DE GRUYTER Open Math. 2019; 17:385–401

Open Mathematics

Research Article

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Dynamics of two-species delayed competitive stage-structured model described by differential-difference equations

https://doi.org/10.1515/math-2019-0030
Received July 4, 2018; accepted February 20, 2019

Abstract: Over the last few years, by utilizing Mawhin's continuation theorem of coincidence degree theory and Lyapunov functional, many scholars have been concerned with the global asymptotical stability of positive periodic solutions for the non-linear ecosystems. In the real world, almost periodicity is usually more realistic and more general than periodicity, but there are scarcely any papers concerning the issue of the global asymptotical stability of positive almost periodic solutions of non-linear ecosystems. In this paper we consider a kind of delayed two-species competitive model with stage structure. By means of Mawhin's continuation theorem of coincidence degree theory, some sufficient conditions are obtained for the existence of at least one positive almost periodic solutions for the above model with nonnegative coefficients. Furthermore, the global asymptotical stability of positive almost periodic solution of the model is also studied. The work of this paper extends and improves some results in recent years. An example and simulations are employed to illustrate the main results of this paper.

Keywords: Stage structure, Almost periodic solution, Coincidence degree, Competitive model, Stability

MSC: 34K13; 34K20; 92B05; 92D25

1 Introduction

In [1], Zeng et al. proposed the following two-species competitive model with stage structure:

$$\begin{cases} \dot{x}_1(t) = -a_1(t)x_1(t) + b_1(t)x_2(t), \\ \dot{x}_2(t) = a_2(t)x_2(t) - b_2(t)x_2(t) - c(t)x_2^2(t) - \beta_1(t)x_2(t)x_3(t), \\ \dot{x}_3(t) = x_3(t)\left[a_3(t) - b_3(t)x_3(t) - \beta_2(t)x_2(t)\right], \end{cases} \tag{1.1}$$

where x_1 and x_2 are immature and mature population densities of one species, respectively, and x_3 represents the population density of another species. The competition is between x_2 and x_3. By means of the fixed point theory and Lyapunv functional, Zeng et al. studied the existence and uniqueness of globally attractive positive T-periodic solution of system (1.1).

In real world, almost periodicity is more realistic and more general than periodicity. Therefore, more and more scholars have focused on the study of almost periodic dynamics of non-linear ecosystem [2-6]. Moreover, population models with delays have attracted much attention in recent years. Time delays represent an

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additional level of complexity that can be incorporated in a more detailed analysis of a particular system. So, this article is to consider the following delayed two-species almost periodic competitive model with stage structure:

$$\begin{cases} \dot{x}_1(t) = -a_1(t)x_1(t) + b_1(t)x_2(t - \tau(t)), \\ \dot{x}_2(t) = a_2(t)x_2(t) - b_2(t)x_2(t) - c(t)x_2^2(t) - \beta_1(t)x_2(t)x_3(t - \delta(t)), \\ \dot{x}_3(t) = x_3(t)\left[a_3(t) - b_3(t)x_3(t) - \beta_2(t)x_2(t - \sigma(t))\right], \end{cases} \tag{1.2}$$

where a_i, b_i, β_j, c, τ, δ and σ are all nonnegative almost periodic functions, $i = 1, 2, 3$, $j = 1, 2$.

Let $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}^+$ denote the sets of real numbers, integers and positive integers, respectively, $C(\mathbb{X}, \mathbb{Y})$ and $C^1(\mathbb{X}, \mathbb{Y})$ be the space of continuous functions and continuously differential functions which map $\mathbb{X}$ into $\mathbb{Y}$, respectively. Especially, $C(\mathbb{X}) := C(\mathbb{X}, \mathbb{X})$, $C^1(\mathbb{X}) := C^1(\mathbb{X}, \mathbb{X})$. In relation to a continuous bounded function f, we use the following notations:

$$f^- = \inf_{s\in\mathbb{R}} f(s), \quad f^+ = \sup_{s\in\mathbb{R}} f(s), \quad |f|_\infty = \sup_{s\in\mathbb{R}} |f(s)|, \quad \bar{f} = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(s)\,\mathrm{d}s.$$

Throughout this paper, we always make the following assumption for system (1.2):

$(H_1)\bar{a}_1 > 0$, $\bar{a}_2 > \bar{b}_2$, $\bar{a}_3 > 0$, $\bar{b}_1 > 0$ and $\bar{b}_3 > 0$.

The main purpose of this article is to study the existence and global asymptotic stability of positive almost periodic solution of system (1.2) by using the coincidence degree theory and Lyapunov functional. Finally, a example and some simulations are also given to illustrate the main results.

2 Preliminaries

Definition 2.1. ([7, 8]) $x \in C(\mathbb{R}, \mathbb{R}^n)$ is called almost periodic, if for any $\epsilon > 0$, it is possible to find a real number $l = l(\epsilon) > 0$, for any interval with length $l(\epsilon)$, there exists a number $\tau = \tau(\epsilon)$ in this interval such that $\|x(t + \tau) - x(t)\| < \epsilon$, $\forall t \in \mathbb{R}$, where $\|\cdot\|$ is arbitrary norm of $\mathbb{R}^n$. τ is called to the ϵ-almost period of x, $T(x, \epsilon)$ denotes the set of ϵ-almost periods for x and $l(\epsilon)$ is called to the length of the inclusion interval for $T(x, \epsilon)$. The collection of those functions is denoted by $AP(\mathbb{R}, \mathbb{R}^n)$. Let $AP(\mathbb{R}) := AP(\mathbb{R}, \mathbb{R})$.

Lemma 2.1. ([5]) *If $x \in AP(\mathbb{R})$ is differentiable, then for $\forall\epsilon > 0$, it follows*:

(1) *there exists $\xi_\epsilon \in [0, +\infty)$ such that $x(\xi_\epsilon) \in [x^+ - \epsilon, x^+]$ and $x'(\xi_\epsilon) = 0$;*
(2) *there exists $\eta_\epsilon \in [0, +\infty)$ such that $x(\eta_\epsilon) \in [x^-, x^- + \epsilon]$ and $x'(\eta_\epsilon) = 0$.*

Lemma 2.2. ([6]) *If $x \in AP(\mathbb{R})$ is differentiable, for any interval $[a, b]$ with $b - a > 0$, let $\xi, \eta \in [a, b]$ and*

$$I_1 = \left\{s \in [\xi, b] : \dot{x}(s) \geq 0\right\}, \quad I_2 = \left\{s \in [\eta, b] : \dot{x}(s) \leq 0\right\},$$

then

$$x(t) \leq x(\xi) + \int_{I_1} \dot{x}(s)\,\mathrm{d}s, \quad \forall t \in [\xi, b], \quad x(t) \geq x(\eta) + \int_{I_2} \dot{x}(s)\,\mathrm{d}s, \quad \forall t \in [\eta, b].$$

Lemma 2.3. ([6]) *Assume that $x \in AP(\mathbb{R})$, there exist $\xi \in [a, b]$, $\underline{\xi} \in (-\infty, a]$ and $\bar{\xi} \in [b, +\infty)$ such that*

$$x(\underline{\xi}) = x(\bar{\xi}) \quad \text{and} \quad x(\xi) \leq x(s), \quad \forall s \in [\underline{\xi}, \bar{\xi}].$$

Lemma 2.4. ([6]) *Assume that $x \in AP(\mathbb{R})$, there exist $\eta \in [a, b]$, $\underline{\eta} \in (-\infty, a]$ and $\bar{\eta} \in [b, +\infty)$ such that*

$$x(\underline{\eta}) = x(\bar{\eta}) \quad \text{and} \quad x(\eta) \geq x(s), \quad \forall s \in [\underline{\eta}, \bar{\eta}].$$

Lemma 2.5. ([7]) *If* $f \in AP(\mathbb{R})$ *and* $\bar{f} = m(f) = \frac{1}{T}\int_0^T f(s)\,\mathrm{d}s > 0$, *then we have*

$$\frac{1}{T}\int_a^{a+T} f(s)\,\mathrm{d}s \in \left[\frac{\bar{f}}{2}, \frac{3\bar{f}}{2}\right], \quad \forall T \geq T_0,$$

where a is an arbitrary real constant and $T_0 > 0$ is a constant independent of a.

The method to be used in this paper involves the applications of the continuation theorem of coincidence degree.

