzmaszewska [19]) Let* $\{X_n, n \geq 1\}$ *be a sequence of random variables which is weakly mean dominated (or weakly upper bounded) by a random variable* X. *If* $\mathbb{E}|X|^p < \infty$ *for some* $p > 0$, *then for any* $t > 0$ *and* $n \geq 1$, *the following statements hold:*

$$\frac{1}{n}\sum_{k=1}^{n}\mathbb{E}|X_k|^p \leq C\mathbb{E}|X|^p, \tag{3.1}$$

$$\frac{1}{n}\sum_{k=1}^{n}\mathbb{E}|X_k|^p I(|X_k| \leq t) \leq C\left[\mathbb{E}|X|^p I(|X| \leq t) + t^p\mathbb{P}(|X| > t)\right] \tag{3.2}$$

and

$$\frac{1}{n}\sum_{k=1}^{n}\mathbb{E}|X_k|^p I(|X_k| > t) \leq C\mathbb{E}|X|^p I(|X| > t). \tag{3.3}$$

Lemma 3.2. *(Lan [5]) Let* $\{X_n, n \geq 1\}$ *be a sequence of* $\widetilde{\rho}$*-mixing random variables, then there exists a positive constant* C *such that for any* $x \geq 0$ *and all* $n \geq 1$,

$$\left(\frac{1}{2} - \mathbb{P}\left(\max_{1\leq k\leq n}|X_k| > x\right)\right)\sum_{k=1}^{n}\mathbb{P}(|X_k| > x) \leq \left(\frac{C}{2}+1\right)\mathbb{P}\left(\max_{1\leq k\leq n}|X_k| > x\right). \tag{3.4}$$

Lemma 3.3. *(Sung [20]) Let* $\{Y_n, n \geq 1\}$ *and* $\{Z_n, n \geq 1\}$ *be sequences of random variables, then for any* $q > 1$, $\epsilon > 0$ *and all* $a > 0$, *we have*

$$\begin{aligned}&\mathbb{E}\left[\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}(Y_i+Z_i)\right| - \epsilon a\right]^+ \\ &\leq \left(\frac{1}{\epsilon^q}+\frac{1}{q-1}\right)\frac{1}{a^{q-1}}\mathbb{E}\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}Y_i\right|^q\right) + \mathbb{E}\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}Z_i\right|\right).\end{aligned} \tag{3.5}$$

Lemma 3.4. *(Utev and Peligrad [9]) For a positive integer* $N \geq 1$ *and positive real numbers* $q \geq 2$ *and* $0 \leq r < 1$, *there is a positive constant* $C = C(q, N, r)$ *such that if* $\{X_n, n \geq 1\}$ *is a sequence of random variables with* $\widetilde{\rho}_N \leq r$, *with* $\mathbb{E}X_k = 0$ *and* $\mathbb{E}|X_k|^q < \infty$ *for every* $k \geq 1$, *then for all* $n \geq 1$,

$$\mathbb{E}\left[\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}X_i\right|^q\right] \leq C\left(\sum_{i=1}^{n}\mathbb{E}|X_i|^q + \left(\sum_{i=1}^{n}\mathbb{E}X_i^2\right)^{q/2}\right). \tag{3.6}$$

Lemma 3.5. *(Zhou [21]) Let* $l(x)$ *be slowly varying at infinity, then we have*

$$(i)\ \sum_{k=1}^{n}k^p l(k) \leq Cn^{p+1}l(n),\ \text{for } p > -1 \text{ and positive integer } n;$$

$$(ii)\ \sum_{k=n}^{\infty}k^p l(k) \leq Cn^{p+1}l(n),\ \text{for } p < -1 \text{ and positive integer } n.$$

Lemma 3.6. *(Bai [22]) Let* $l(x)$ *be slowly varying at infinity, then we have*

(i) $\lim_{x\to\infty}\frac{l(kx)}{l(x)} = 1$, *for any* $k > 0$; $\lim_{x\to\infty}\frac{l(x+u)}{l(x)} = 1$, *for any* $u > 0$;

