DE GRUYTER Open Math. 2019; 17:439–451

Open Mathematics

Research Article

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Complete convergence for arrays of ratios of order statistics

https://doi.org/10.1515/math-2019-0035
Received January 22, 2019; accepted March 14, 2019

Abstract: Let $\{X_{n,k}, 1 \le k \le m_n, n \ge 1\}$ be an array of independent random variables from the Pareto distribution. Let $X_{n(k)}$ be the kth largest order statistic from the nth row of the array and set $R_{n,i_n,j_n} = X_{n(j_n)}/X_{n(i_n)}$ where $j_n < i_n$. The aim of this paper is to study the complete convergence of the ratios $\{R_{n,i_n,j_n}, n \ge 1\}$.

Keywords: Ratios; Pareto distribution; complete convergence.

MSC: 60F15, 62G30

1 Introduction

Let $\{X_n, n \ge 1\}$ be a sequence of independent and identically distributed random variables and let $X_{n(n)} \le X_{n(n-1)} \le \cdots \le X_{n(1)}$ be the order statistics. During the past many years, the influence of order statistics has attracted considerable attention. This topic is relevant in various practical situations like in (re)insurance applications when a significant proportion of the sum of claims is consumed by a small number of claims (or even by a single claim) due to earthquakes, floods, hurricanes, terrorism, etc. Numerous authors gave necessary and/or sufficient conditions for the convergence of certain ratios of extreme terms (order statistics and terms of maximum modulus) and sums. Smid and Stam [1] proved that the successive quotients of the order statistics in decreasing order are asymptotically independent with some distribution functions, under certain conditions. O'Brien [2] studied the ratio variate $X_{n(1)}/S_n$, where $S_n = X_1 + \cdots + X_n$, which is a quantity arising in the analysis of process speedup and the performance of scheduling. Balakrishnan and Stepanov [3] discussed the asymptotic properties of ratios $X_{n(j)}/X_{n(i)}$ for $i \ne j$. Furthermore, there are many scholars who have studied the more-refined properties for some specific distributions. For example, Malik and Trudel [4] found the distribution of the quotient of and two order statistics from the Pareto, Power and Weibull distributions, by using the Mellin transform technique.

In the present paper, we want to study some limit theorems for the order statistics from the Pareto distribution. Pareto distribution describes the income of individuals, and since its introduction, it has found wide applications in many fields of studies such as economics, insurance premium, population distributions, stock market analysis, and queuing theory. Johnson et al. [5] discussed some potential applications of this distribution in different subject fields. Mann et al. [6] studied different estimation procedures for the unknown

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parameters of the Pareto distribution. Miao et al. [7] establish two large deviations for the Pareto distribution, and discussed the maxima of sums of the two-tailed Pareto random variables.

Let X be a random variable with the Pareto distribution, i.e., the probability density function of X is

$$f(x) = kx_m^k/x^{k+1}I(x \geq x_m)$$

where x_m is the (necessarily positive) minimum possible value of X, and k is a positive parameter. In the present paper, we consider an array of independent random variables $\{X_{n,k}, 1 \leq k \leq m_n, n \geq 1\}$ with density

$$f(x) = p_n x^{-p_n-1} I(x \geq 1)$$

where $p_n > 0$. Let $X_{n(k_n)}$ be the k_nth largest order statistic from each row of the array. Hence we have

$$X_{n(m_n)} \leq X_{n(m_n-1)} \leq \cdots \leq X_{n(2)} \leq X_{n(1)}.$$

Next let us define the ratios of the order statistics

$$R_n := R_{n,i_n,j_n} = \frac{X_{n(j_n)}}{X_{n(i_n)}}, \quad j_n < i_n$$

which implies that $R_n \geq 1$. It is not difficult to check that the density of R_n is

$$f_{R_n}(r) = \frac{p_n(i_n-1)!}{(i_n-j_n-1)!(j_n-1)!} r^{-p_n j_n - 1}(1-r^{-p_n})^{i_n-j_n-1} I(r \geq 1).$$

It is obvious that the density of R_n is free of m_n. Adler [8–10] studied the limit behaviors for the weighted sums of the ratios. In [9], Adler assumed that the parameters i_n, j_n were fixed. In [8], Adler was allowed to let m_n and i_n grow, but j_n was fixed. The paper [10] was a natural extension of Adler [8, 9] and allow all subscripts to grow, but the distance between i_n and j_n was fixed, i.e., $\Delta := i_n - j_n$ was fixed. Adler [10] obtained the following results. The first theorem establishes an unusual strong law where $\Delta = 1$.

Theorem 1.1. *[10] If $p_n j_n = 1, \Delta = 1$ and $\alpha > -2$, then*

$$\lim_{N\to\infty} \frac{\sum_{n=1}^N ((\log n)^\alpha/n) R_n}{(\log N)^{\alpha+2}} = \frac{1}{\alpha+2}, \quad a.s.$$

The following results are to consider the case $\Delta > 1$.

Theorem 1.2. *[10] If $p_n j_n = 1, j_n = o(\log n), \Delta \geq 2$ and $\alpha > -2$, then*

$$\lim_{N\to\infty} \frac{\sum_{n=1}^N ((\log n)^\alpha/(n j_n^{\Delta-1})) R_n}{(\log N)^{\alpha+2}} = \frac{1}{(\alpha+2)(\Delta-1)!}, \quad a.s.$$

Theorem 1.3. *[10] If $p_n j_n = 1, j_n \sim \log n, \Delta \geq 2$ and $\alpha > -2$, then*

$$\lim_{N\to\infty} \frac{\sum_{n=1}^N ((\log n)^\alpha/(n j_n^{\Delta-1})) R_n}{(\log N)^{\alpha+2}} = \frac{\gamma_\Delta}{(\alpha+2)(\Delta-1)!}, \quad a.s.$$

where

$$\gamma_\Delta = \sum_{k=1}^{\Delta-1} \frac{\binom{\Delta-1}{k}(-1)^{k+1}e^{-k}}{k} - \sum_{k=2}^{\Delta-1} \frac{1}{k},$$

or, if one wishes

$$\gamma_\Delta = 1 + \sum_{k=1}^{\Delta-1} \frac{\binom{\Delta-1}{k}(-1)^k(1-e^{-k})}{k},$$

where naturally, if $\Delta = 2$ we have $\sum_{k=2}^{\Delta-1} \frac{1}{k} = 0$.