Let $\mathbb{X}$ and $\mathbb{Y}$ be real Banach spaces, $L : \mathrm{Dom}L \subseteq \mathbb{X} \to \mathbb{Y}$ be a linear mapping and $N : \mathbb{X} \to \mathbb{Y}$ be a continuous mapping. The mapping L is called a Fredholm mapping of index zero if $\mathrm{Im}L$ is closed in $\mathbb{Y}$ and $\mathrm{dimKer}L = \mathrm{codimIm}L < +\infty$. If L is a Fredholm mapping of index zero and there exist continuous projectors $P : \mathbb{X} \to \mathbb{X}$ and $Q : \mathbb{Y} \to \mathbb{Y}$ such that $\mathrm{Im}P = \mathrm{Ker}L$, $\mathrm{Ker}Q = \mathrm{Im}L = \mathrm{Im}(I - Q)$. It follows that $L_P = L|_{\mathrm{Dom}L\cap\mathrm{Ker}P} : (I - P)\mathbb{X} \to \mathrm{Im}L$ is invertible and its inverse is denoted by K_P. If Ω is an open bounded subset of $\mathbb{X}$, the mapping N will be called L-compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_P(I - Q)N : \bar{\Omega} \to \mathbb{X}$ is compact. Since $\mathrm{Im}Q$ is isomorphic to $\mathrm{Ker}L$, there exists an isomorphism $J : \mathrm{Im}Q \to \mathrm{Ker}L$.

Lemma 2.6. ([9]) *Let* $\Omega \subseteq \mathbb{X}$ *be an open bounded set*, L *be a Fredholm mapping of index zero and* N *be* L*-compact on* $\bar{\Omega}$. *If all the following conditions hold*:

(a) $Lx \neq \lambda Nx$, $\forall x \in \partial\Omega \cap \mathrm{Dom}L$, $\lambda \in (0, 1)$;

(b) $QNx \neq 0$, $\forall x \in \partial\Omega \cap \mathrm{Ker}L$;

(c) $\deg\{JQN, \Omega \cap \mathrm{Ker}L, 0\} \neq 0$, *where* $\deg(\cdot, \cdot, \cdot)$ *is the Brouwer degree.*

Then $Lx = Nx$ *has a solution on* $\bar{\Omega} \cap \mathrm{Dom}L$.

For $f \in AP(\mathbb{R})$, we denote by

$$\Lambda(f) = \left\{\varpi \in \mathbb{R} : \lim_{T\to\infty} \frac{1}{T}\int_0^T f(s)e^{-\mathrm{i}\varpi s}\mathrm{d}s \neq 0\right\},$$

$$\mathrm{mod}(f) = \left\{\sum_{j=1}^{m} n_j\varpi_j : n_j \in \mathbb{Z}, m \in \mathbb{N}, \varpi_j \in \Lambda(f), j = 1, 2 \ldots, m\right\}$$

the set of Fourier exponents and the module of f, respectively.

3 Almost periodic solution

Let

$$f_2^+ := \ln\left[\frac{3(\bar{a}_2 - \bar{b}_2)}{\bar{c}}\right] + \frac{3(\bar{a}_2 - \bar{b}_2)\omega}{2}, \quad f_3^+ := \ln\left[\frac{3\bar{a}_3}{\bar{b}_3}\right] + \frac{3\bar{a}_3\omega}{2},$$

$$\mu(s) = a_2(s) - b_2(s) - \beta_1(s)e^{f_3^+}, \quad \nu(s) = a_3(s) - \beta_2(s)e^{f_2^+}, \quad \forall s \in \mathbb{R},$$

where ω is defined as that in (3.3).

Theorem 3.1. *Assume that* (H_1) *holds. Suppose further that*
$(H_2)\bar{\mu} > 0$ *and* $\bar{\nu} > 0$,

then system (1.2) *admits at least one positive almost periodic solution.*

Proof. Under the invariant transformation $(x_1, x_2, x_3)^T = (e^{y_1}, e^{y_2}, e^{y_3})^T$, system (1.2) reduces to

$$\begin{cases} \dot{y}_1(t) = -a_1(t) + \frac{b_1(t)e^{y_2(t-\tau(t))}}{e^{y_1(t)}} := F_1(t), \\ \dot{y}_2(t) = a_2(t) - b_2(t) - c(t)e^{y_2(t)} - \beta_1(t)e^{y_3(t-\delta(t))} := F_2(t), \\ \dot{y}_3(t) = a_3(t) - b_3(t)e^{y_3(t)} - \beta_2(t)e^{y_2(t-\sigma(t))} := F_3(t). \end{cases} \tag{3.0}$$

It is easy to see that if system (3.0) has one almost periodic solution $(y_1, y_2, y_3)^T$, then $(x_1, x_2, x_3)^T = (e^{y_1}, e^{y_2}, e^{y_3})^T$ is a positive almost periodic solution of system (1.2). Therefore, to complete the proof it suffices to show that system (3.0) has one almost periodic solution.

Take $\mathbb{X} = \mathbb{Y} = \mathbb{V}_1 \bigoplus \mathbb{V}_2$, where

$$\mathbb{V}_1 = \left\{ z = (y_1, y_2, y_3)^T \in AP(\mathbb{R}, \mathbb{R}^3) : \operatorname{mod}(y_i) \subseteq \operatorname{mod}(L_i), \forall \varpi \in \Lambda(y_i), |\varpi| \geq \theta_0, i = 1, 2, 3 \right\},$$

$$\mathbb{V}_2 = \left\{ z = (y_1, y_2, y_3)^T \equiv (k_1, k_2, k_3)^T, k_1, k_2, k_3 \in \mathbb{R} \right\},$$

where

$$L_1 = L_1(t, \varphi) = -a_1(t) + \frac{b_1(t)e^{\varphi_2(-\tau(0))}}{e^{\varphi_1(0)}},$$

$$L_2 = L_2(t, \varphi) = a_2(t) - b_2(t) - c(t)e^{\varphi_2(0)} - \beta_1(t)e^{\varphi_3(-\delta(0))},$$

$$L_3 = L_3(t, \varphi) = a_3(t) - b_3(t)e^{\varphi_3(0)} - \beta_2(t)e^{\varphi_2(-\sigma(0))},$$

$\varphi = (\varphi_1, \varphi_2, \varphi_3)^T \in C([-\tau_0, 0], \mathbb{R}^3)$, $\tau_0 = \max_{i=1,2}\{\tau^+, \delta^+, \sigma^+\}$, θ_0 is a given positive constant. Define the norm

$$\|z\| = \max \left\{ \sup_{s\in\mathbb{R}} |y_1(s)|, \sup_{s\in\mathbb{R}} |y_2(s)|, \sup_{s\in\mathbb{R}} |y_3(s)| \right\}, \quad \forall z = (y_1, y_2, y_3)^T \in \mathbb{X} = \mathbb{Y},$$

then $\mathbb{X}$ and $\mathbb{Y}$ are Banach spaces with the norm $\|\cdot\|$. Set

$$L : \operatorname{Dom}L \subseteq \mathbb{X} \to \mathbb{Y}, \quad Lz = L(y_1, y_2, y_3)^T = (y_1', y_2', y_3')^T,$$

where $\operatorname{Dom}L = \{z = (y_1, y_2, y_3)^T \in \mathbb{X} : y_1, y_2, y_3 \in C^1(\mathbb{R})\}$ and

$$N : \mathbb{X} \to \mathbb{Y}, \quad Nz = N \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix} = \begin{bmatrix} F_1(t) \\ F_2(t) \\ F_3(t) \end{bmatrix}.$$

With these notations, system (3.0) can be written in the form

$$Lz = Nz, \quad \forall z \in \mathbb{X}.$$

It is not difficult to verify that $\operatorname{Ker}L = \mathbb{V}_2$, $\operatorname{Im}L = \mathbb{V}_1$ is closed in $\mathbb{Y}$ and $\dim \operatorname{Ker}L = 3 = \operatorname{codim} \operatorname{Im}L$. Therefore, L is a Fredholm mapping of index zero (see Lemma 2.12 in [6]). Now define two projectors $P : \mathbb{X} \to \mathbb{X}$ and $Q : \mathbb{Y} \to \mathbb{Y}$ as

$$Pz = P \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} m(y_1) \\ m(y_2) \\ m(y_3) \end{bmatrix} = Qz, \quad \forall z = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \in \mathbb{X} = \mathbb{Y}.$$

Then P and Q are continuous projectors such that $\operatorname{Im}P = \operatorname{Ker}L$ and $\operatorname{Im}L = \operatorname{Ker}Q = \operatorname{Im}(I - Q)$.

Furthermore, through an easy computation we find that the inverse $K_P : \operatorname{Im}L \to \operatorname{Ker}P \cap \operatorname{Dom}L$ of L_P has the form

$$K_P z = K_P \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \int_0^t y_1(s)\,\mathrm{d}s - m\left[\int_0^t y_1(s)\,\mathrm{d}s\right] \\ \int_0^t y_2(s)\,\mathrm{d}s - m\left[\int_0^t y_2(s)\,\mathrm{d}s\right] \\ \int_0^t y_3(s)\,\mathrm{d}s - m\left[\int_0^t y_3(s)\,\mathrm{d}s\right] \end{bmatrix}, \quad \forall z = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \in \operatorname{Im}L.$$

Then $QN : \mathbb{X} \to \mathbb{Y}$ and $K_P(I - Q)N : \mathbb{X} \to \mathbb{X}$ read

$$QNz = QN \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} m(F_1) \\ m(F_2) \\ m(F_3) \end{bmatrix},$$

$$K_P(I - Q)Nz = K_P(I - Q)N \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} f[y_1(t)] - Qf[y_1(t)] \\ f[y_2(t)] - Qf[y_2(t)] \\ f[y_3(t)] - Qf[y_3(t)] \end{bmatrix}, \quad \forall z \in \operatorname{Im}L,$$

where $f(x)$ is defined by $f[x(t)] = \int_0^t \left[Nx(s) - QNx(s)\right] \mathrm{d}s$.ąď Then N is L-compact on $\bar{\Omega}$ (see Lemma 2.13 in [6]).