(ii) $\lim_{x\to\infty} x^{\alpha} l(x)=\infty$; $\lim_{x\to\infty} x^{-\alpha} l(x)=0$, *for any* $\alpha>0$;

(iii) $\lim_{n\to\infty} \sup_{2^n\le x<2^{n+1}} \frac{l(x)}{l(2^n)}=1$;

(iv) $C_1 2^{nr} l(\epsilon 2^n) \le \sum_{k=1}^{n} 2^{kr} l(\epsilon 2^k) \le C_2 2^{nr} l(\epsilon 2^n)$ *for every* $r>0$, $\epsilon>0$ *and positive integer* n.

3.2 Proof of Theorem 2.1

Without loss of generality, we can assume that $a_{ni}>0$ for all $1\le i\le n$, $n\ge 1$. For fixed n, let

$$X_{ni}=X_i I(|X_i|\le n^{\alpha}),\quad X'_{ni}=X_i I(|X_i|>n^{\alpha}),\quad i\ge 1.$$

We will consider the following three cases, $p>1$, $p=1$ and $0<p<1$ respectively.

(i) Let $p>1$. It is easy to see that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2} l(n)\mathbb{P}\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}X_i\right|>\varepsilon\right)\\
\le&\sum_{n=1}^{\infty} n^{\alpha p-2} l(n)\mathbb{P}\left(\bigcup_{i=1}^{n}(|X_i|>n^{\alpha})\right)\\
&+\sum_{n=1}^{\infty} n^{\alpha p-2} l(n)\mathbb{P}\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}X_{ni}\right|>\varepsilon\right)\\
\triangleq& I+J.
\end{aligned}\tag{3.7}$$

In order to prove (2.1), it need only to show that $I<\infty$ and $J<\infty$. From (1.3), Lemma 3.5 and Markov's inequality, we have

$$\begin{aligned}
I=&\sum_{n=1}^{\infty} n^{\alpha p-2} l(n)\mathbb{P}\left(\bigcup_{i=1}^{n}(|X_i|>n^{\alpha})\right)\\
\le&\sum_{n=1}^{\infty} n^{\alpha p-2} l(n)\sum_{i=1}^{n}\mathbb{P}(|X_i|>n^{\alpha})\\
\le&C\sum_{n=1}^{\infty} n^{\alpha p-1} l(n)\mathbb{P}(|X|>n^{\alpha})\\
\le&C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} l(n)\mathbb{E}[|X|I(|X|>n^{\alpha})]\\
\le&C\sum_{n=1}^{\infty} n^{\alpha p-\alpha-1} l(n)\sum_{k=n}^{\infty}\mathbb{E}[|X|I(k^{\alpha}\le|X|<(k+1)^{\alpha})]\\
\le&C\sum_{k=1}^{\infty}\mathbb{E}[|X|I(k^{\alpha}\le|X|<(k+1)^{\alpha})]\sum_{n=1}^{k} n^{\alpha p-\alpha-1} l(n)\\
\le&C\sum_{k=1}^{\infty}\mathbb{E}[|X|I(k^{\alpha}\le|X|<(k+1)^{\alpha})]k^{\alpha p-\alpha} l(k)\\
\le&C\sum_{k=1}^{\infty}\mathbb{E}[|X|^p l(|X|^{1/\alpha})I(k^{\alpha}\le|X|<(k+1)^{\alpha})]\\
\le&C\mathbb{E}[|X|^p l(|X|^{1/\alpha})]<\infty.
\end{aligned}\tag{3.8}$$

Note that $\alpha p \geq 1$, $\mathbb{E}X_n = 0$, by Lemma 3.1, then

$$\begin{aligned}
&\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_{ni}\mathbb{E}X_{ni}\right| \leq \max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_{ni}\mathbb{E}X_i I(|X_i|>n^{\alpha})\right| \\
\leq&\sum_{i=1}^{n} a_{ni}\mathbb{E}|X_i|I(|X_i|>n^{\alpha}) \leq C\frac{1}{n^{\alpha}}\sum_{i=1}^{n}\mathbb{E}|X_i|I(|X_i|>n^{\alpha}) \\
\leq& C\frac{1}{n^{\alpha p-1}}\mathbb{E}|X|^{p}I(|X|>n^{\alpha}) \to 0.
\end{aligned} \tag{3.9}$$