Theorem 1.4. *[10] If $p_n j_n = 1$, $j_n \sim (\log n)^a$, where $1 < a < 2$, $\Delta = 2$ and $\alpha > -3$, then*

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N}((\log n)^{\alpha}/n)R_n}{(\log N)^{\alpha+3}} = \frac{1}{2(\alpha+3)}, \quad a.s.$$

There is some literature concerning the limit theorems, for example, Miao et al. [11, 12] established some limit theorems for the ratios of order statistics from exponentials and uniform distribution. In the present paper, we are interested in the complete convergence for the ratios R_n. In order to prove the complete convergence, since the expectation of R_n dose not exists, we need to deal with the tailed term by using the results in Sung et al. [13]. Throughout the paper, we use the constant C to denote a generic real number that is not necessarily the same in each appearance, and we define $\log x = \log(\max\{e, x\})$.

2 Complete convergence theorems

In this section, we consider the complete convergence theorems for the weighted sums of the ratios R_n. The concept of complete convergence of a sequence of random variables was introduced by Hsu and Robbins [14] as follows. A sequence $\{U_n, n \geq 1\}$ is said to converge completely to a constant C if

$$\sum_{n=1}^{\infty} P(|U_n - C| > \varepsilon) < \infty, \quad \text{for all } \varepsilon > 0.$$

By using Borel-Cantelli lemma, this implies that $U_n \to C$ almost surely.

2.1 Several lemmas

Before giving the main results, we need to recall the following some lemmas.

Lemma 2.1. *[13] Let $\{X_{n,k}, 1 \leq k \leq m_n, n \geq 1\}$ be an array of rowwise independent random variables and $\{a_n, n \geq 1\}$ a sequence of positive constants such that*

$$\sum_{n=1}^{\infty} a_n = \infty.$$

Suppose that for every $r > 0$ and some $\varepsilon > 0$:

(i) $\sum_{n=1}^{\infty} a_n \sum_{i=1}^{m_n} P(|X_{n,i}| > r) < \infty$,

(ii) there exists $J \geq 2$ such that

$$\sum_{n=1}^{\infty} a_n \left(\sum_{i=1}^{m_n} EX_{n,i}^2 I(|X_{n,i}| \leq \varepsilon)\right)^J < \infty,$$

(iii) $\sum_{i=1}^{m_n} EX_{n,i} I(|X_{n,i}| \leq \varepsilon) \to 0$ as $n \to \infty$.

Then we have

$$\sum_{n=1}^{\infty} a_n P\left(\left|\sum_{i=1}^{m_n} X_{n,i}\right| > r\right) < \infty \text{ for all } r > 0.$$

Lemma 2.2. *[10] For $\Delta > 1$ and $p_n j_n = 1$, we have*

$$\frac{p_n(i_n-1)!}{(i_n-j_n-1)!(j_n-1)!} \sim \frac{j_n^{\Delta-1}}{(\Delta-1)!}$$

where "$a_n \sim b_n$" denotes $\lim_{n\to\infty} a_n/b_n = 1$.

Lemma 2.3. *[10] If $1 < a < 2$, then*

$$\lim_{x\to\infty} \frac{1 + 3x^{-1}\log x + x^{a-1}[(e^x x^3)^{-1/x^a} - 1]}{x^{1-a}} = 1/2.$$

2.2 Main results

In the subsection, we give the complete convergence of the weighted sums with different forms. The distance between i_n and j_n is fixed, i.e., Δ is fixed. We consider the most interesting case and assume that $p_n j_n = 1$. The growth of j_n can't be very fast, so we need a logarithmic growth rate for j_n. The conclusion breaks into three cases for different forms.
Case 1: $\Delta = 1$ and $\alpha > -2$;
Case 2: $j_n = o(\log n)$, $\Delta \geq 2$ and $\alpha > -2$;
Case 3: $j_n \sim (\log n)^a$, $\Delta = 2$ and $\alpha > -3$, where $1 < a < 2$.

Theorem 2.1. *Let $\{b_N, N \geq 1\}$ and $\{r_N, N \geq 1\}$ be sequences of positive numbers satisfying*

$$\sum_{N=1}^{\infty} b_N = \infty, \quad \frac{(\log\log N)(\log\log N + \log r_N)}{r_N \log N} \to 0 \tag{2.1}$$

and

$$\sum_{N=1}^{\infty} \frac{b_N \log\log N}{r_N \log N} \max\left\{1, \frac{1}{r_N}\right\} < \infty. \tag{2.2}$$

If $p_n j_n = 1, \Delta = 1$ and $\alpha > -2$, then

$$\sum_{N=1}^{\infty} b_N P\left(\max_{1\leq k\leq N}\left|\sum_{n=1}^{k} \frac{(\log n)^{\alpha}}{n}\left(R_n - ER_n I(1 \leq R_n \leq c_n)\right)\right| \geq r_N (\log N)^{\alpha+2}\right) < \infty \tag{2.3}$$

where $c_n = n(\log n)^2$.

Remark 2.1. *The convergence of the series in (2.3) holds trivially for any $\sum_{N=1}^{\infty} b_N < \infty$.*

Theorem 2.2. *Let $\{b_N, N \geq 1\}$ and $\{r_N, N \geq 1\}$ be sequences of positive numbers satisfying (2.1) and (2.2). If $p_n j_n = 1, j_n = o(\log n), \Delta \geq 2$ and $\alpha > -2$, then*

$$\sum_{N=1}^{\infty} b_N P\left(\max_{1\leq k\leq N}\left|\sum_{n=1}^{k} \frac{(\log n)^{\alpha}}{n j_n^{\Delta-1}}\left(R_n - ER_n I(1 \leq R_n \leq c_n)\right)\right| \geq r_N (\log N)^{\alpha+2}\right) < \infty \tag{2.4}$$

where $c_n = n j_n^{\Delta-1}(\log n)^2$.