In order to apply Lemma 2.6, we need to search for an appropriate open-bounded subset Ω.

Corresponding to the operator equation $Lz = \lambda z$, $\lambda \in (0, 1)$, we have

$$\begin{cases} \dot{y}_1(t) = \lambda \left[-a_1(t) + \frac{b_1(t)e^{y_2(t-\tau(t))}}{e^{y_1(t)}}\right], \\ \dot{y}_2(t) = \lambda \left[a_2(t) - b_2(t) - c(t)e^{y_2(t)} - \beta_1(t)e^{y_3(t-\delta(t))}\right], \\ \dot{y}_3(t) = \lambda \left[a_3(t) - b_3(t)e^{y_3(t)} - \beta_2(t)e^{y_2(t-\sigma(t))}\right]. \end{cases} \tag{3.1}$$

Suppose that $z = (y_1, y_2, y_3)^T \in \operatorname{Dom}L \subseteq \mathbb{X}$ is a solution of system (3.1) for some $\lambda \in (0, 1)$, where $\operatorname{Dom}L = \{z = (y_1, y_2, y_3)^T \in \mathbb{X} : y_i \in C^1(\mathbb{R}), i = 1, 2, 3\}$.

There must exist $\{\alpha_n^{(i)} : n \in \mathbb{N}^+\}$ such that

$$y_i(\alpha_n^{(i)}) \in \left[y_i^+ - \frac{1}{n}, y_i^+\right], \quad y_i^+ = \sup_{s\in\mathbb{R}} y_i(s), \quad i = 1, 2, 3. \tag{3.2}$$

From (H_1) and Lemma 2.5, $\forall t_0 \in \mathbb{R}$, there exists a constant $\omega \in (0, +\infty)$ independent of t_0 such that

$$\begin{cases} \frac{1}{T}\int\limits_{t_0}^{t_0+T} a_i(s)\,\mathrm{d}s \in \left[\frac{\bar{a}_i}{2}, \frac{3\bar{a}_i}{2}\right], \quad \frac{1}{T}\int\limits_{t_0}^{t_0+T} b_i(s)\,\mathrm{d}s \in \left[\frac{\bar{b}_i}{2}, \frac{3\bar{b}_i}{2}\right], \\ \frac{1}{T}\int\limits_{t_0}^{t_0+T} [a_2(s) - b_2(s)]\,\mathrm{d}s \in \left[\frac{\bar{a}_2 - \bar{b}_2}{2}, \frac{3(\bar{a}_2 - \bar{b}_2)}{2}\right], \quad \forall T \geq \omega, \quad i = 1, 3. \end{cases} \tag{3.3}$$

For $\forall n_0 \in \mathbb{N}^+$, we consider $\left[\alpha_{n_0}^{(i)} - \omega, \alpha_{n_0}^{(i)}\right]$ $(i = 1, 2, 3)$, where ω is defined as that in (3.3). By Lemma 2.3, there exist $\xi_i \in \left[\alpha_{n_0}^{(i)} - \omega, \alpha_{n_0}^{(i)}\right]$, $\underline{\xi}_i \in \left(-\infty, \alpha_{n_0}^{(i)} - \omega\right]$ and $\bar{\xi}_i \in \left[\alpha_{n_0}^{(i)}, +\infty\right)$ such that

$$y_i(\underline{\xi}_i) = y_i(\bar{\xi}_i), \quad y_i(\xi_i) \leq y_i(s), \quad \forall s \in \left[\underline{\xi}_i, \bar{\xi}_i\right], \quad i = 1, 2, 3. \tag{3.4}$$

By (3.4), we obtain from system (3.1) that

$$\begin{cases} 0 = \int\limits_{\underline{\xi}_1}^{\bar{\xi}_1} \left[-a_1(s) + \frac{b_1(s)e^{y_2(s-\tau(s))}}{e^{y_1(s)}}\right] \mathrm{d}s, \\ 0 = \int\limits_{\underline{\xi}_2}^{\bar{\xi}_2} \left[a_2(s) - b_2(s) - c(s)e^{y_2(s)} - \beta_1(s)e^{y_3(s-\delta(s))}\right] \mathrm{d}s, \\ 0 = \int\limits_{\underline{\xi}_3}^{\bar{\xi}_3} \left[a_3(s) - b_3(s)e^{y_3(s)} - \beta_2(s)e^{y_2(s-\sigma(s))}\right] \mathrm{d}s. \end{cases} \tag{3.5}$$

From the second equation of system (3.5), it follows from (3.3)-(3.4) that

$$\frac{\bar{c}}{2}(\bar{\xi}_2 - \underline{\xi}_2)e^{y_2(\xi_2)} \leq \int\limits_{\underline{\xi}_2}^{\bar{\xi}_2} c(s)e^{y_2(s)}\,\mathrm{d}s$$

$$\begin{aligned}
&= \int_{\underline{\xi}_2}^{\bar{\xi}_2} \left[a_2(s) - b_2(s) - \beta_1(s)e^{y_3(s-\delta(s))}\right] \mathrm{d}s \\
&\leq \int_{\underline{\xi}_2}^{\bar{\xi}_2} [a_2(s) - b_2(s)] \,\mathrm{d}s \\
&\leq \frac{3(\bar{a}_2 - \bar{b}_2)}{2}(\bar{\xi}_2 - \underline{\xi}_2),
\end{aligned}$$

which implies that

$$y_2(\xi_2) \leq \ln\left[\frac{3(\bar{a}_2 - \bar{b}_2)}{\bar{c}}\right]. \tag{3.6}$$

Let $I_i = \left\{s \in \left[\xi_i, \alpha_{n_0}^{(i)}\right] : \dot{y}_i(s) \geq 0\right\}$ $(i = 1, 2, 3)$. It follows from the first equation of system (3.1) and (3.3) that

$$\begin{aligned}
\int_{I_2} \dot{y}_2(s)\,\mathrm{d}s &= \int_{I_2} \lambda\left[a_2(s) - b_2(s) - c(s)e^{y_2(s)} - \beta_1(s)e^{y_3(s-\delta(s))}\right] \mathrm{d}s \\
&\leq \int_{I_2} \left[a_2(s) - b_2(s)\right] \mathrm{d}s \leq \int_{\alpha_{n_0}^{(2)}-\omega}^{\alpha_{n_0}^{(2)}} \left[a_2(s) - b_2(s)\right] \mathrm{d}s \\
&\leq \frac{3(\bar{a}_2 - \bar{b}_2)\omega}{2}.
\end{aligned} \tag{3.7}$$

By Lemma 2.2, it follows from (3.6)-(3.7) that

$$y_2(t) \leq y_2(\xi_2) + \int_{I_2} \dot{y}_2(s)\,\mathrm{d}s \leq \ln\left[\frac{3(\bar{a}_2 - \bar{b}_2)}{\bar{c}}\right] + \frac{3(\bar{a}_2 - \bar{b}_2)\omega}{2} := f_2^+, \quad \forall t \in [\xi_2, \alpha_{n_0}^{(2)}],$$

which implies that

$$y_2(\alpha_{n_0}^{(2)}) \leq f_2^+.$$

In view of (3.2), letting $n_0 \to +\infty$ in the above inequality leads to

$$y_2^+ = \lim_{n_0 \to +\infty} y_2(\alpha_{n_0}^{(2)}) \leq f_2^+. \tag{3.8}$$

Similar to the arguments as that in (3.6)-(3.7), we can obtain from the third equation of system (3.5) that

$$y_3(\xi_3) \leq \ln\left[\frac{3\bar{a}_3}{\bar{b}_3}\right] \quad \text{and} \quad \int_{I_3} \dot{y}_3(s)\,\mathrm{d}s \leq \frac{3\bar{a}_3\omega}{2}. \tag{3.9}$$

By a similar discussion as that in (3.8), it follows from (3.9) that

$$y_3^+ \leq \ln\left[\frac{3\bar{a}_3}{\bar{b}_3}\right] + \frac{3\bar{a}_3\omega}{2} := f_3^+. \tag{3.10}$$

