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Research Article

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Parity results for broken 11-diamond partitions

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Abstract: Recently, Dai proved new infinite families of congruences modulo 2 for broken 11-diamond partition functions by using Hecke operators. In this note, we establish new parity results for broken 11-diamond partition functions. In particular, we generalize the congruences due to Dai by utilizing an identity due to Newman. Furthermore, we prove some strange congruences modulo 2 for broken 11-diamond partition functions. For example, we prove that if p is a prime with $p \neq 23$ and $\Delta_{11}(2p+1) \equiv 1 \pmod 2$, then for any $k \geq 0$, $\Delta_{11}(2p^{3k+3}+1) \equiv 1 \pmod 2$, where $\Delta_{11}(n)$ is the number of broken 11-diamond partitions of n.

Keywords: broken k-diamond partition, congruence, parity results

MSC: 11P83, 05A17

1 Introduction

A combinatorial study guided by MacMahon's Partition Analysis led Andrews and Paule [1] to the construction of a new class of directed graphs called broken k-diamond partitions. For a fixed positive integer k, let $\Delta_k(n)$ denote the number of broken k-diamond partitions of n. Moreover, the following generating functions for $\Delta_k(n)$ was established by Andrews and Paule [1]:

$$\sum_{n=0}^{\infty} \Delta_k(n) q^n = \frac{(q^2;q^2)_\infty (q^{2k+1};q^{2k+1})_\infty}{(q;q)_\infty^3 (q^{4k+2};q^{4k+2})_\infty},$$

where

$$(q;q)_\infty := \prod_{n=1}^{\infty}(1-q^n).$$

Andrews and Paule [1] also proved that for $n \geq 0$,

$$\Delta_1(2n+1) \equiv 0 \pmod 3$$

by employing generating function manipulations. In addition, they also gave three conjectures modulo 2, 5 and 25 for $\Delta_k(n)$. Since then, a number of congruences for $\Delta_k(n)$ for small integers k have been proved. See, for example, [2–12].

In 2010, by using theory of modular forms, Radu and Sellers [4] proved some parity results for $\Delta_k(n)$, where $k \in \{2, 3, 5, 6, 8, 9, 11\}$. For example, they proved that for $n \geq 0$,

$$\Delta_{11}(46n+j) \equiv 0 \pmod 2.$$

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where $j \in \{11, 15, 21, 23, 29, 31, 35, 39, 41, 43, 45\}$. Recently, Yao [12] proved several infinite families of congruences modulo 2 for $\Delta_{11}(n)$ which generalize Radu and Sellers' congruences on $\Delta_{11}(n)$. Ahmed and Baruah [13] also proved some congruences modulo 2 for $\Delta_{11}(n)$. Very recently, Dai [2] established more parity results for $\Delta_{11}(n)$. Let p be a prime with $p \neq 23$ and let $n,\ k \geq 0$ be integers. He proved that if $\Delta_{11}(2p+1)$ is even, then

$$\Delta_{11}(2p^{2k+2}n + 2rp^{2k+1} + 1) \equiv 0 \pmod 2 \tag{1.1}$$

for $1 \leq r \leq p-1$ and if $\Delta_{11}(2p+1)$ is odd, then

$$\Delta_{11}(2p^{3k+3}n + 2rp^{3k+2} + 1) \equiv 0 \pmod 2 \tag{1.2}$$

for $1 \leq r \leq p-1$.

In this paper, we prove new congruences modulo 2 for $\Delta_{11}(n)$ by employing an identity due to Newman [14]. In particular, our results generalize (1.1) and (1.2). Moreover, we also prove some strange congruences modulo 2 for $\Delta_{11}(n)$. The main results of this paper can be stated as follows.