In order to prove $J < \infty$, from (3.9), it is enough to check

$$\sum_{n=1}^{\infty} n^{\alpha p-2}l(n)\mathbb{P}\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni}(X_{ni}-\mathbb{E}X_{ni})\right| > \frac{\varepsilon}{2}\right) < \infty.$$

By taking $q > \max\{p, 2, \frac{\alpha p}{\alpha-\frac{1}{2}}\}$ and from Lemma 3.4, we have that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2}l(n)\mathbb{P}\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni}(X_{ni}-\mathbb{E}X_{ni})\right| > \frac{\varepsilon}{2}\right) \\
\leq& C\sum_{n=1}^{\infty} n^{\alpha p-2}l(n)\mathbb{E}\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni}(X_{ni}-\mathbb{E}X_{ni})\right|\right)^{q} \\
\leq& C\sum_{n=1}^{\infty} n^{\alpha p-2}l(n)\left[\sum_{i=1}^{n} a_{ni}^{q}\mathbb{E}|X_{ni}-\mathbb{E}X_{ni}|^{q} + \left(\sum_{i=1}^{n} a_{ni}^{2}\mathbb{E}(X_{ni}-\mathbb{E}X_{ni})^{2}\right)^{q/2}\right] \\
\triangleq& J_1 + J_2.
\end{aligned} \tag{3.10}$$

Note that $q > p$, by (3.8) and Lemma 3.1, Lemma 3.5, then

$$\begin{aligned}
J_1 \leq& C\sum_{n=1}^{\infty} n^{\alpha p-2}l(n)\sum_{i=1}^{n} a_{ni}^{q}\mathbb{E}|X_{ni}|^{q} \\
\leq& C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q}l(n)\sum_{i=1}^{n}\mathbb{E}|X_i|^{q}I(|X_i|\leq n^{\alpha}) \\
\leq& C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha q}l(n)\left[\mathbb{E}|X|^{q}I(|X|\leq n^{\alpha}) + n^{\alpha q}\mathbb{P}(|X|>n^{\alpha})\right] \\
\leq& C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha q}l(n)\sum_{k=1}^{n}\mathbb{E}|X|^{q}I((k-1)^{\alpha}<|X|\leq k^{\alpha}) \\
&+ C\sum_{n=1}^{\infty} n^{\alpha p-1}l(n)\mathbb{P}(|X|>n^{\alpha}) \\
\leq& C\sum_{k=1}^{\infty}\mathbb{E}|X|^{q}I((k-1)^{\alpha}<|X|\leq k^{\alpha})\sum_{n=k}^{\infty} n^{\alpha p-1-\alpha q}l(n) \\
\leq& C\sum_{k=1}^{\infty}\mathbb{E}|X|^{q}I((k-1)^{\alpha}<|X|\leq k^{\alpha})k^{\alpha p-\alpha q}l(k) \\
\leq& C\mathbb{E}\left[|X|^{p}l(|X|^{1/\alpha})\right] < \infty.
\end{aligned} \tag{3.11}$$

In order to get $J_2 < \infty$, we consider the following cases.

case 1: $p \geq 2$, $\alpha p > 1$. From $q > (\alpha p - 1)/(\alpha - \frac{1}{2})$ and Lemma 3.6, we can get that

$$\begin{aligned} J_2 &= \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) \left(\sum_{i=1}^{n} a_{ni}^2 \mathbb{E}(X_{ni} - \mathbb{E}X_{ni})^2 \right)^{q/2} \\ &\leq C \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q} l(n) \left(\sum_{i=1}^{n} \mathbb{E}X_{ni}^2 \right)^{q/2} \\ &\leq C \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q} n^{q/2} l(n) \left(\mathbb{E}X^2 \right)^{q/2} \\ &\leq C \sum_{n=1}^{\infty} n^{\alpha p-2-\alpha q+q/2} l(n) < \infty. \end{aligned} \tag{3.12}$$