On the other hand, from (H_2) and Lemma 2.5, $\forall t_0 \in \mathbb{R}$, there exists a constant $l \in [\omega, +\infty)$ independent of t_0 such that

$$\frac{1}{T}\int_{t_0}^{t_0+T} \mu(s)\,\mathrm{d}s \in \left[\frac{\bar{\mu}}{2}, \frac{3\bar{\mu}}{2}\right], \quad \frac{1}{T}\int_{t_0}^{t_0+T} \nu(s)\,\mathrm{d}s \in \left[\frac{\bar{\nu}}{2}, \frac{3\bar{\nu}}{2}\right], \quad \forall T \geq l. \tag{3.11}$$

For $\forall n_0 \in \mathbb{Z}$, by Lemma 2.4, there exist $\eta_i \in [n_0 l, n_0 l + l]$, $\underline{\eta}_i \in (-\infty, n_0 l]$ and $\bar{\eta}_i \in [n_0 l + l, +\infty)$ such that

$$y_i(\underline{\eta}_i) = y_i(\bar{\eta}_i), \quad y_i(\eta_i) \geq y_i(s), \quad \forall s \in \left[\underline{\eta}_i, \bar{\eta}_i\right], \quad i = 2, 3. \tag{3.12}$$

By (3.12), we obtain from system (3.1) that

$$\begin{cases} 0 = \int\limits_{\underline{\eta}_2}^{\bar{\eta}_2} \left[a_2(s) - b_2(s) - c(s)e^{y_2(s)} - \beta_1(s)e^{y_3(s-\delta(s))}\right] ds, \\ 0 = \int\limits_{\underline{\eta}_3}^{\bar{\eta}_3} \left[a_3(s) - b_3(s)e^{y_3(s)} - \beta_2(s)e^{y_2(s-\sigma(s))}\right] ds. \end{cases} \tag{3.13}$$

By the first equation of system (3.13), we have from (3.3), (3.11) and (3.12) that

$$\begin{aligned} \frac{3\bar{c}}{2}(\bar{\eta}_2 - \underline{\eta}_2)e^{y_2(\eta_2)} &\geq \int\limits_{\underline{\eta}_2}^{\bar{\eta}_2} c(s)e^{y_2(s)}\, ds \\ &= \int\limits_{\underline{\eta}_2}^{\bar{\eta}_2} \left[a_2(s) - b_2(s) - \beta_1(s)e^{y_3(s-\delta(s))}\right] ds \\ &\geq \int\limits_{\underline{\eta}_2}^{\bar{\eta}_2} \left[a_2(s) - b_2(s) - \beta_1(s)e^{f_3^+}\right] ds \\ &= \int\limits_{\underline{\eta}_2}^{\bar{\eta}_2} \mu(s)\, ds \\ &\geq \frac{\bar{\mu}}{2}(\bar{\eta}_2 - \underline{\eta}_2), \end{aligned}$$

which yields that

$$y_2(\eta_2) \geq \ln\left[\frac{\bar{\mu}}{3\bar{c}}\right]. \tag{3.14}$$

Furthermore, by the second equation of system (3.1), it follows that

$$\begin{aligned} \int\limits_{n_0 l}^{n_0 l+l} |\dot{y}_2(s)|\, ds &= \int\limits_{n_0 l}^{n_0 l+l} \lambda \left|a_2(s) - b_2(s) - c(s)e^{y_2(s)} - \beta_1(s)e^{y_3(s-\delta(s))}\right| ds \\ &\leq \left(a_2^+ + b_2^+ + c^+ e^{f_2^+} + \beta_1^+ e^{f_3^+}\right) l. \end{aligned} \tag{3.15}$$

Then

$$\begin{aligned} y_2(t) &\geq y_2(\eta_2) - \int\limits_{n_0 l}^{n_0 l+l} |\dot{y}_2(s)|\, ds \\ &\geq \ln\left[\frac{\bar{\mu}}{3\bar{c}}\right] - \left(a_2^+ + b_2^+ + c^+ e^{f_2^+} + \beta_1^+ e^{f_3^+}\right) l := f_2^-, \quad \forall t \in [n_0 l, n_0 l + l]. \end{aligned} \tag{3.16}$$

Obviously, f_2^- is a constant independent of n_0. So it follows from (3.16) that

$$y_2^- = \inf_{s\in\mathbb{R}} y_2(s) = \inf_{n_0\in\mathbb{Z}} \left\{ \min_{s\in[n_0 l, n_0 l+l]} y_2(s) \right\} \geq \inf_{n_0\in\mathbb{Z}} \{f_2^-\} = f_2^-. \tag{3.17}$$

Similar to the arguments as that in (3.14)-(3.15), we can obtain from the third equation of systems (3.1) and (3.13) that

$$y_3(\eta_3) \geq \ln\left[\frac{\bar{\nu}}{3\bar{b}_3}\right] \tag{3.18}$$

and

$$\int_{n_0 l}^{n_0 l+l} |\dot{y}_3(s)|\,\mathrm{d}s = \int_{n_0 l}^{n_0 l+l} \lambda \left| a_3(s) - b_3(s)e^{y_3(s)} - \beta_2(s)e^{y_2(s-\sigma(s))} \right| \mathrm{d}s \le \left(a_3^+ + b_3^+ e^{f_3^+} + \beta_2^+ e^{f_2^+} \right) l. \tag{3.19}$$

It follows from (3.18)-(3.19) that

$$y_3(t) \ge y_3(\eta_3) - \int_{n_0 l}^{n_0 l+l} |\dot{y}_3(s)|\,\mathrm{d}s \ge \ln\left[\frac{\bar{v}}{3\bar{b}_3}\right] - \left(a_3^+ + b_3^+ e^{f_3^+} + \beta_2^+ e^{f_2^+} \right) l := f_3^-, \quad \forall t \in [n_0 l, n_0 l + l]. \tag{3.20}$$

Obviously, f_3^- is a constant independent of n_0. So it follows from (3.20) that

$$y_3^- = \inf_{s\in\mathbb{R}} y_3(s) = \inf_{n_0\in\mathbb{Z}} \left\{ \min_{s\in[n_0 l, n_0 l+l]} y_3(s) \right\} \ge \inf_{n_0\in\mathbb{Z}} \{f_3^-\} = f_3^-. \tag{3.21}$$

Finally, there must exist $\{\theta_n : n \in \mathbb{N}^+\}$ such that

$$y_1(\theta_n) \in \left[y_1^-, y_1^- + \frac{1}{n} \right], \tag{3.22}$$

where $y_1^- = \inf_{s\in\mathbb{R}} y_1(s)$. For $\forall n_0 \in \mathbb{N}^+$, we consider $[\theta_{n_0} - \omega, \theta_{n_0}]$, where ω is defined as that in (3.3). By Lemma 2.4, there exist $\eta_1 \in [\theta_{n_0} - \omega, \theta_{n_0}]$, $\underline{\eta}_1 \in (-\infty, \theta_{n_0} - \omega]$ and $\bar{\eta}_1 \in [\theta_{n_0}, +\infty)$ such that

$$y_1(\underline{\eta}_1) = y_1(\bar{\eta}_1), \quad y_1(\eta_1) \ge y_1(s), \quad \forall s \in \left[\underline{\eta}_1, \bar{\eta}_1 \right]. \tag{3.23}$$

In view of the first equation of system (3.1), we get

$$0 = \int_{\underline{\eta}_1}^{\bar{\eta}_1} \left[-a_1(s) + \frac{b_1(s)e^{y_2(s-\tau(s))}}{e^{y_1(s)}} \right] \mathrm{d}s,$$

which implies from (3.2) and (3.23) that

$$\frac{3\bar{a}_1}{2}(\bar{\eta}_1 - \underline{\eta}_1) \ge \int_{\underline{\eta}_1}^{\bar{\eta}_1} a_1(s)\,\mathrm{d}s = \int_{\underline{\eta}_1}^{\bar{\eta}_1} \frac{b_1(s)e^{y_2(s-\tau(s))}}{e^{y_1(s)}}\,\mathrm{d}s \ge \frac{\bar{b}_1 e^{f_2^-}}{2e^{y_1(\eta_1)}}(\bar{\eta}_1 - \underline{\eta}_1),$$

which yields that

$$y_1(\eta_1) \ge \ln\left[\frac{\bar{b}_1 e^{f_2^-}}{3\bar{a}_1}\right]. \tag{3.24}$$

Let $J = \left\{ s \in [\eta_1, \theta_{n_0}] : \dot{y}_1(s) \le 0 \right\}$. Then we have from the first equation of system (3.1) and (3.2) that

$$\int_J \dot{y}_1(s)\,\mathrm{d}s = \int_J \lambda \left[-a_1(s) + \frac{b_1(s)e^{y_2(s-\tau(s))}}{e^{y_1(s)}} \right] \mathrm{d}s \ge -\int_J a_1(s)\,\mathrm{d}s \ge -\int_{\theta_{n_0}-\omega}^{\theta_{n_0}} a_1(s)\,\mathrm{d}s \ge -\frac{3\bar{a}_1\omega}{2}. \tag{3.25}$$