Theorem 1.1. *Let k be a nonnegative integer and let p_i $(1 \leq i \leq k+1)$ be primes with $p_i \neq 23$ (p_i may equal to 2). For $n \geq 0$,*

$$\Delta_{11}(2p_1^{\alpha(p_1)} \cdots p_k^{\alpha(p_k)} p_{k+1}^{\alpha(p_{k+1})} n + 2rp_1^{\alpha(p_1)} \cdots p_k^{\alpha(p_k)} p_{k+1}^{\alpha(p_{k+1})-1} + 1) \equiv 0 \pmod 2, \tag{1.3}$$

where $1 \leq r \leq p_{k+1} - 1$ *and*

$$\alpha(p_i) := \begin{cases} 2, & \text{if } \Delta_{11}(2p_i+1) \text{ is even,} \\ 3, & \text{if } \Delta_{11}(2p_i+1) \text{ is odd.} \end{cases} \tag{1.4}$$

Moreover,

$$\Delta_{11}(2p_1^{\alpha(p_1)} \cdots p_k^{\alpha(p_k)} p_{k+1}^{\alpha(p_{k+1})} + 1) \equiv 1 \pmod 2. \tag{1.5}$$

If we set $p_1 = p_2 = \cdots = p_{k+1} = p$ in (1.3), we get (1.1) and (1.2). Moreover, if we set $k = 1$, $p_1 = 3$ and $p_2 = 5$ in (1.3), we find that for $n \geq 0$,

$$\Delta_{11}(1350n + 270r + 1) \equiv 0 \pmod 2, \tag{1.6}$$

where $1 \leq r \leq 4$. Note that Dai's congruences (1.1) and (1.2) do not imply (1.6). Therefore, (1.3) is a generalization of (1.1) and (1.2). Moreover, if set $p_1 = p_2 = \cdots = p_{k+1} = p$ in (1.5), we deduce that if $\Delta_{11}(2p+1) \equiv 0 \pmod 2$, then for $k \geq 0$,

$$\Delta_{11}(2p^{2k+2} + 1) \equiv 1 \pmod 2$$

and if $\Delta_{11}(2p+1) \equiv 1 \pmod 2$, then for $k \geq 0$,

$$\Delta_{11}(2p^{3k+3} + 1) \equiv 1 \pmod 2.$$

2 Two Lemmas

In order to prove Theorem 1.1, we prove the following two lemmas in this section.

Lemma 2.1. *Let $c(n)$ be defined by*

$$\sum_{n=0}^{\infty} c(n)q^n := (q;q)_\infty (q^{23};q^{23})_\infty \tag{2.1}$$

and let p be an odd prime with $p \neq 23$*. If* $c(p-1) \equiv 0 \pmod 2$*, then for* $n \geq 0$*,*

$$c(p^2n + p^2 - 1) \equiv c(n) \pmod 2 \tag{2.2}$$

and

$$c(p^2n + rp - 1) \equiv 0 \pmod 2, \tag{2.3}$$

where r is an integer and $1 \leq r \leq p-1$*.*

Proof. Newman [14] proved that

$$c(pn + p - 1) = c(p-1)c(n) - (-1)^{p-1}\left(\frac{23}{p}\right)c((n-p+1)/p), \tag{2.4}$$

where p is an odd prime with $p \neq 23$ and $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol. Identity (2.4) implies that

$$c(pn + p - 1) \equiv c(p-1)c(n) + c((n-p+1)/p) \pmod 2. \tag{2.5}$$

Therefore, if $c(p-1) \equiv 0 \pmod 2$, then

$$c(pn + p - 1) \equiv c((n-p+1)/p) \pmod 2. \tag{2.6}$$

Replacing n by $pn + p - 1$ in (2.6), we arrive at (2.2). Note that for $0 \leq j \leq p-2$, $\frac{pn+j-p+1}{p}$ is not an integer and

$$c((pn + j - p + 1)/p) = 0. \tag{2.7}$$

Replacing n by $pn + j$ $(0 \leq j \leq p-2)$ in (2.6) and using (2.7), we deduce that for $n \geq 0$,

$$c(p^2n + (j+1)p - 1) \equiv 0 \pmod 2, \qquad 0 \leq j \leq p-2,$$

which is nothing but (2.3). This completes the proof. □

Lemma 2.2. *Let p be an odd prime with* $p \neq 23$*. If* $c(p-1) \equiv 1 \pmod 2$*, then for* $n \geq 0$*,*