case 2: $1 < p < 2$, $\alpha p > 1$. Take $q = 2$, by Lemma 3.1 and (3.8), then

$$\begin{aligned} J_2 &= \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) \sum_{i=1}^{n} a_{ni}^2 \mathbb{E}(X_{ni} - \mathbb{E}X_{ni})^2 \\ &\leq C \sum_{n=1}^{\infty} n^{\alpha p-2-2\alpha} l(n) \sum_{i=1}^{n} \mathbb{E}X_i^2 I(|X_i| \leq n^{\alpha}) \\ &\leq C \sum_{n=1}^{\infty} n^{\alpha p-1-2\alpha} l(n) [\mathbb{E}X^2 I(|X| \leq n^{\alpha}) + n^{2\alpha} \mathbb{P}(|X| > n^{\alpha})] \\ &\leq C \sum_{n=1}^{\infty} n^{\alpha p-1-2\alpha} l(n) \sum_{k=1}^{n} \mathbb{E}X^2 I((k-1)^{\alpha} < |X| \leq k^{\alpha}) + \sum_{n=1}^{\infty} n^{\alpha p-1} l(n) \mathbb{P}(|X| > n^{\alpha})] \\ &\leq C \sum_{k=1}^{\infty} \mathbb{E}X^2 I((k-1)^{\alpha} < |X| \leq k^{\alpha}) \sum_{n=k}^{\infty} n^{\alpha p-1-2\alpha} l(n) \\ &\leq C \sum_{k=1}^{\infty} \mathbb{E}X^2 I((k-1)^{\alpha} < |X| \leq k^{\alpha}) k^{\alpha p-2\alpha} l(k) \\ &\leq C \mathbb{E}\left[|X|^p l(|X|^{1/\alpha}) \right] < \infty. \end{aligned} \tag{3.13}$$

case 3: $p > 1$, $\alpha p = 1$, $\alpha > \frac{1}{2}$. Take $q = 2$, similar to the proof of (3.13), it following that $J_2 < \infty$.
From the above discussions, we can get (2.1) for the case $p > 1$.
(ii) Let $p = 1$. By the similar proof of (3.9), we have

$$\max_{1 \leq j \leq n} \left| \sum_{i=1}^{j} a_{ni} \mathbb{E}X_{ni} \right| \to 0. \tag{3.14}$$

So in order to get (2.1), it is enough to show

$$\sum_{n=1}^{\infty} n^{\alpha-2} l(n) \mathbb{P}\left(\bigcup_{i=1}^{n} (|X_i| > n^{\alpha}) \right) < \infty \tag{3.15}$$

and

$$\sum_{n=1}^{\infty} n^{\alpha-2} l(n) \mathbb{P}\left(\max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} a_{ni} X_{ni} - \sum_{i=1}^{k} a_{ni} \mathbb{E}X_{ni} \right| > \frac{\varepsilon}{2} \right) < \infty. \tag{3.16}$$

By the condition (1.3) and Lemma 3.5, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha-2} l(n)\mathbb{P}\left(\bigcup_{i=1}^{n}(|X_i| > n^{\alpha})\right)\\
\leq &\sum_{n=1}^{\infty} n^{\alpha-2} l(n) \sum_{i=1}^{n} \mathbb{P}(|X_i| > n^{\alpha})\\
\leq &C\sum_{n=1}^{\infty} n^{\alpha-1} l(n)\mathbb{P}(|X| > n^{\alpha})\\
\leq &C\sum_{n=1}^{\infty} n^{\alpha-1} l(n) \sum_{k=n}^{\infty} \mathbb{P}(k^{\alpha} < |X| \leq (k+1)^{\alpha})\\
\leq &C\sum_{k=1}^{\infty} \mathbb{P}(k^{\alpha} < |X| \leq (k+1)^{\alpha}) \sum_{n=1}^{k} n^{\alpha-1} l(n)\\
\leq &C\sum_{k=1}^{\infty} \mathbb{P}(k^{\alpha} < |X| \leq (k+1)^{\alpha}) k^{\alpha} l(k)\\
\leq &C\sum_{k=1}^{\infty} \mathbb{E}[|X| l(|X|^{1/\alpha}) I(k^{\alpha} < |X| \leq (k+1)^{\alpha})]\\
\leq &C\mathbb{E}[|X| l(|X|^{1/\alpha})] < \infty.
\end{aligned} \tag{3.17}$$