By Lemma 2.2, we get from (3.24)-(3.25) that

$$y_1(t) \geq y_1(\eta_1) + \int_J \dot{y}_1(s)\,ds \geq \ln\left[\frac{\bar{b}_1 e^{f_2^-}}{3\bar{a}_1}\right] - \frac{3\bar{a}_1\omega}{2} := f_1^-, \quad \forall t \in [\eta_1, \theta_{n_0}].$$

So

$$y_1(\theta_{n_0}) \geq f_1^- \Longrightarrow y_1^- \geq f_1^- - \frac{1}{n_0}.$$

Letting $n_0 \to +\infty$ in the last inequality leads to

$$y_1^- \geq f_1^-. \tag{3.26}$$

Furthermore, by the first equation of system (3.5), we have from (3.3)-(3.4) that

$$\frac{\bar{a}_1}{2}(\bar{\xi}_1 - \underline{\xi}_1) \leq \int_{\underline{\xi}_1}^{\bar{\xi}_1} a_1(s)\,ds = \int_{\underline{\xi}_1}^{\bar{\xi}_1}\left[\frac{b_1(s)e^{y_2(s-\tau(s))}}{e^{y_1(s)}}\right]ds \leq \frac{3\bar{b}_1 e^{f_2^+}}{2e^{y_1(\xi_1)}}(\bar{\xi}_1 - \underline{\xi}_1),$$

which yields that

$$y_1(\xi_1) \leq \ln\left[\frac{3\bar{b}_1 e^{f_2^+}}{\bar{a}_1}\right]. \tag{3.27}$$

And

$$\begin{aligned}\int_{I_1} \dot{y}_1(s)\,ds &= \int_{I_1}\left[-a_1(s) + \frac{b_1(s)e^{y_2(s-\tau(s))}}{e^{y_1(s)}}\right]ds \\ &\leq \int_{\alpha_{n_0}^{(1)}-\omega}^{\alpha_{n_0}^{(1)}} \frac{b_1(s)e^{y_2(s-\tau(s))}}{e^{y_1(s)}}\,ds \\ &\leq \frac{3\bar{b}_1 e^{f_2^+}\omega}{2e^{f_1^-}}.\end{aligned} \tag{3.28}$$

Similar to the argument as that in (3.8), we obtain from (3.27)-(3.28) that

$$y_1^+ \leq \ln\left[\frac{3\bar{b}_1 e^{f_2^+}}{\bar{a}_1}\right] + \frac{3\bar{b}_1 e^{f_2^+}\omega}{2e^{f_1^-}} := f_1^+. \tag{3.29}$$

Set $C = \sum_{i=1}^{3}(|f_i^-| + |f_i^+|) + 1$. Clearly, C is independent of $\lambda \in (0, 1)$. Let $\Omega = \{z \in \mathbb{X} : \|z\|_{\mathbb{X}} < C\}$. Therefore, Ω satisfies condition (a) of Lemma 2.6.

Now we show that condition (b) of Lemma 2.6 holds, i.e., we prove that $QNz \neq 0$ for all $z = (y_1, y_2, y_3)^T \in \partial\Omega \cap \mathrm{Ker}L = \partial\Omega \cap \mathbb{R}^3$. If it is not true, then there exists at least one constant vector $z_0 = (y_1^0, y_2^0, y_3^0)^T \in \partial\Omega$ such that

$$\begin{cases} 0 = m\left[-a_1(t) + \frac{b_1(t)e^{y_2^0}}{e^{y_1^0}}\right], \\ 0 = m\left[a_2(t) - b_2(t) - c(t)e^{y_2^0} - \beta_1(t)e^{y_3^0}\right], \\ 0 = m\left[a_3(t) - b_3(t)e^{y_3^0} - \beta_2(t)e^{y_2^0}\right]. \end{cases}$$

Similar to the argument as that in (3.8), (3.10), (3.17), (3.21), (3.26) and (3.29), it follows that

$$f_1^- \leq y_1^0 \leq f_1^+, \quad f_2^- \leq y_2^0 \leq f_2^+, \quad f_3^- \leq y_3^0 \leq f_3^+.$$

Then $z_0 \in \Omega \cap \mathbb{R}^3$. This contradicts the fact that $z_0 \in \partial\Omega$. This proves that condition (b) of Lemma 2.6 holds.

Finally, we will show that condition (c) of Lemma 2.6 is satisfied. Let us consider the homotopy

$$H(\iota, z) = \iota QNz + (1-\iota)\Phi z, \quad (\iota, z) \in [0,1] \times \mathbb{R}^3,$$

where

$$\Phi z = \Phi \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -\bar{a}_1 + \frac{\bar{b}_1 e^{\bar{f}_2}}{e^{y_1}} \\ \bar{a}_2 - \bar{b}_2 - \bar{c}e^{y_2} \\ \bar{a}_3 - \bar{b}_3 e^{y_3} \end{pmatrix}.$$

From the above discussion it is easy to verify that $H(\iota, z) \neq 0$ on $\partial\Omega \cap \mathrm{Ker}L$, $\forall \iota \in [0,1]$. Further, $\Phi z = 0$ has a solution:

$$(y_1^+, y_2^+, y_3^+)^T = \left(\ln\left[\frac{\bar{b}_1 e^{\bar{f}_2}}{\bar{a}_1}\right], \ln\left[\frac{\bar{a}_2 - \bar{b}_2}{\bar{c}}\right], \ln\left[\frac{\bar{a}_3}{\bar{b}_3}\right]\right)^T \in \Omega.$$

A direct computation yields

$$\deg\left(\Phi, \Omega \cap \mathrm{Ker}L, 0\right) = \mathrm{sign}\begin{vmatrix} -\frac{\bar{b}_1 e^{\bar{f}_2}}{e^{y_1^+}} & 0 & 0 \\ 0 & -\bar{c}e^{y_2^+} & 0 \\ 0 & 0 & -\bar{b}_3 e^{y_3^+} \end{vmatrix} = -1.$$

By the invariance property of homotopy, we have

$$\deg\left(JQN, \Omega \cap \mathrm{Ker}L, 0\right) = \deg\left(QN, \Omega \cap \mathrm{Ker}L, 0\right) = \deg\left(\Phi, \Omega \cap \mathrm{Ker}L, 0\right) \neq 0,$$

where J is the identity mapping since $\mathrm{Im}Q = \mathrm{Ker}L$. Obviously, all the conditions of Lemma 2.6 are satisfied. Therefore, system (3.0) has at least one almost periodic solution, that is, system (1.2) has at least one positive almost periodic solution. This completes the proof. □

Remark 3.1. In [2], Chen also obtained the existence of positive almost periodic solutions of system (1.1). However, the result in [2] required the following condition:
$(F_1) a_i^- > 0$, $b_i^- > 0$, $\beta_j^- > 0$ and $c^- > 0$, $i = 1, 2, 3$, $j = 1, 2$.
$(F_2) a_1^- > \frac{a_2^+}{m}$, $c^- > b_1^+ + \beta_2^+$ and $b_3^- > \beta_1^+$, where m is some positive constant.

Obviously, (H_1) in Theorem 3.1 is weaker than (F_1). Further, by Theorem 3.1, it is easy to obtain the existence of positive almost periodic of system (1.1) without (F_2). So the work of this paper extends and improves the result in [2].

Corollary 3.1. *Assume that* (H_1)-(H_2) *hold. Suppose further that* a_i, b_i, β_j, c, τ, σ *and* δ *of system* (1.2) *are continuous nonnegative periodic functions with periods* α_i, ζ_i, η_j, ξ, ρ, ϱ *and* ς, *respectively,* $i = 1, 2, 3$, $j = 1, 2$, *then system* (1.2) *has at least one positive almost periodic solution.*

By Corollary 3.1, we obtain

Corollary 3.2. *Assume that* (H_1)-(H_2) *hold. Suppose further that* a_i, b_i, β_j, c, τ, σ *and* δ *of system* (1.2) *are continuous nonnegative* ω*-periodic functions,* $i = 1, 2, 3$, $j = 1, 2$, *then system* (1.2) *has at least one positive* ω*-periodic solution.*

Let

$$M_1 := \frac{(a_2 - b_2)^+}{c^-}, \quad M_2 := \frac{a_3^+}{b_3^-},$$

$$\mu_0(s) = a_2(s) - b_2(s) - \beta_1(s)M_2, \quad \nu_0(s) = a_3(s) - \beta_2(s)M_1, \quad \forall s \in \mathbb{R}.$$

In order to complement or improve (H_2) in Theorem 3.1, we give the following result:

Theorem 3.2. *Assume that* (H_1) *holds. Suppose further that*
$(H_3) a_1^- > 0$, $b_3^- > 0$ *and* $c^- > 0$,

$(H_4)\bar{\mu}_0 > 0$ *and* $\bar{v}_0 > 0$,

then system (1.2) *admits at least one positive almost periodic solution.*

Proof. Proceeding as in the proof of Theorem 3.1, in order to use Lemma 2.6, it remains to search for an appropriate open and bounded subset $\Omega \subseteq \mathbb{X}$.