$$c(p^3n + p^3 - 1) \equiv c(n) \pmod 2 \tag{2.8}$$

and

$$c(p^3n + rp^2 - 1) \equiv 0 \pmod 2, \tag{2.9}$$

where r is an integer and $1 \leq r \leq p-1$*.*

Proof. If $c(p-1) \equiv 1 \pmod 2$, then we can rewrite (2.5) as

$$c(pn + p - 1) \equiv c(n) + c((n-p+1)/p) \pmod 2. \tag{2.10}$$

Replacing n by $pn + p - 1$ in (2.10) yields

$$c(p^2n + p^2 - 1) \equiv c(pn + p - 1) + c(n) \pmod 2. \tag{2.11}$$

Substituting (2.10) into (2.11) yields

$$c(p^2n + p^2 - 1) \equiv c((n-p+1)/p) \pmod 2. \tag{2.12}$$

Replacing n by $pn+p-1$ in (2.12), we arrive at (2.8). Replacing n by $pn+j$ $(0 \leq j \leq p-2)$ in (2.12) and employing (2.7), we deduce that for $n \geq 0$,

$$c(p^3n + (j+1)p^2 - 1) \equiv 0 \pmod 2, \qquad 0 \leq j \leq p-2,$$

which yields (2.9). The proof is complete. □

3 Proof of Theorem 1.1

In [12], Yao established the generating function for $\Delta_{11}(2n+1)$ modulo 2:

$$\sum_{n=0}^{\infty} \Delta_{11}(2n+1)q^n \equiv 1 + q(q;q)_\infty (q^{23};q^{23})_\infty \pmod 2. \tag{3.1}$$

In view of (2.1) and (3.1),

$$\sum_{n=0}^{\infty} \Delta_{11}(2n+1)q^n \equiv 1 + \sum_{n=0}^{\infty} c(n)q^{n+1} \pmod 2.$$

Therefore, for $n \geq 0$,

$$\Delta_{11}(2n+3) \equiv c(n) \pmod 2. \tag{3.2}$$

Yao [12] also proved that for $n \geq 0$,

$$\Delta_{11}(16n+1) \equiv \Delta_{11}(2n+1) \pmod 2. \tag{3.3}$$

and

$$\Delta_{11}(16n+9) \equiv 0 \pmod 2. \tag{3.4}$$

Let k be a nonnegative integer and let p_i $(1 \leq i \leq k+1)$ be odd primes with $p_i \neq 23$. By (3.2) and Lemmas 2.1 and 2.2, we see that for $n \geq 0$

$$\Delta_{11}(2p_i^{\alpha(p_i)}n + 2p_i^{\alpha(p_i)} + 1) \equiv \Delta_{11}(2n+3) \pmod 2 \tag{3.5}$$

and

$$\Delta_{11}(2p_{t+1}^{\alpha(p_{t+1})}n + 2rp_{t+1}^{\alpha(p_{t+1})-1} + 1) \equiv 0 \pmod 2, \tag{3.6}$$

where $1 \leq r \leq p_{t+1} - 1$ and $\alpha(p_i)$ is defined by (1.4). By (3.3), (3.4) and the fact that $\Delta_{11}(5) \equiv 1 \pmod 2$, we find that (3.5) holds when $p_i = 2$ and (3.6) is also true when $p_{t+1} = 2$. Replacing n by $n-1$ in (3.5), we find that for $n \geq 1$,

$$\Delta_{11}(2p_i^{\alpha(p_i)}n + 1) \equiv \Delta_{11}(2n+1) \pmod 2. \tag{3.7}$$

Therefore, by (3.7) and iterative method, we see that for $n \geq 0$,

$$\Delta_{11}(2p_1^{\alpha(p_i)}p_2^{\alpha(p_2)} \cdots p_k^{\alpha(p_k)}n + 1) \equiv \Delta_{11}(2n+1) \pmod 2, \tag{3.8}$$

where p_i are primes with $p_i \neq 23$ (p_i may equal to 2) for $1 \leq i \leq k$. Replacing n by $p_{k+1}^{\alpha(p_{k+1})} + rp_{k+1}^{\alpha(p_{k+1})-1}$ in (3.8) and employing (3.6), we arrive at (1.3).

Setting $n = 0$ and $i = k+1$ in (3.5) and using the fact that $\Delta_{11}(3) \equiv 1 \pmod 2$, we have

$$\Delta_{11}(2p_{k+1}^{\alpha(p_{k+1})} + 1) \equiv 1 \pmod 2. \tag{3.9}$$

Setting $n = p_{k+1}^{\alpha(p_{k+1})}$ in (3.8) and using (3.9), we get (1.5). This completes the proof. □

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