Furthermore, from (3.17), Lemma 3.4 and Lemma 3.5, we get

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha-2} l(n)\mathbb{P}\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni}X_{ni} - \sum_{i=1}^{k} a_{ni}\mathbb{E}X_{ni}\right| > \frac{\varepsilon}{2}\right)\\
\leq &C\sum_{n=1}^{\infty} n^{\alpha-2} l(n)\mathbb{E}\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni}(X_{ni} - \mathbb{E}X_{ni})\right|\right)^2\\
\leq &C\sum_{n=1}^{\infty} n^{-\alpha-2} l(n) \sum_{i=1}^{n} \mathbb{E}|X_{ni} - \mathbb{E}X_{ni}|^2\\
\leq &C\sum_{n=1}^{\infty} n^{-\alpha-1} l(n) \left[\mathbb{E}|X|^2 I(|X| \leq n^{\alpha}) + n^{2\alpha}\mathbb{P}(|X| > n^{\alpha})\right]\\
\leq &C\sum_{n=1}^{\infty} n^{-\alpha-1} l(n) \sum_{k=1}^{n} \mathbb{E}|X|^2 I((k-1)^{\alpha} < |X| \leq k^{\alpha})\\
\leq &C\sum_{k=1}^{\infty} \mathbb{E}|X|^2 I((k-1)^{\alpha} < |X| \leq k^{\alpha}) \sum_{n=k}^{\infty} n^{-\alpha-1} l(n)\\
\leq &C\sum_{k=1}^{\infty} \mathbb{E}|X|^2 I((k-1)^{\alpha} < |X| \leq k^{\alpha}) k^{-\alpha} l(k)\\
\leq &C\mathbb{E}|X| l(|X|^{1/\alpha}) < \infty.
\end{aligned} \tag{3.18}$$

Based on the above discussions, we can get (2.1) for the case $p = 1$.

(iii) Let $0 < p < 1$. Since $X_i = X_{ni} + X'_{ni}$ for all $i \geq 1$, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2} l(n) \mathbb{P}\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni} X_i\right| > \varepsilon\right) \\
&\leq \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) \mathbb{P}\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni} X_{ni}\right| > \frac{\varepsilon}{2}\right) \\
&\quad + \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) \mathbb{P}\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni} X'_{ni}\right| > \frac{\varepsilon}{2}\right) \\
&\triangleq I^{\star} + J^{\star}.
\end{aligned} \tag{3.19}$$