Consider the operator equations (3.1). By Lemma 2.1, for $\forall \epsilon \in (0, 1)$, there exist $\xi_i = \xi_i(\epsilon)$ such that

$$\dot{y}_i(\xi_i) = 0, \quad y_i(\xi_i) \in [y_i^+ - \epsilon, y_i^+], \quad y_i^+ = \sup_{s\in\mathbb{R}} y_i(s), \quad i = 1, 2, 3. \tag{3.30}$$

From system (3.1), it follows from (3.30) that

$$\begin{cases} 0 = -a_1(\xi_1) + \frac{b_1(\xi_1)e^{y_2(\xi_1-\tau(\xi_1))}}{e^{y_1(\xi_1)}}, \\ 0 = a_2(\xi_2) - b_2(\xi_2) - c(\xi_2)e^{y_2(\xi_2)} - \beta_1(\xi_2)e^{y_3(\xi_2-\delta(\xi_2))}, \\ 0 = a_3(\xi_3) - b_3(\xi_3)e^{y_3(\xi_3)} - \beta_2(\xi_3)e^{y_2(\xi_3-\sigma(\xi_3))}. \end{cases} \tag{3.31}$$

By the second equation of system (3.31), we have from (3.30) that

$$c^- e^{y_2^+-\epsilon} \le c(\xi_2)e^{y_2(\xi_2)} = a_2(\xi_2) - b_2(\xi_2) - \beta_1(\xi_2)e^{y_3(\xi_2-\sigma(\xi_2))} \le (a_2 - b_2)^+ = \sup_{s\in\mathbb{R}}[a_2(s) - b_2(s)],$$

which yields that

$$y_2^+ \le \ln\left[\frac{(a_2 - b_2)^+ e^\epsilon}{c^-}\right].$$

Letting $\epsilon \to 0$ in the above inequality, we get

$$y_2^+ \le \ln\left[\frac{(a_2 - b_2)^+}{c^-}\right] := g_2^+. \tag{3.32}$$

So we can obtain from the first equation of system (3.31) that

$$a_1^- e^{y_1^+-\epsilon} \le a_1(\xi_1)e^{y_1(\xi_1)} = b_1(\xi_1)e^{y_2(\xi_1-\tau(\xi_1))} \le b_1^+ e^{y_2^+} \le \frac{b_1^+(a_2 - b_2)^+ e^\epsilon}{c^-}.$$

Letting $\epsilon \to 0$ in the above inequality, we get

$$y_1^+ \le \ln\left[\frac{b_1^+(a_2 - b_2)^+}{a_1^- c^-}\right] := g_1^+. \tag{3.33}$$

Similar to the arguments as that in (3.32)-(3.33), it follows from the third equation of system (3.31) that

$$y_3^+ \le \ln\left[\frac{a_3^+}{b_3^-}\right] := g_3^+. \tag{3.34}$$

By the similar discussions as that in (3.17), (3.21) and (3.26), we could find g_i^- such that

$$y_i^- \ge g_i^-, \quad i = 1, 2, 3.$$

Set $C_0 = \sum_{i=1}^3(|g_i^-| + |g_i^+|) + 1$. Clearly, C_0 is independent of $\lambda \in (0, 1)$. Let $\Omega = \{z \in \mathbb{X} : \|z\|_{\mathbb{X}} < C_0\}$. From the proof in Theorem 3.1, it is easy to verify that Ω satisfies conditions (a)-(c) of Lemma 2.6. Obviously, all the conditions of Lemma 2.6 are satisfied. Therefore, system (3.0) has one almost periodic solution, that is, system (1.2) has at least one positive almost periodic solution. This completes the proof. □

Corollary 3.3. *Assume that* (H_1), (H_3) *and* (H_4) *hold. Suppose further that* a_i, b_i, β_j, c, τ, σ *and* δ *of system* (1.2) *are continuous nonnegative periodic functions with periods* α_i, ζ_i, η_j, ξ, ρ, ϱ *and* ς, *respectively,* $i = 1, 2, 3$, $j = 1, 2$, *then system* (1.2) *has at least one positive almost periodic solution.*

From Corollary 3.3, we have

Corollary 3.4. *Assume that* (H_1), (H_3) *and* (H_4) *hold. Suppose further that* a_i, b_i, β_j, c, τ, σ *and* δ *of system* (1.2) *are continuous nonnegative* ω*-periodic functions,* $i = 1, 2, 3$, $j = 1, 2$, *then system* (1.2) *has at least one positive* ω*-periodic solution.*

4 Global asymptotical stability

Theorem 4.1. *Assume that* (H_1)-(H_2) *hold. Suppose further that*
$(H_5)\tau, \sigma, \delta \in C^1(\mathbb{R})$, $\sup_{s\in\mathbb{R}}\{\dot{\tau}(s), \dot{\sigma}(s), \dot{\delta}(s)\} < 1$, $a_1^- > 0$ *and*

$$\inf_{s\in\mathbb{R}}\left\{c(s) - \frac{b_1(\xi^{-1}(s))}{1-\dot{\tau}(\xi^{-1}(s))} - \frac{\beta_2(\varphi^{-1}(s))}{1-\dot{\sigma}(\varphi^{-1}(s))}\right\} > 0, \quad \inf_{s\in\mathbb{R}}\left\{b_3(s) - \frac{\beta_1(\psi^{-1}(s))}{1-\dot{\delta}(\psi^{-1}(s))}\right\} > 0,$$

where ξ^{-1}, φ^{-1} *and* ψ^{-1} *are the inverse functions of* $\xi(t) = t - \tau(t)$, $\varphi(t) = t - \sigma(t)$ *and* $\psi(t) = t - \delta(t)$, *respectively,* $\forall t \in \mathbb{R}$.

Then system (1.2) *has a unique positive almost periodic solution, which is globally asymptotically stable.*

Proof. By Theorem 3.1, we know that system (1.2) has at least one positive almost periodic solution $(x_1^\star, x_2^\star, x_3^\star)^T$. Suppose that $(x_1, x_2, x_3)^T$ is another positive solution of system (1.2).

From (H_5), there must exist $0 < \theta < a_1^-$ such that

$$\inf_{s\in\mathbb{R}}\left\{c(s) - \frac{b_1(\xi^{-1}(s))}{1-\dot{\tau}(\xi^{-1}(s))} - \frac{\beta_2(\varphi^{-1}(s))}{1-\dot{\sigma}(\varphi^{-1}(s))}\right\} > \theta, \quad \inf_{s\in\mathbb{R}}\left\{b_3(s) - \frac{\beta_1(\psi^{-1}(s))}{1-\dot{\delta}(\psi^{-1}(s))}\right\} > \theta.$$

Define

$$V(t) = \sum_{i=1}^{6} V_i(t),$$

where

$$V_1(t) = |x_1(t) - x_1^\star(t)|, \quad V_i(t) = |\ln x_i(t) - \ln x_i^\star(t)|, \quad i = 2, 3,$$

$$V_4(t) = \int_{t-\tau(t)}^{t} \frac{b_1(\xi^{-1}(s))}{1-\dot{\tau}(\xi^{-1}(s))}|x_2(s) - x_2^\star(s)|\,ds,$$

$$V_5(t) = \int_{t-\sigma(t)}^{t} \frac{\beta_2(\varphi^{-1}(s))}{1-\dot{\sigma}(\varphi^{-1}(s))}|x_2(s) - x_2^\star(s)|\,ds,$$

$$V_6(t) = \int_{t-\delta(t)}^{t} \frac{\beta_1(\psi^{-1}(s))}{1-\dot{\delta}(\psi^{-1}(s))}|x_3(s) - x_3^\star(s)|\,ds.$$