Remark 2.2. *The conclusion of Theorem 2.2 can also be obtained when $j_n \sim \log n$ and $c_n = n(\log n)^{\Delta+1}$.*

Theorem 2.3. *Let $\{b_N, N \geq 1\}$ and $\{r_N, N \geq 1\}$ be sequences of positive numbers satisfying*

$$\sum_{N=1}^{\infty} b_N = \infty, \quad \frac{(\log\log N)(\log\log N + \log r_N)}{r_N (\log N)^{2-a}} \to 0 \tag{2.5}$$

and

$$\sum_{N=1}^{\infty} \frac{b_N \log\log N}{r_N (\log N)^{2-a}} \max\left\{1, \frac{1}{r_N}\right\} < \infty \tag{2.6}$$

where $1 < a < 2$. If $p_n j_n = 1, j_n \sim (\log n)^a, \Delta = 2$ and $\alpha > -3$, then

$$\sum_{N=1}^{\infty} b_N P\left(\max_{1\leq k\leq N}\left|\sum_{n=1}^{k} \frac{(\log n)^{\alpha}}{n}\left(R_n - ER_n I(1 \leq R_n \leq c_n)\right)\right| \geq r_N (\log N)^{\alpha+3}\right) < \infty$$

where $c_n = n(\log n)^3$.

2.3 Proofs of main results

Before giving the proofs, we want to show the outlines of the proofs. For any k, we partition $\sum_{n=1}^{k} t_n \left(R_n - ER_nI(1 \le R_n \le c_n)\right)$ into the following two terms:

$$\begin{aligned}\sum_{n=1}^{k} t_n \left(R_n - ER_nI(1 \le R_n \le c_n)\right) &= \sum_{n=1}^{k} t_n[R_nI(1 \le R_n \le c_n) - ER_nI(1 \le R_n \le c_n)] \\ &\quad + \sum_{n=1}^{k} t_nR_nI(R_n > c_n) =: I_k + II_k,\end{aligned} \tag{2.7}$$

where t_n are our weights. Next we discuss separately the complete convergence of the above terms. The complete convergence of the first term can be verified by the different density of R_n, Lemma 2.2 and Kolmogorov's inequality. Since the expectation of R_n dose not exists, we will deal with the tailed term by using the results in Sung et al. [13]. Therefore, the complete convergence of the second term can be obtained by Lemma 2.1.

Proof of Theorem 2.1. Firstly we partition $\sum_{n=1}^{k}((\log n)^\alpha/n)\left(R_n - ER_nI(1 \le R_n \le c_n)\right)$ into the following two terms:

$$\begin{aligned}\sum_{n=1}^{k}((\log n)^\alpha/n)\left(R_n - ER_nI(1 \le R_n \le c_n)\right) &= \sum_{n=1}^{k}((\log n)^\alpha/n)[R_nI(1 \le R_n \le c_n) - ER_nI(1 \le R_n \le c_n)] \\ &\quad + \sum_{n=1}^{k}((\log n)^\alpha/n)R_nI(R_n > c_n) \\ &=: I_k + II_k.\end{aligned} \tag{2.8}$$

Note that the density for the ratio of our adjacent order statistics, that is, $\Delta = 1$, is

$$f_{R_n}(x) = x^{-2}I(x \ge 1),$$

then it is not difficult to see that

$$ER_n^2I(1 \le R_n \le c_n) = c_n - 1 = n(\log n)^2 - 1.$$

By denoting

$$\tilde{R}_n := R_nI(1 \le R_n \le c_n) - ER_nI(1 \le R_n \le c_n),$$

and by the Kolmogorov's inequality, we have

$$\begin{aligned}P\left(\max_{1\le k\le N}\left|\sum_{n=1}^{k}((\log n)^\alpha/n)\tilde{R}_n\right| > r_N(\log N)^{\alpha+2}\right) &\le \frac{1}{r_N^2(\log N)^{2(\alpha+2)}} Var\left[\sum_{n=1}^{N}((\log n)^\alpha/n)\tilde{R}_n\right] \\ &\le \sum_{n=1}^{N}\frac{((\log n)^\alpha/n)^2}{r_N^2(\log N)^{2(\alpha+2)}} Var[R_nI(1 \le R_n \le c_n)] \\ &\le \sum_{n=1}^{N}\frac{((\log n)^\alpha/n)^2}{r_N^2(\log N)^{2(\alpha+2)}} n(\log n)^2 \le \frac{\log\log N}{r_N^2\log N}.\end{aligned} \tag{2.9}$$

Here we use the fact

$$\sum_{n=1}^{N}\frac{(\log n)^{(2\alpha+2)}}{n(\log N)^{2(\alpha+2)}} \le \begin{cases}\frac{C}{\log N}, & \text{for } \alpha > -2, \alpha \ne -\frac{3}{2} \\ C\frac{\log\log N}{\log N}, & \text{for } \alpha = -\frac{3}{2}\end{cases}, \tag{2.10}$$

for all N large enough. From the condition (2.2), we can get

$$\sum_{N=1}^{\infty} b_NP\left(\max_{1\le k\le N}|I_k| \ge r_N(\log N)^{\alpha+2}\right) < \infty. \tag{2.11}$$

Next, we consider the complete convergence of the term II_k, from using Lemma 2.1. By denoting

$$X_{N,n} := \frac{((\log n)^{\alpha}/n)R_n I(R_n > c_n)}{r_N(\log N)^{\alpha+2}},$$

then by the similar discussions in (2.10), for any $r > 0$, we have

$$\begin{aligned}\sum_{n=1}^{N} P\left(X_{N,n} \geq r\right) &\leq \sum_{n=1}^{N} P\left(R_n \geq \max\left\{c_n, nr \cdot r_N(\log N)^{\alpha+2}/(\log n)^{\alpha}\right\}\right)\\ &\leq C\sum_{n=1}^{N} \max\left\{\frac{1}{n(\log n)^2}, \frac{(\log n)^{\alpha}}{nr_N(\log N)^{\alpha+2}}\right\} \leq \frac{C\log\log N}{r_N \log N}\end{aligned}$$

which implies

$$\sum_{N=1}^{\infty} b_N \sum_{n=1}^{N} P\left(X_{N,n} \geq r\right) < \infty. \tag{2.12}$$

