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Xu Kong*

A concise proof to the spectral and nuclear norm bounds through tensor partitions

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Abstract: On estimations of the lower and upper bounds for the spectral and nuclear norm of a tensor, Li established neat bounds for the two norms based on regular tensor partitions, and proposed a conjecture for the same bounds to be hold based on general tensor partitions [Z. Li, Bounds on the spectral norm and the nuclear norm of a tensor based on tensor partition, SIAM J. Matrix Anal. Appl., 37 (2016), pp. 1440-1452]. Later, Chen and Li provided a solution to the conjecture [Chen B., Li Z., On the tensor spectral p-norm and its dual norm via partitions]. In this short paper, we present a concise and different proof for the validity of the conjecture, which also offers a new and simpler proof to the bounds of the spectral and nuclear norms established by Li for regular tensor partitions. Two numerical examples are provided to illustrate tightness of these bounds.

Keywords: tensor norm, spectral norm, nuclear norm, tensor partition

MSC: 15A60, 15A69

1 Introduction

Tensor is the main subject in multilinear algebra [1–6]. Let $\mathbb{R}$ be the field of real numbers. Specially, a tensor $\mathcal{T} = (t_{i_1 i_2 \cdots i_d}) \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_d}$ is a d-way array, i.e., its entries $t_{i_1 i_2 \cdots i_d}$ are represented via d indices, say $i_1, i_2, \cdots, i_d$ with each index ranging from 1 to n_j, $1 \le j \le d$. $\mathcal{T}$ is also called as an $n_1 \times n_2 \times \cdots \times n_d$ tensor. Similar to the definition of the submatrix of a matrix, a $p_1 \times p_2 \times \cdots \times p_d$ subtensor of a tensor $\mathcal{T} \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_d}$ is a $p_1 \times p_2 \times \cdots \times p_d$ tensor formed by taking a block of the entries from the original tensor $\mathcal{T}$.

Some general notations are in place: tensors are denoted by the calligraphic letters (e.g. $\mathcal{T}$ or $\mathcal{X}$), scalars are denoted by plain letters, and matrices and vectors are denoted by bold letters (e.g. $\mathbf{X}$ and $\mathbf{x}$).

Let $\|\cdot\|_1$, $\|\cdot\|_2$, and $\|\cdot\|_\infty$ denote the conventional 1-norm, 2-norm, and ∞-norm of a vector, respectively, i.e.,

$$\|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i|,$$
$$\|\mathbf{x}\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2},$$

and

$$\|\mathbf{x}\|_\infty = \max_{1 \le i \le n}\{|x_i|\},$$

where $\mathbf{x} = (x_1, x_2, \cdots, x_n)^T \in \mathbb{R}^n$.

***Corresponding Author: Xu Kong:** School of Mathematical Sciences, Liaocheng University, Liaocheng 252000, China; E-mail: xu.kong@hotmail.com

Definition 1.1. *Let $\mathcal{T} \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_d}$. The spectral norm of $\mathcal{T}$ denoted by $\|\mathcal{T}\|_\sigma$ is defined as*

$$\|\mathcal{T}\|_\sigma = \max_{\boldsymbol{x}_j \in \mathbb{R}^{n_j}, \|\boldsymbol{x}_j\|_2 = 1, 1 \le j \le d} \left\{ \langle \mathcal{T}, \boldsymbol{x}_1 \circ \boldsymbol{x}_2 \circ \cdots \circ \boldsymbol{x}_d \rangle^{1/2} \right\},$$

where $\langle \cdot, \cdot \rangle$ is the classical Euclidean inner product, and the symbol "$\circ$" denotes the outer product operation of vectors such that the entries of $\boldsymbol{x}_1 \circ \boldsymbol{x}_2 \circ \cdots \circ \boldsymbol{x}_d$ are $x_{i_1 1} x_{i_2 2} \cdots x_{i_d d}$, and $\boldsymbol{x}_k = (x_{1k}, x_{2k}, \cdots, x_{n_k k})^T$, $1 \le i_k \le n_k$, and $1 \le k \le d$.

Definition 1.2. *Let $\mathcal{T} \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_d}$. The nuclear norm of $\mathcal{T}$ denoted by $\|\mathcal{T}\|_\star$ is defined as*

$$\begin{aligned} \|\mathcal{T}\|_\star &= \max_{\mathcal{X} \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_d}, \|\mathcal{X}\|_\sigma \le 1} \{\langle \mathcal{T}, \mathcal{X} \rangle\} \\ &= \max_{\mathcal{X} \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_d}, \|\mathcal{X}\|_\sigma = 1} \{\langle \mathcal{T}, \mathcal{X} \rangle\}. \end{aligned}$$

With the wide applications of the spectral and nuclear norm of a matrix, the research on the tensor spectral and nuclear norm has also attracted much attention recently. Unlike the computation of the spectral and nuclear norm of a matrix that can be done easily, the tensor spectral and nuclear norm are both NP-hard to compute; see [7] and [8]. Therefore, estimating these bounds, especially the polynomial-time approximation bounds has been a hot issue [7–14].

In [15], Li proposed an efficient way for the estimation of the tensor spectral and nuclear norms based on tensor partitions, which is defined as follows.