By Lemma 3.1 and Lemma 3.5, we obtain that

$$\begin{aligned}
I^{\star} &= \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) \mathbb{P}\left(\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} a_{ni} X_{ni}\right| > \frac{\varepsilon}{2}\right) \\
&\leq C \sum_{n=1}^{\infty} n^{\alpha p-\alpha-2} l(n) \sum_{i=1}^{n} \mathbb{E}\,|X_{ni}| \\
&\leq C \sum_{n=1}^{\infty} n^{\alpha p-\alpha-1} l(n) \left[\mathbb{E}|X| I(|X| \leq n^{\alpha}) + n^{\alpha} \mathbb{P}(|X| > n^{\alpha})\right] \\
&\leq C \sum_{n=1}^{\infty} n^{\alpha p-\alpha-1} l(n) \sum_{k=1}^{n} \mathbb{E}|X| I((k-1)^{\alpha} < |X| \leq k^{\alpha}) \\
&\quad + C \sum_{n=1}^{\infty} n^{\alpha p/2-1} l(n) \sum_{k=n}^{\infty} \mathbb{E}|X|^{p/2} I(k^{\alpha} \leq |X| < (k+1)^{\alpha}) \\
&\leq C \sum_{k=1}^{\infty} \mathbb{E}|X| I((k-1)^{\alpha} < |X| \leq k^{\alpha}) \sum_{n=k}^{\infty} n^{\alpha p-\alpha-1} l(n) \\
&\quad + C \sum_{k=1}^{\infty} \mathbb{E}|X|^{p/2} I(k^{\alpha} \leq |X| < (k+1)^{\alpha}) \sum_{n=1}^{k} n^{\alpha p/2-1} l(n) \\
&\leq C \sum_{k=1}^{\infty} \mathbb{E}|X| I((k-1)^{\alpha} < |X| \leq k^{\alpha}) k^{\alpha p-\alpha} l(k) \\
&\quad + C \sum_{k=1}^{\infty} \mathbb{E}|X|^{p/2} I(k^{\alpha} \leq |X| < (k+1)^{\alpha}) k^{\alpha p/2} l(k) \\
&\leq C \mathbb{E}|X|^{p} l(|X|^{1/\alpha}) < \infty.
\end{aligned} \tag{3.20}$$

and

$$
\begin{aligned}
J^{*} =& \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) \mathbb{P}\left(\max_{1\le k\le n} \left| \sum_{i=1}^{k} a_{ni} X'_{ni} \right| > \frac{\varepsilon}{2} \right) \\
\le& C \sum_{n=1}^{\infty} n^{\alpha p/2-2} l(n) \mathbb{E} \left(\sum_{i=1}^{n} |X'_{ni}| \right)^{p/2} \\
\le& C \sum_{n=1}^{\infty} n^{\alpha p/2-2} l(n) \sum_{i=1}^{n} \mathbb{E}|X'_{ni}|^{p/2} \\
\le& C \sum_{n=1}^{\infty} n^{\alpha p/2-1} l(n) \mathbb{E}|X|^{p/2} I(|X| > n^{\alpha}) \\
\le& C \sum_{n=1}^{\infty} n^{\alpha p/2-1} l(n) \sum_{k=n}^{\infty} \mathbb{E}|X|^{p/2} I(k^{\alpha} \le |X| < (k+1)^{\alpha}) \\
\le& C \sum_{k=1}^{\infty} \mathbb{E}|X|^{p/2} I(k^{\alpha} \le |X| < (k+1)^{\alpha}) k^{\alpha p/2} l(k) \\
\le& C \mathbb{E}|X|^{p} l(|X|^{1/\alpha}) < \infty.
\end{aligned} \tag{3.21}
$$

From (3.19)-(3.21), we can see that (2.1) holds for $0 < p < 1$.

Conversely, we take $a_{ni} = n^{-\alpha}$ for all $1 \le i \le n,\ n \ge 1$, then

$$
\begin{aligned}
& \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) \mathbb{P}\left(\max_{1\le i\le n} |X_i| > \epsilon n^{\alpha} \right) \\
\le& C \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) \mathbb{P}\left(2 \max_{1\le j\le n} \left| \sum_{i=1}^{j} X_i \right| > \epsilon n^{\alpha} \right) \\
\le& C \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) \mathbb{P}\left(\max_{1\le j\le n} \left| \sum_{i=1}^{j} a_{ni} X_i \right| > \frac{\epsilon}{2} \right) < \infty.
\end{aligned} \tag{3.22}
$$

Because of $\alpha p \ge 1$, we have

$$
\lim_{n\to\infty} \mathbb{P}\left(\max_{1\le i\le n} |X_i| > \epsilon n^{\alpha} \right) \to 0. \tag{3.23}
$$

Thus, for all n large enough, it follows that

$$
\mathbb{P}\left(\max_{1\le i\le n} |X_i| > \epsilon n^{\alpha} \right) < \frac{1}{4}. \tag{3.24}
$$