Furthermore, for any $\varepsilon > 0$, we have

$$\begin{aligned}\sum_{n=1}^{N} EX_{N,n} I\left(X_{N,n} \leq \varepsilon\right) &\leq C\frac{\log\log N + \log r_N}{r_N(\log N)^{\alpha+2}} \sum_{n=1}^{N} \frac{(\log n)^{\alpha}}{n}\\ &\leq C\frac{(\log\log N)(\log\log N + \log r_N)}{r_N \log N} \to 0\end{aligned} \tag{2.13}$$

as $N \to \infty$, since the condition (2.1). At last

$$\sum_{n=1}^{N} EX_{N,n}^2 I\left(X_{N,n} \leq \varepsilon\right) \leq C\sum_{n=1}^{N} \frac{(\log n)^{\alpha}/n}{r_N(\log N)^{\alpha+2}} \leq \frac{C\log\log N}{r_N \log N},$$

then we get

$$\sum_{N=1}^{\infty} b_N \left(\sum_{n=1}^{N} EX_{N,n}^2 I\left(X_{N,n} \leq \varepsilon\right)\right)^2 < \infty. \tag{2.14}$$

By Lemma 2.1 and (2.12)-(2.14), we have

$$\sum_{N=1}^{\infty} b_N P\left(\max_{1\leq k\leq N} |II_k| \geq r \cdot r_N(\log N)^{\alpha+2}\right) = \sum_{N=1}^{\infty} b_N P\left(\frac{((\log n)^{\alpha}/n)R_n I(R_n > c_n)}{r_N(\log N)^{\alpha+2}} \geq r\right) < \infty. \tag{2.15}$$

Therefore, the desired result can be obtained by (2.11) and (2.15). □

Proof of Theorem 2.2. As the proof of Theorem 2.1, we partition $\sum_{n=1}^{k}((\log n)^{\alpha}/(nj_n^{\Delta-1}))(R_n - ER_n I(1 \leq R_n \leq c_n))$ into the following two terms:

$$\begin{aligned}&\sum_{n=1}^{k}((\log n)^{\alpha}/(nj_n^{\Delta-1}))(R_n - ER_n I(1 \leq R_n \leq c_n))\\ &= \sum_{n=1}^{k}((\log n)^{\alpha}/(nj_n^{\Delta-1}))[R_n I(1 \leq R_n \leq c_n) - ER_n I(1 \leq R_n \leq c_n)]\\ &\quad + \sum_{n=1}^{k}((\log n)^{\alpha}/(nj_n^{\Delta-1}))R_n I(R_n > c_n)\\ &=: I_k + II_k.\end{aligned} \tag{2.16}$$

For the case $\Delta \geq 2$, $p_n j_n = 1$ and by Lemma 2.2, the density for the ratio of our adjacent order statistics is

$$\begin{aligned}f_{R_n}(x) &= \frac{p_n(i_n - 1)!}{(i_n - j_n - 1)!(j_n - 1)!} x^{-p_n j_n - 1}(1 - x^{-p_n})^{i_n - j_n - 1} I(x \geq 1)\\ &\sim \frac{j_n^{\Delta-1}}{(\Delta - 1)!} x^{-2}(1 - x^{-1/j_n})^{\Delta-1} I(x \geq 1).\end{aligned}$$

Then it follows that

$$\begin{aligned} ER_nI(1 \le R_n \le c_n) &\sim \frac{j_n^{\Delta-1}}{(\Delta-1)!}\int_1^{c_n} x^{-1}(1-x^{-1/j_n})^{\Delta-1}dx \\ &= \frac{j_n^{\Delta-1}}{(\Delta-1)!}\int_1^{c_n} x^{-1}\sum_{k=0}^{\Delta-1}\binom{\Delta-1}{k}(-1)^k x^{-k/j_n}dx \\ &= \frac{j_n^{\Delta-1}}{(\Delta-1)!}\left[\log c_n + \sum_{k=1}^{\Delta-1}\binom{\Delta-1}{k}(-1)^k(k/j_n)^{-1}(1-c_n^{-k/j_n})\right] \\ &\sim \frac{j_n^{\Delta-1}\log n}{(\Delta-1)!} \end{aligned}$$

and

$$ER_n^2I(1 \le R_n \le c_n) \le Cj_n^{\Delta-1}c_n = Cnj_n^{2(\Delta-1)}(\log n)^2.$$

By the Kolmogorov's inequality, we have

$$\begin{aligned} &P\left(\max_{1\le k\le N}\left|\sum_{n=1}^{k}((\log n)^\alpha/(nj_n^{\Delta-1}))\tilde{R}_n\right| > r_N(\log N)^{\alpha+2}\right) \\ &\le \sum_{n=1}^{N}\frac{((\log n)^\alpha/(nj_n^{\Delta-1}))^2}{r_N^2(\log N)^{2(\alpha+2)}}Var[R_nI(1 \le R_n \le c_n)] \\ &\le \sum_{n=1}^{N}\frac{((\log n)^\alpha/(nj_n^{\Delta-1}))^2}{r_N^2(\log N)^{2(\alpha+2)}}nj_n^{2(\Delta-1)}(\log n)^2 \le \frac{\log\log N}{r_N^2\log N} \end{aligned} \tag{2.17}$$

where

$$\tilde{R}_n := R_nI(1 \le R_n \le c_n) - ER_nI(1 \le R_n \le c_n).$$

From the condition (2.2), we can get

$$\sum_{N=1}^{\infty} b_NP\left(\max_{1\le k\le N}|I_k| \ge r_N(\log N)^{\alpha+2}\right) < \infty. \tag{2.18}$$