By calculating the upper right derivative of $V_i (i = 1, 2, 3, 4, 5, 6)$ along the positive solution of system (1.2), it follows that

$$\begin{aligned} D^+V_1(t) &= \operatorname{sign}[x_1(t) - x_1^\star(t)]\Big\{\Big[-a_1(t)x_1(t) + b_1(t)x_2(t-\tau(t))\Big] \\ &\quad - \Big[-a_1(t)x_1^\star(t) + b_1(t)x_2^\star(t-\tau(t))\Big]\Big\} \\ &\le -a_1(t)|x_1(t) - x_1^\star(t)| + b_1(t)|x_2(t-\tau(t)) - x_2^\star(t-\tau(t))|, \end{aligned} \tag{4.1}$$

$$D^+V_2(t) = \operatorname{sign}[x_2(t) - x_2^\star(t)]\Big\{\Big[a_2(t) - b_2(t) - c(t)x_2(t) - \beta_1(t)x_3(t-\delta(t))\Big]$$

$$
\begin{aligned}
&-\Big[a_2(t)-b_2(t)-c(t)x_2^\star(t)-\beta_1(t)x_3^\star(t-\delta(t))\Big]\Big\}\\
&\le -c(t)|x_2(t)-x_2^\star(t)|+\beta_1(t)|x_3(t-\delta(t))-x_3^\star(t-\delta(t))|,
\end{aligned} \tag{4.2}
$$

$$
\begin{aligned}
D^+V_3(t) &= \operatorname{sign}[x_3(t)-x_3^\star(t)]\Big\{\Big[a_3(t)-b_3(t)x_3(t)-\beta_2(t)x_2(t-\sigma(t))\Big]\\
&\quad -\Big[a_3(t)-b_3(t)x_3^\star(t)-\beta_2(t)x_2^\star(t-\sigma(t))\Big]\Big\}\\
&\le -b_3(t)|x_3(t)-x_3^\star(t)|+\beta_2(t)|x_2(t-\sigma(t))-x_2^\star(t-\sigma(t))|,
\end{aligned} \tag{4.3}
$$

$$
D^+V_4(t) = \frac{b_1(\xi^{-1}(t))}{1-\dot\tau(\xi^{-1}(t))}|x_2(t)-x_2^\star(t)|-b_1(t)|x_2(t-\tau(t))-x_2^\star(t-\tau(t))|, \tag{4.4}
$$

$$
D^+V_5(t) = \frac{\beta_2(\varphi^{-1}(t))}{1-\dot\sigma(\varphi^{-1}(t))}|x_2(t)-x_2^\star(t)|-\beta_2(t)|x_2(t-\sigma(t))-x_2^\star(t-\sigma(t))| \tag{4.5}
$$

and

$$
D^+V_6(t) = \frac{\beta_1(\psi^{-1}(t))}{1-\dot\delta(\psi^{-1}(t))}|x_3(t)-x_3^\star(t)|-\beta_1(t)|x_3(t-\delta(t))-x_3^\star(t-\delta(t))|. \tag{4.6}
$$

Together with (4.1)-(4.6), it follows that

$$
\begin{aligned}
D^+V(t) &\le -a_1(t)|x_1(t)-x_1^\star(t)|\\
&\quad -\Big\{c(t)-\frac{b_1(\xi^{-1}(t))}{1-\dot\tau(\xi^{-1}(t))}-\frac{\beta_2(\varphi^{-1}(t))}{1-\dot\sigma(\varphi^{-1}(t))}\Big\}|x_2(t)-x_2^\star(t)|\\
&\quad -\Big\{b_3(t)-\frac{\beta_1(\psi^{-1}(t))}{1-\dot\delta(\psi^{-1}(t))}\Big\}|x_3(t)-x_3^\star(t)|\\
&\le -\theta\sum_{i=1}^{3}|x_i(t)-x_i^\star(t)|, \quad \forall t\ge 0.
\end{aligned}
$$

Therefore, V is non-increasing. Integrating the last inequality from 0 to t leads to

$$
V(t)+\theta\sum_{i=1}^{3}\int_0^t |x_i(s)-x_i^\star(s)|\,\mathrm{d}s \le V(0)<+\infty, \quad \forall t\ge 0,
$$

that is,

$$
\sum_{i=1}^{3}\int_0^{+\infty}|x_i(s)-x_i^\star(s)|\,\mathrm{d}s<+\infty,
$$

which implies that

$$
\sum_{i=1}^{3}\lim_{s\to+\infty}|x_i(s)-x_i^\star(s)|=0.
$$

Thus, the almost periodic solution of system (1.2) is globally exponentially stable.

Next, we show that there is only one positive almost periodic solution of system (1.2). For any two positive almost periodic solutions $(x_1,x_2,x_3)^T$ and $(\bar x_1,\bar x_2,\bar x_3)^T$ of system (1.2), we claim that $x_i(t)\equiv\bar x_i(t)$, $\forall t\in\mathbb{R}$, $i=1,2,3$. If not, without loss of generality, there must be at least one $t_0\in\mathbb{R}$ such that $x_1(t_0)\ne\bar x_1(t_0)$, i.e., $|x_1(t_0)-\bar x_1(t_0)|:=l>0$. The global asymptotical stability implies that there exists $t_1>t_0$ such that

$$
|x_1(t)-\bar x_1(t)|<\frac{l}{4}, \quad \forall t\ge t_1. \tag{4.7}
$$

By the almost periodicity of x_1 and $\bar{x}_1$, there must exist $l_1 > 0$ and $\tau_0 \in [t_1 - t_0, t_1 - t_0 + l_1]$ such that

$$|x_1(t+\tau_0) - x_1(t)| < \frac{l}{4}, \quad |\bar{x}_1(t+\tau_0) - \bar{x}_1(t)| < \frac{l}{4}, \quad \forall t \in \mathbb{R}. \tag{4.8}$$

So we can easily know from (4.7)-(4.8) that

$$\begin{aligned} l = |x_1(t_0) - \bar{x}_1(t_0)| &\le |x_1(t_0) - x_1(t_0+\tau_0)| + |x_1(t_0+\tau_0) - \bar{x}_1(t_0+\tau_0)| + |\bar{x}_1(t_0+\tau_0) - \bar{x}_1(t_0)| \\ &< \frac{l}{4} + \frac{l}{4} + \frac{l}{4} = \frac{3l}{4}, \end{aligned}$$

which is a contradiction. Then $x_1(t) \equiv \bar{x}_1(t)$, $\forall t \in \mathbb{R}$. Similarly, we can prove $x_i(t) \equiv \bar{x}_i(t)$, $\forall t \in \mathbb{R}$, $i = 2, 3$. Therefore, the almost periodic solution of system (1.2) is unique. This completes the proof. □

Together with Theorem 3.2, we can easily show that

Theorem 4.2. *Assume that (H_1),(H_3), (H_4) and (H_5) hold, then system (1.2) has a unique positive almost periodic solution, which is globally asymptotically stable.*

5 Example and simulations

Example 5.1. Considering the following delayed two-species competitive model with stage structure and different periods:

$$\begin{cases} \dot{x}_1(t) = -[1 + |\sin(\sqrt{2}t)|]x_1(t) + 0.1|\cos(\sqrt{3}t)|x_2(t-1), \\ \dot{x}_2(t) = x_2(t) - |\cos(\sqrt{3}t)|x_2(t) - x_2^2(t) - 0.1\sin^2(\sqrt{3}t)x_2(t)x_3(t-1), \\ \dot{x}_3(t) = x_3(t)\left[|\cos(\sqrt{3}t)| - [1 + 0.99\sin(\sqrt{2}t)]x_3(t) - 0.1|\cos(\sqrt{3}t)|x_2(t-2)\right]. \end{cases} \tag{5.1}$$

Then system (5.1) has a unique positive almost periodic solution, which is globally asymptotically stable.

Proof. Corresponding to system (1.2), we have

$$\begin{pmatrix} a_1(s) \\ a_2(s) \\ a_3(s) \end{pmatrix} = \begin{pmatrix} 1 + |\sin(\sqrt{2}s)| \\ 1 \\ |\cos(\sqrt{3}s)| \end{pmatrix}, \quad \begin{pmatrix} b_1(s) \\ b_2(s) \\ b_3(s) \end{pmatrix} = \begin{pmatrix} 0.1|\cos(\sqrt{3}s)| \\ |\cos(\sqrt{3}s)| \\ 1 + \sin(\sqrt{2}s) \end{pmatrix},$$

$$\begin{pmatrix} \beta_1(s) \\ \beta_2(s) \end{pmatrix} = \begin{pmatrix} 0.1\sin^2(\sqrt{3}s) \\ 0.1|\cos(\sqrt{3}s)| \end{pmatrix}, \quad c(s) = 1, \quad \forall s \in \mathbb{R}.$$

Obviously, (H_1) in Theorem 3.1 holds. Further, $\bar{a}_2 = \bar{c} = 1$, $\bar{a}_3 = \bar{b}_2 = \frac{2}{\pi}$, $\bar{b}_3 = 1 + \frac{1.98}{\pi}$ and $\beta_1^+ = \beta_2^+ = 0.1$.