Using Lemma 3.2 and (3.24), we get

$$
n C_1 \mathbb{P}(|X| > \epsilon n^{\alpha}) \le \sum_{k=1}^{n} \mathbb{P}(|X_k| > \epsilon n^{\alpha}) \le (2C+4) \mathbb{P}\left(\max_{1\le k\le n} |X_k| > \epsilon n^{\alpha} \right). \tag{3.25}
$$

Hence by $l \in \mathcal{L}$, we obtain

$$\begin{aligned}
\infty >& \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) \mathbb{P}\left(\max_{1\le i\le n} |X_i| > n^{\alpha}\right) \\
\ge& C \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) n \mathbb{P}(|X| > n^{\alpha}) \\
=& C \sum_{n=1}^{\infty} n^{\alpha p-1} l(n) \sum_{k=n}^{\infty} \mathbb{P}(k^{\alpha} < |X| \le (k+1)^{\alpha}) \\
\ge& C \sum_{k=1}^{\infty} \mathbb{P}(k^{\alpha} < |X| \le (k+1)^{\alpha}) \sum_{n=1}^{k} n^{\alpha p-1} l(n) \\
\ge& C \sum_{k=1}^{\infty} \mathbb{P}(k^{\alpha} < |X| \le (k+1)^{\alpha}) k^{\alpha p} l(k) \\
\ge& C \mathbb{E}\left(|X|^p l(|X|^{1/\alpha})\right),
\end{aligned} \tag{3.26}$$

which implies the desired results.

3.3 Proof of Theorem 2.2

First we assume that the claim (2.4) holds, then it follows that

$$\begin{aligned}
\infty >& \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) \mathbb{E}\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni} X_i\right| - \epsilon\right)^{+} \\
=& \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) \int_0^{\infty} \mathbb{P}\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni} X_i\right| - \epsilon > t\right) dt \\
\ge& \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) \int_0^{\epsilon} \mathbb{P}\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni} X_i\right| > \epsilon + t\right) dt \\
\ge& C \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) \mathbb{P}\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni} X_i\right| > 2\epsilon\right).
\end{aligned} \tag{3.27}$$

From Theorem 2.1, we know that $\mathbb{E}|X|^p l(|X|^{1/\alpha}) < \infty$ is equivalent to

$$\sum_{n=1}^{\infty} n^{\alpha p-2} l(n) \mathbb{P}\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni} X_i\right| > \epsilon\right) < \infty.$$

Conversely, if $\mathbb{E}|X|^p l(|X|^{1/\alpha}) < \infty$, then from Lemma 3.3, we have that

$$\begin{aligned}
& \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) \mathbb{E}\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni} X_i\right| - \epsilon\right)^{+} \\
\le& \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) \mathbb{E}\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}[(X_{ni} - \mathbb{E}X_{ni}) + (X'_{ni} - \mathbb{E}X'_{ni})]\right| - \epsilon\right)^{+} \\
\le& C \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) \mathbb{E}\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}(X_{ni} - \mathbb{E}X_{ni})\right|\right)^{q} \\
& + \sum_{n=1}^{\infty} n^{\alpha p-2} l(n) \mathbb{E}\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k} a_{ni}(X'_{ni} - \mathbb{E}X'_{ni})\right|\right) \\
\triangleq& \widetilde{I} + \widetilde{J}.
\end{aligned} \tag{3.28}$$

From (3.10)-(3.13), we have that $\widetilde{I} < \infty$. Furthermore, by (3.8) and Lemma 3.1, we obtain

$$\begin{aligned}\widetilde{J} &\le C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n)\mathbb{E}\left(\max_{1\le k\le n}\left|\sum_{i=1}^{k}(X'_{ni}-\mathbb{E}X'_{ni})\right|\right)\\ &\le C\sum_{n=1}^{\infty} n^{\alpha p-2-\alpha} l(n)\sum_{i=1}^{n}\mathbb{E}|X_i|I(|X_i|>n^{\alpha})\\ &\le C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} l(n)\mathbb{E}|X|I(|X|>n^{\alpha})<\infty.\end{aligned} \tag{3.29}$$

From (3.28) and (3.29), we can get (2.4).

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