Next, we consider the complete convergence of the term II_k, from using Lemma 2.1. By denoting

$$X_{N,n} := \frac{((\log n)^\alpha/(nj_n^{\Delta-1}))R_nI(R_n > c_n)}{r_N(\log N)^{\alpha+2}},$$

then for any $r > 0$, we have

$$\begin{aligned} \sum_{n=1}^{N}P\left(X_{N,n} \ge r\right) &\le \sum_{n=1}^{N}P\left(R_n \ge \max\left\{c_n, nr\cdot r_Nj_n^{\Delta-1}(\log N)^{\alpha+2}/(\log n)^\alpha\right\}\right) \\ &\le C\sum_{n=1}^{N}\max\left\{\frac{1}{n(\log n)^2}, \frac{(\log n)^\alpha}{nr_N(\log N)^{\alpha+2}}\right\} \le \frac{C\log\log N}{r_N\log N} \end{aligned}$$

which implies

$$\sum_{N=1}^{\infty} b_N\sum_{n=1}^{N}P\left(X_{N,n} \ge r\right) < \infty. \tag{2.19}$$

Furthermore, for any $\varepsilon > 0$, we have

$$\begin{aligned} \sum_{n=1}^{N}EX_{N,n}I\left(X_{N,n} \le \varepsilon\right) &\le \frac{C}{r_N(\log N)^{\alpha+2}}\sum_{n=1}^{N}\frac{(\log n)^\alpha}{nj_n^{\Delta-1}}\frac{j_n^{\Delta-1}(\log\log N + \log r_N)}{(\Delta-1)!} \\ &\le C\frac{\log\log N + \log r_N}{r_N(\log N)^{\alpha+2}}\sum_{n=1}^{N}\frac{(\log n)^\alpha}{n} \\ &\le C\frac{(\log\log N)(\log\log N + \log r_N)}{r_N\log N} \to 0 \end{aligned} \tag{2.20}$$

as $N \to \infty$. At last, since

$$\begin{aligned}\sum_{n=1}^{N} EX_{N,n}^2 I\left(X_{N,n}^2 \le \varepsilon\right) &\le C \sum_{n=1}^{N} \frac{j_n^{\Delta-1}}{(\Delta-1)!} \frac{\varepsilon(\log n)^\alpha}{n j_n^{\Delta-1} r_N (\log N)^{\alpha+2}} \\ &\le C \sum_{n=1}^{N} \frac{(\log n)^\alpha / n}{r_N (\log N)^{\alpha+2}} \\ &\le \frac{C \log\log N}{r_N \log N},\end{aligned}$$

we get

$$\sum_{N=1}^{\infty} b_N \left(\sum_{n=1}^{N} EX_{N,n}^2 I\left(X_{N,n} \le \varepsilon\right) \right)^2 < \infty. \tag{2.21}$$

By Lemma 2.1 and (2.19)-(2.21), we have

$$\sum_{N=1}^{\infty} b_N P\left(\max_{1\le k\le N} |II_k| \ge r \cdot r_N (\log N)^{\alpha+2} \right) = \sum_{N=1}^{\infty} b_N P\left(\frac{((\log n)^\alpha / (n j_n^{\Delta-1})) R_n I(R_n > c_n)}{r_N (\log N)^{\alpha+2}} \ge r \right) < \infty. \tag{2.22}$$

Therefore, the desired result can be obtained by (2.18) and (2.22). □

Proof of Theorem 2.3. As the proof of Theorem 2.1, we partition $\sum_{n=1}^{k}((\log n)^\alpha/n)(R_n - ER_n I(1 \le R_n \le c_n))$ into the following two terms:

$$\begin{aligned}&\sum_{n=1}^{k}((\log n)^\alpha/n)(R_n - ER_n I(1 \le R_n \le c_n)) \\ &= \sum_{n=1}^{k}((\log n)^\alpha/n)[R_n I(1 \le R_n \le c_n) - ER_n I(1 \le R_n \le c_n)] \\ &\quad + \sum_{n=1}^{k}((\log n)^\alpha/n) R_n I(R_n > c_n) \\ &=: I_k + II_k.\end{aligned} \tag{2.23}$$

For the case $\Delta = 2$, $p_n j_n = 1$ and by Lemma 2.2, the density for the ratio of our order statistics is

$$\begin{aligned}f_{R_n}(x) &= \frac{p_n (i_n - 1)!}{(i_n - j_n - 1)!(j_n - 1)!} x^{-p_n j_n - 1} (1 - x^{-p_n})^{i_n - j_n - 1} I(x \ge 1) \\ &\sim j_n x^{-2} (1 - x^{-1/j_n}) I(x \ge 1).\end{aligned}$$

Then it follows that

$$ER_n^2 I(1 \le R_n \le c_n) \sim j_n \int_1^{c_n} (1 - x^{-1/j_n}) dx \le C j_n c_n \sim C n (\log n)^{\alpha+3}.$$

and

$$\begin{aligned}&P\left(\max_{1\le k\le N} \left| \sum_{n=1}^{k} ((\log n)^\alpha/n) \tilde{R}_n \right| > r_N (\log N)^{\alpha+3} \right) \\ &\le \sum_{n=1}^{N} \frac{((\log n)^\alpha/n)^2}{r_N^2 (\log N)^{2(\alpha+3)}} Var[R_n I(1 \le R_n \le c_n)] \\ &\le \sum_{n=1}^{N} \frac{((\log n)^\alpha/n)^2}{r_N^2 (\log N)^{2(\alpha+3)}} n (\log n)^{\alpha+3} \le C \frac{\log\log N}{r_N^2 (\log N)^{2-\alpha}}\end{aligned} \tag{2.24}$$

where

$$\tilde{R}_n := R_n I(1 \le R_n \le c_n) - ER_n I(1 \le R_n \le c_n).$$

From the condition (2.6), we can get

$$\sum_{N=1}^{\infty} b_N P\left(\max_{1\le k\le N} |I_k| \ge r_N(\log N)^{\alpha+3}\right) < \infty. \tag{2.25}$$