Let $e_1(t) = |\sin(\sqrt{2}t)|$ and $e_2(t) = |\cos(\sqrt{3}t)|$, $\forall t > 0$. Note that $\bar{e}_i = \frac{2}{\pi} \approx 0.64$, $i = 1, 2$. For $\forall t_0 \in \mathbb{R}$, we have

$$E_i(t) = \frac{\int_{t_0}^{t_0+t} e_i(s)\,ds}{t} \in \left[\frac{1}{\pi}, \frac{3}{\pi}\right] \approx [0.32, 0.955], \quad \forall t \ge 1, \quad i = 1, 2.$$

Therefore, we can choose $\omega = 1$ so that (3.3) holds. By an easy calculation, we obtain that

$$f_2^+ \approx 0.62, \quad f_3^+ \approx 1.12, \quad \bar{\mu} > 0.06 > 0, \quad \bar{\nu} > 0.472 > 0,$$

which implies that (H_2) in Theorem 3.1 holds. Further, it is easy to verify that (H_5) in Theorem 4.1 is satisfied. Therefore, all the conditions of Theorems 3.1-4.1 are satisfied. By Theorems 3.1-4.1, system (5.1) has a unique positive almost periodic solution, which is globally asymptotically stable (see Figures 2-5). This completes the proof. □

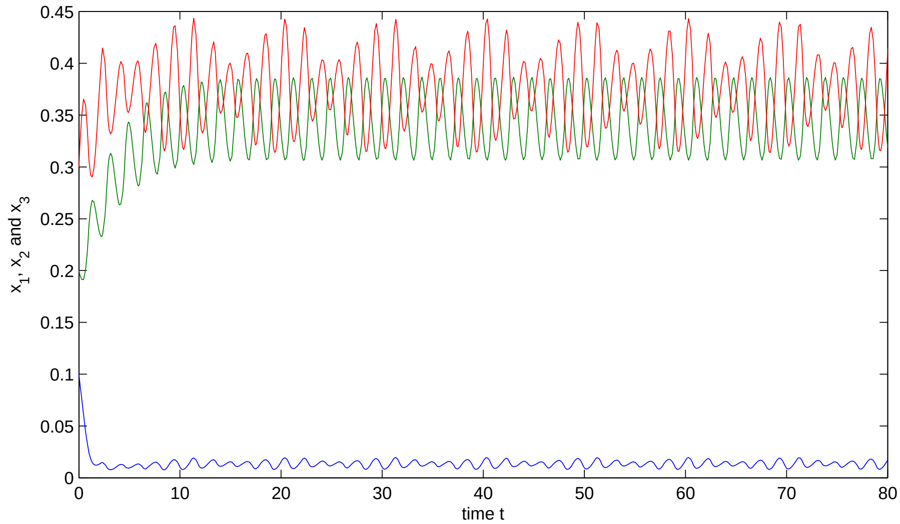

Figure 1: Almost periodic oscillations of system (5.1)

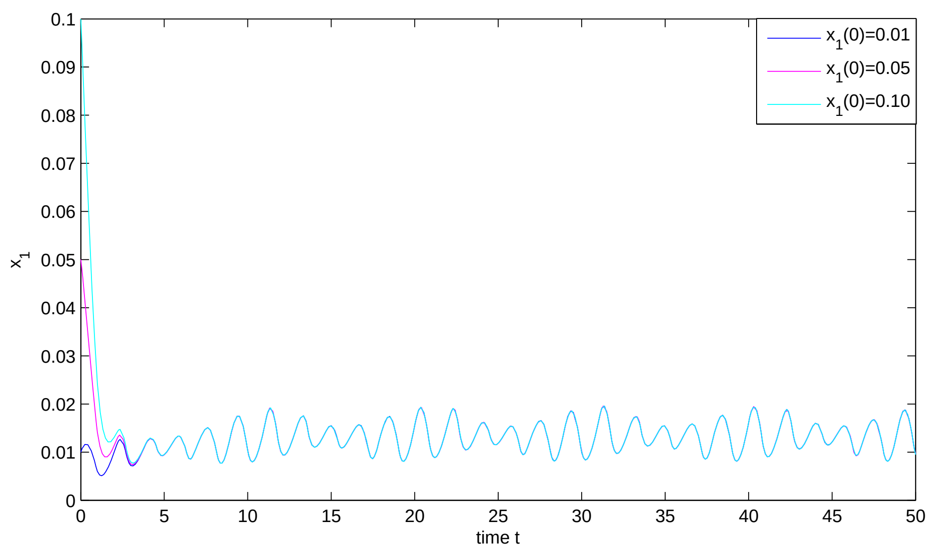

Figure 2: Global asymptotical stability of state variable x_1 of system (5.1)

Remark 5.1. Clearly, system (5.1) is with some nonnegative coefficients. Thus, condition (F_1) in Remark 3.1 is not satisfied. Further, corresponding to system (1.1), $b_3^- = 0.01 < 0.1 = \beta_1^+$, which implies that condition (F_2) in Remark 3.1 is invalid. Therefore, by the main result obtained by paper [2], it is impossible to obtain the existence and global asymptotical stability of positive almost periodic solutions of system (5.1). So the work of this paper extends and improves the result in [2].

6 Conclusions

The stage-structured models have been studied extensively, and many important phenomena have been observed in recent years. In this paper we study an almost periodic nonautonomous delayed two-species competitive model with stage structure, and this motivation comes from a nonautonomous delayed two-

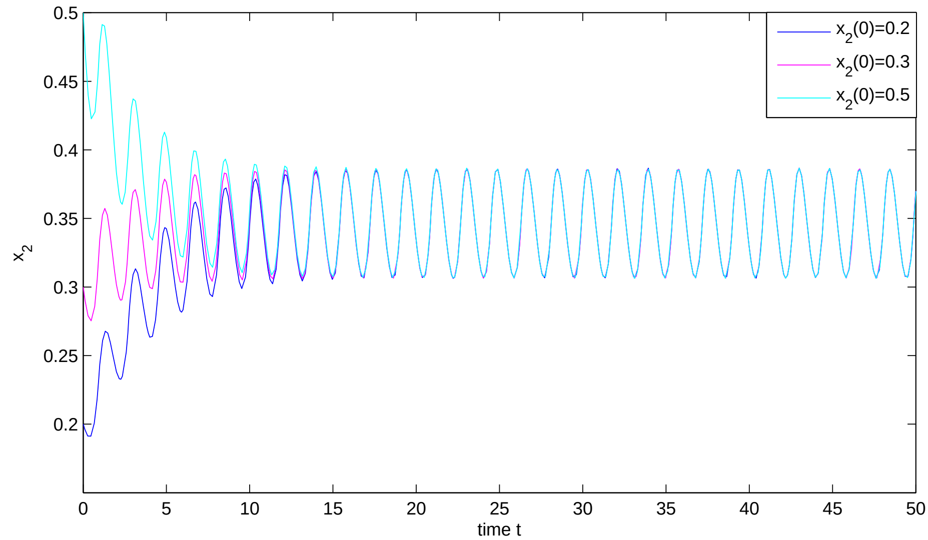

Figure 3: Global asymptotical stability of state variable x_2 of system (5.1)

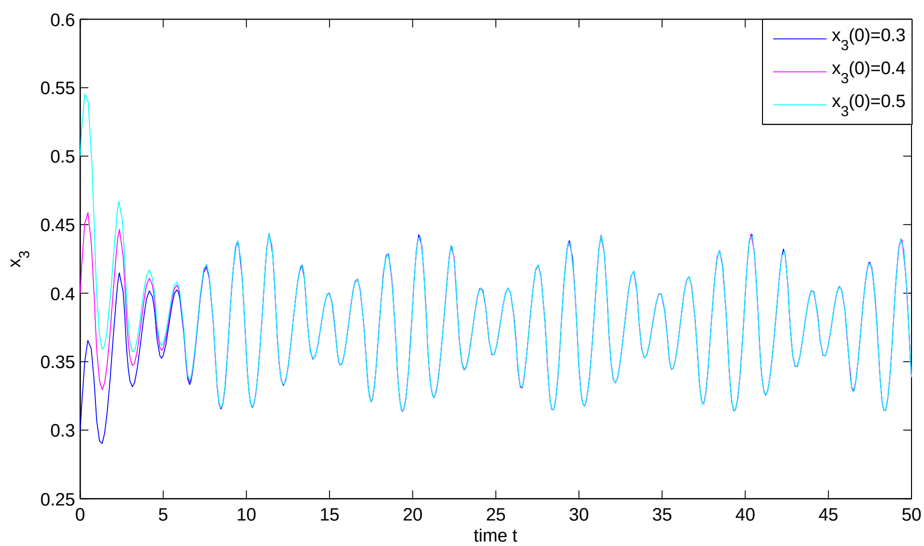

Figure 4: Global asymptotical stability of state variable x_3 of system (5.1)

species competitive model. We obtain easily verifiable sufficient criteria for the existence and globally asymptotic stability of positive almost periodic solutions of the above model. In order to obtain a more accurate description of the ecological system perturbed by human exploitation activities such as planting and harvesting and so on, we need to consider the impulsive differential equations. In this paper, we only studied system (1.2) without impulses. Whether system (1.2) with impulses can be discussed in the same methods or not is still an open problem. We will continue to study this problem in the future.

Acknowledgement: This work are supported by National Nature Science Foundation under Grant No. 11461082, 11601474 and 61472093, Key laboratory of numerical simulation of Sichuan Province under Grant No. 2017KF002.

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