Next, we consider the complete convergence of the term II_k, from using Lemma 2.1. By denoting

$$X_{N,n} := \frac{((\log n)^{\alpha}/n)R_n I(R_n > c_n)}{r_N(\log N)^{\alpha+3}},$$

then for any $r > 0$, we have

$$\begin{aligned}\sum_{n=1}^{N} P\left(X_{N,n} \ge r\right) &\le \sum_{n=1}^{N} P\left(R_n \ge \max\left\{c_n, nr \cdot r_N(\log N)^{\alpha+3}/(\log n)^{\alpha}\right\}\right)\\ &\le C\sum_{n=1}^{N} \max\left\{\frac{j_n}{c_n}, \frac{(\log n)^{\alpha+a}}{nr_N(\log N)^{\alpha+3}}\right\}\\ &\le C\sum_{n=1}^{N} \max\left\{\frac{1}{n(\log n)^{3-a}}, \frac{(\log n)^{\alpha+a}}{nr_N(\log N)^{\alpha+3}}\right\}\\ &\le \frac{C\log\log N}{r_N(\log N)^{2-a}}\end{aligned}$$

which implies

$$\sum_{N=1}^{\infty} b_N \sum_{n=1}^{N} P\left(X_{N,n} \ge r\right) < \infty. \tag{2.26}$$

Furthermore, for any $\varepsilon > 0$, we have

$$\begin{aligned}\sum_{n=1}^{N} EX_{N,n} I\left(X_{N,n} \le \varepsilon\right) &\le C\frac{\log\log N + \log r_N}{r_N(\log N)^{\alpha+3}} \sum_{n=1}^{N} \frac{(\log n)^{\alpha+a}}{n}\\ &\le C\frac{(\log\log N)(\log\log N + \log r_N)}{r_N(\log N)^{2-a}} \to 0\end{aligned} \tag{2.27}$$

as $N \to \infty$, since the condition (2.5). Then

$$\sum_{n=1}^{N} EX^2_{N,n} I\left(X_{N,n} \le \varepsilon\right) \le C\sum_{n=1}^{N} \frac{(\log n)^{\alpha+a}/n}{r_N(\log N)^{\alpha+3}} \le \frac{C\log\log N}{r_N(\log N)^{2-a}},$$

and

$$\sum_{N=1}^{\infty} b_N \left(\sum_{n=1}^{N} EX^2_{N,n} I\left(X_{N,n} \le \varepsilon\right)\right)^2 < \infty. \tag{2.28}$$

By Lemma 2.1 and (2.26)-(2.28), we get

$$\begin{aligned}&\sum_{N=1}^{\infty} b_N P\left(\max_{1\le k\le N} |II_k| \ge r \cdot r_N(\log N)^{\alpha+3}\right)\\ &= \sum_{N=1}^{\infty} b_N P\left(\frac{((\log n)^{\alpha}/n)R_n I(R_n > c_n)}{r_N(\log N)^{\alpha+3}} \ge r\right) < \infty.\end{aligned} \tag{2.29}$$

Therefore, the desired result can be obtained by (2.25) and (2.29). □

3 Corollaries and examples

In this section, we investigate some corollaries and examples for the complete convergence theorems in Section 2. Adler [10] examined strong laws involving weighted sums of $\{R_n, n \ge 1\}$, and get some unusual limit theorems. Our corollaries show the complete convergence about them. And then, we search some examples to enhance the persuasive of the conclusion for specific r_N and b_N, $N \ge 1$.

Corollary 3.1. *Let $\{b_N, N \geq 1\}$ and $\{r_N, N \geq 1\}$ satisfy the condition (2.1) and (2.2). If $p_n j_n = 1, \Delta = 1$ and $\alpha > -2$, then*

$$\sum_{N=1}^{\infty} b_N P\left(\left|\sum_{n=1}^{N} \frac{((\log n)^{\alpha}/n)R_n}{(\log N)^{\alpha+2}} - \frac{1}{\alpha+2}\right| \geq r_N\right) < \infty. \tag{3.1}$$

Proof. We can check directly the conditions in the proof of Theorem 2.1 to get the corollary, since

$$ER_n I(1 \leq R_n \leq c_n) = \log c_n \sim \log n$$

and

$$\frac{\sum_{n=1}^{N}((\log n)^{\alpha}/n)ER_n I(1 \leq R_n \leq c_n)}{(\log N)^{\alpha+2}} \sim \frac{1}{\alpha+2}.$$

□

Example 3.1. *For $p_n j_n = 1, \Delta = 1$ and $\alpha > -2$, let $r_N = N$ and $b_N = \frac{1}{(\log N)^{\delta}}$ for any $\delta > 0$. Then we can get*

$$\sum_{N=1}^{\infty} \frac{1}{(\log N)^{\delta}} P\left(\left|\sum_{n=1}^{N} \frac{((\log n)^{\alpha}/n)R_n}{(\log N)^{\alpha+2}} - \frac{1}{\alpha+2}\right| \geq N\right) < \infty. \tag{3.2}$$

Example 3.2. *For $p_n j_n = 1, \Delta = 1$ and $\alpha > -2$, let $r_N = \frac{1}{\log\log N}$ and $b_N = \frac{1}{N(\log\log N)^{\delta+4}}$ for any $\delta > 0$. Then we can get*

$$\sum_{N=1}^{\infty} \frac{1}{N(\log\log N)^{\delta+4}} P\left(\left|\sum_{n=1}^{N} \frac{((\log n)^{\alpha}/n)R_n}{(\log N)^{\alpha+2}} - \frac{1}{\alpha+2}\right| \geq \frac{1}{\log\log N}\right) < \infty. \tag{3.3}$$

Example 3.3. *If $p_n j_n = 1, \Delta = 1$ and $\alpha > -2$, let $r_N = r$ be a constant and $b_N = \frac{1}{N(\log N)^{\delta}}$, where $0 < \delta \leq 1$. Then for $r > 0$, we have*

$$\sum_{N=1}^{\infty} \frac{1}{N(\log N)^{\delta}} P\left(\left|\sum_{n=1}^{N} \frac{((\log n)^{\alpha}/n)R_n}{(\log N)^{\alpha+2}} - \frac{1}{\alpha+2}\right| \geq r\right) < \infty. \tag{3.4}$$

Corollary 3.2. *For $p_n j_n = 1, \Delta \geq 2$ and $\alpha > -2$, let $\{b_N, N \geq 1\}$ and $\{r_N, N \geq 1\}$ satisfy the condition (2.1) and (2.2). If $j_n = o(\log n)$, then*

$$\sum_{N=1}^{\infty} b_N P\left(\left|\sum_{n=1}^{N} \frac{((\log n)^{\alpha}/(nj_n^{\Delta-1}))R_n}{(\log N)^{\alpha+2}} - \frac{1}{(\alpha+2)(\Delta-1)!}\right| \geq r_N\right) < \infty. \tag{3.5}$$

If $j_n \sim \log n$, then

$$\sum_{N=1}^{\infty} b_N P\left(\left|\sum_{n=1}^{N} \frac{((\log n)^{\alpha}/(nj_n^{\Delta-1}))R_n}{(\log N)^{\alpha+2}} - \frac{\gamma_{\Delta}}{(\alpha+2)(\Delta-1)!}\right| \geq r_N\right) < \infty \tag{3.6}$$

where

$$\gamma_{\Delta} = 1 + \sum_{k=1}^{\Delta-1} \frac{\binom{\Delta-1}{k}(-1)^k(1-e^{-k})}{k}.$$

Proof. For the case $j_n = o(\log n)$ and $c_n = nj_n^{\Delta-1}(\log n)^2$, we have

$$ER_n I(1 \leq R_n \leq c_n) \sim \frac{j_n^{\Delta-1}\log n}{(\Delta-1)!}$$

and

$$\frac{\sum_{n=1}^{N}((\log n)^{\alpha}/(nj_n^{\Delta-1}))ER_n I(1 \leq R_n \leq c_n)}{(\log N)^{\alpha+2}} \sim \frac{1}{(\alpha+2)(\Delta-1)!}.$$

By checking directly the conditions in the proof of Theorem 2.2, the conclusion (3.5) can be obtained.

For the case $j_n \sim \log n$, let $c_n = n(\log n)^{\Delta+1}$, then it is not difficult to check that $c_n^{-1/j_n} \to e^{-1}$ as $n \to \infty$. Hence we get

$$
\begin{aligned}
ER_nI(1 \le R_n \le c_n) &\sim \frac{j_n^{\Delta-1}}{(\Delta-1)!}\int_1^{c_n} x^{-1}(1-x^{-1/j_n})^{\Delta-1}\,dx \\
&= \frac{j_n^{\Delta-1}}{(\Delta-1)!}\int_1^{c_n} x^{-1}\sum_{k=0}^{\Delta-1}\binom{\Delta-1}{k}(-1)^k x^{-k/j_n}\,dx \\
&= \frac{j_n^{\Delta-1}}{(\Delta-1)!}\left[\log c_n + \sum_{k=1}^{\Delta-1}\binom{\Delta-1}{k}(-1)^k (k/j_n)^{-1}(1-c_n^{-k/j_n})\right] \\
&\sim \frac{(\log n)^{\Delta-1}}{(\Delta-1)!}\left[\log n + \sum_{k=1}^{\Delta-1}\binom{\Delta-1}{k}(-1)^k \frac{\log n}{k}(1-e^{-k})\right] \\
&= \frac{(\log n)^{\Delta}}{(\Delta-1)!}\left[1 + \sum_{k=1}^{\Delta-1}\frac{1}{k}\binom{\Delta-1}{k}(-1)^k (1-e^{-k})\right] \\
&= \frac{\gamma_\Delta(\log n)^{\Delta}}{(\Delta-1)!}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\sum_{n=1}^N((\log n)^\alpha/(nj_n^{\Delta-1}))ER_nI(1 \le R_n \le c_n)}{(\log N)^{\alpha+2}} &\sim \frac{\sum_{n=1}^N((\log n)^\alpha/(nj_n^{\Delta-1}))\gamma_\Delta(\log n)^\Delta}{(\log N)^{\alpha+2}(\Delta-1)!} \\
&\sim \frac{\gamma_\Delta}{(\alpha+2)(\Delta-1)!}.
\end{aligned}
$$

By checking the conditions in the proof of Theorem 2.2 where $j_n \sim \log n$, the conclusion (3.6) can be obtained.

□

Example 3.4. *For $p_nj_n = 1, \Delta \ge 2$ and $\alpha > -2$, let $r_N = N$ and $b_N = \frac{1}{(\log N)^\delta}$ for any $\delta > 0$. If $j_n = o(\log n)$, then*

$$\sum_{N=1}^\infty \frac{1}{(\log N)^\delta} P\left(\left|\sum_{n=1}^N \frac{((\log n)^\alpha/(nj_n^{\Delta-1}))R_n}{(\log N)^{\alpha+2}} - \frac{1}{(\alpha+2)(\Delta-1)!}\right| \ge N\right) < \infty.$$

If $j_n \sim \log n$, then

$$\sum_{N=1}^\infty \frac{1}{(\log N)^\delta} P\left(\left|\sum_{n=1}^N \frac{((\log n)^\alpha/(nj_n^{\Delta-1}))R_n}{(\log N)^{\alpha+2}} - \frac{\gamma_\Delta}{(\alpha+2)(\Delta-1)!}\right| \ge N\right) < \infty$$

where

$$\gamma_\Delta = 1 + \sum_{k=1}^{\Delta-1}\frac{\binom{\Delta-1}{k}(-1)^k(1-e^{-k})}{k}.$$

Example 3.5. *For $p_nj_n = 1, \Delta \ge 2$ and $\alpha > -2$, let $r_N = \frac{1}{\log\log N}$ and $b_N = \frac{1}{N(\log\log N)^{\delta+4}}$ for any $\delta > 0$. If $j_n = o(\log n)$, then*

$$\sum_{N=1}^\infty \frac{1}{N(\log\log N)^{\delta+4}} P\left(\left|\sum_{n=1}^N \frac{((\log n)^\alpha/(nj_n^{\Delta-1}))R_n}{(\log N)^{\alpha+2}} - \frac{1}{(\alpha+2)(\Delta-1)!}\right| \ge \frac{1}{\log\log N}\right) < \infty.$$

If $j_n \sim \log n$, then

$$\sum_{N=1}^\infty \frac{1}{N(\log\log N)^{\delta+4}} P\left(\left|\sum_{n=1}^N \frac{((\log n)^\alpha/(nj_n^{\Delta-1}))R_n}{(\log N)^{\alpha+2}} - \frac{\gamma_\Delta}{(\alpha+2)(\Delta-1)!}\right| \ge \frac{1}{\log\log N}\right) < \infty.$$

Example 3.6. *Let $p_n j_n = 1, \Delta \geq 2$ and $\alpha > -2$. Let $r_N = r$ be a constant and $b_N = \frac{1}{N(\log N)^\delta}$, where $0 < \delta \leq 1$. If $j_n = o(\log n)$, then for $r > 0$,*

$$\sum_{N=1}^{\infty} \frac{1}{N(\log N)^\delta} P\left(\left| \sum_{n=1}^{N} \frac{((\log n)^\alpha/(n j_n^{\Delta-1}))R_n}{(\log N)^{\alpha+2}} - \frac{1}{(\alpha+2)(\Delta-1)!} \right| \geq r \right) < \infty.$$

If $j_n \sim \log n$, then

$$\sum_{N=1}^{\infty} \frac{1}{N(\log N)^\delta} P\left(\left| \sum_{n=1}^{N} \frac{((\log n)^\alpha/(n j_n^{\Delta-1}))R_n}{(\log N)^{\alpha+2}} - \frac{\gamma_\Delta}{(\alpha+2)(\Delta-1)!} \right| \geq r \right) < \infty.$$

Corollary 3.3. *Let $\{b_N, N \geq 1\}$ and $\{r_N, N \geq 1\}$ satisfy the condition (2.5) and (2.6). If $p_n j_n = 1, j_n \sim (\log n)^a$, where $1 < a < 2, \Delta = 2$ and $\alpha > -3$, then*

$$\sum_{N=1}^{\infty} b_N P\left(\left| \sum_{n=1}^{N} \frac{((\log n)^\alpha/n)R_n}{(\log N)^{\alpha+3}} - \frac{1}{2(\alpha+3)} \right| \geq r_N \right) < \infty.$$

Proof. We can check directly the conditions in the proof of Theorem 2.3 to get the corollary. Since

$$\begin{aligned} ER_n I(1 \leq R_n \leq c_n) &\sim j_n \int_1^{c_n} x^{-1}(1 - x^{-1/j_n})dx \\ &= j_n \int_1^{c_n} (x^{-1} - x^{-1-1/j_n})dx \\ &= j_n[\log c_n + j_n(c_n^{-1/j_n} - 1)] \\ &\sim (\log n)^a(\log n + 3\log\log n) + (\log n)^{2a}[(n(\log n)^3)^{-1/(\log n)^a} - 1], \end{aligned}$$

by Lemma 2.3 with $t = \log n$, we have

$$\begin{aligned} ER_n I(1 \leq R_n \leq c_n) &\sim t^a(t + 3\log t) + t^{2a}[(e^t t^3)^{-1/t^a} - 1] \\ &= t^2 t^{a-1}[1 + 3t^{-1}\log t + t^{a-1}((e^t t^3)^{-1/t^a} - 1)] \\ &\sim t^2/2 = (\log n)^2/2. \end{aligned}$$

Thus we have

$$\frac{\sum_{n=1}^{N} ((\log n)^\alpha/n) ER_n I(1 \leq R_n \leq c_n)}{(\log N)^{\alpha+3}} \sim \frac{1}{2(\alpha+3)}. \tag{3.7}$$

Then the conclusion can be obtained. □

Example 3.7. *For $p_n j_n = 1, j_n \sim (\log n)^a$, where $1 < a < 2, \Delta = 2$ and $\alpha > -3$, let $r_N = N$ and $b_N = \frac{1}{(\log N)^\delta}$ for any $\delta > a - 1$. Then we have*

$$\sum_{N=1}^{\infty} \frac{1}{(\log N)^\delta} P\left(\left| \sum_{n=1}^{N} \frac{((\log n)^\alpha/n)R_n}{(\log N)^{\alpha+3}} - \frac{1}{2(\alpha+3)} \right| \geq N \right) < \infty.$$

Example 3.8. *For $p_n j_n = 1, j_n \sim (\log n)^a$, where $1 < a < 2, \Delta = 2$ and $\alpha > -3$, let $r_N = \frac{1}{\log\log N}$ and $b_N = \frac{1}{N(\log N)^{a-1}(\log\log N)^{\delta+4}}$ for any $\delta > 0$. Then we have*

$$\sum_{N=1}^{\infty} \frac{1}{N(\log N)^{a-1}(\log\log N)^{\delta+4}} P\left(\left| \sum_{n=1}^{N} \frac{((\log n)^\alpha/n)R_n}{(\log N)^{\alpha+3}} - \frac{1}{2(\alpha+3)} \right| \geq \frac{1}{\log\log N} \right) < \infty.$$

Example 3.9. *If $p_n j_n = 1, j_n \sim (\log n)^a$, where $1 < a < 2, \Delta = 2$ and $\alpha > -3$, let $r_N = r$ be a constant and $b_N = \frac{1}{N(\log N)^\delta}$, where $a - 1 < \delta \leq 1$. Then for $r > 0$, we have*

$$\sum_{N=1}^{\infty} \frac{1}{N(\log N)^\delta} P\left(\left| \sum_{n=1}^{N} \frac{((\log n)^\alpha/n)R_n}{(\log N)^{\alpha+3}} - \frac{1}{2(\alpha+3)} \right| \geq r \right) < \infty.$$

4 Conclusion

The work examines some limit theorems for the order statistics from the Pareto distribution proposed in current work, and investigates the complete convergence for the ratios of the order statistics. In this paper, we firstly get the complete convergence of the weighted sums with different forms by discussing three different cases. Then we obtain some corollaries. Some examples are also given to support the conclusion. There are more relevant properties regarding order statistics that will be investigated by us in the future.

Acknowledgement: The authors are very grateful to the referees for their valuable reports which improved the presentation of this work.

This work is supported by IRTSTHN (14IRTSTHN023), NSFC (11471104).

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