Definition 1.3. *[15] A partition $\{\mathcal{T}_1, \mathcal{T}_2, \cdots, \mathcal{T}_m\}$ is called a general tensor partition of a tensor $\mathcal{T} \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_d}$ if*

- *every $\mathcal{T}_j$ $(j = 1, 2, \cdots, m)$ is a subtensor of $\mathcal{T}$,*
- *every pair of subtensors $\{\mathcal{T}_i, \mathcal{T}_j\}$ with $i \neq j$ has no common entry of $\mathcal{T}$, and*
- *every entry of $\mathcal{T}$ belongs to one of the subtensors in $\{\mathcal{T}_1, \mathcal{T}_2, \cdots, \mathcal{T}_m\}$.*

Furthermore, let $\mathcal{X} \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_d}$ and $\{\mathcal{X}_1, \mathcal{X}_2, \cdots, \mathcal{X}_m\}$ be a general tensor partition of $\mathcal{X}$. If each $\mathcal{X}_j$ has the same partition way as $\mathcal{T}_j$ for $1 \le j \le m$, then $\mathcal{T}$ and $\mathcal{X}$ are said to have the same partition pattern.

Li also proposed a special tensor partition called regular tensor partition based on which the bounds of tensor norms were established [15]. The partition is obtained via several tensor cuts. We omit the details as it is not relevant to our discussion here. For illustration, Fig. 1 depicts a general tensor partition of a third order tensor.

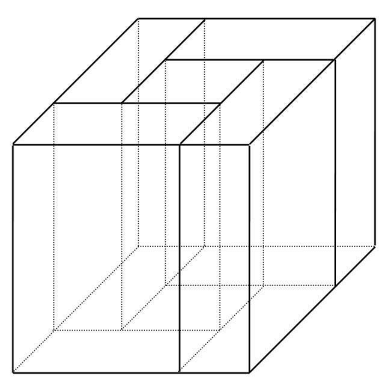

Figure 1: A general tensor partition of a third order tensor.

For a general tensor partition of a tensor, Li presented the following conjecture, which was answered in affirmative via a lengthy proof and also extended to a generalized tensor spectral and nuclear norms in a recent manuscript [11]. Some applications and general tightness results on rank-one tensors are discussed as well.

Conjecture 1.1. *[15] If $\{\mathcal{T}_1, \mathcal{T}_2, \cdots, \mathcal{T}_m\}$ is a general tensor partition of a tensor $\mathcal{T} \in \mathbb{R}^{n_1\times n_2\times\cdots\times n_d}$, then*

$$\|(\|\mathcal{T}_1\|_\sigma, \|\mathcal{T}_2\|_\sigma, \cdots, \|\mathcal{T}_m\|_\sigma)^T\|_\infty \le \|\mathcal{T}\|_\sigma \le \|(\|\mathcal{T}_1\|_\sigma, \|\mathcal{T}_2\|_\sigma, \cdots, \|\mathcal{T}_m\|_\sigma)^T\|_2,$$

and

$$\|(\|\mathcal{T}_1\|_\star, \|\mathcal{T}_2\|_\star, \cdots, \|\mathcal{T}_m\|_\star)^T\|_2 \le \|\mathcal{T}\|_\star \le \|(\|\mathcal{T}_1\|_\star, \|\mathcal{T}_2\|_\star, \cdots, \|\mathcal{T}_m\|_\star)^T\|_1.$$

In the current paper, we, in an independent work[1], propose a much simpler way to prove this conjecture. We also provide some nontrivial examples to show the tightness of these bounds. Since a regular tensor partition is a special type of a general tensor partition, naturally, the way for the solution to the conjecture also offers a new proof to the bounds for the spectral and nuclear norm established in [15].

The rest of this paper is organized as follows: In Section 2, we simply recall some definitions and results required for the subsequent sections. In Section 3, the main results of the paper are presented. A short conclusion is given in Section 4.

2 Preliminaries

The main objective of this section is to review some basic definitions and simple results relating to the tensor.

Definition 2.1. *Let $\mathcal{T} = (t_{i_1 i_2\cdots i_d}) \in \mathbb{R}^{n_1\times n_2\times\cdots\times n_d}$. The Frobenius norm of $\mathcal{T}$ denoted by $\|\mathcal{T}\|_F$ is defined as*

$$\begin{aligned}\|\mathcal{T}\|_F &= \langle \mathcal{T}, \mathcal{T}\rangle^{1/2}\\ &= \left(\sum_{i_1=1}^{n_1}\sum_{i_2=1}^{n_2}\cdots\sum_{i_d=1}^{n_d}(t_{i_1 i_2\cdots i_d})^2\right)^{1/2}.\end{aligned}$$

Definition 2.2. *A tensor $\mathcal{W} = (w_{i_1 i_2\cdots i_d}) \in \mathbb{R}^{n_1\times n_2\times\cdots\times n_d}$ is called a rank-one tensor if there exist nonzero vectors $\boldsymbol{w}_j \in \mathbb{R}^{n_j}$ $(1 \le j \le d)$ such that*

$$\mathcal{W} = \boldsymbol{w}_1 \circ \boldsymbol{w}_2 \circ \cdots \circ \boldsymbol{w}_d.$$

Definition 2.3. *Let $\mathcal{T} = (t_{i_1 i_2\cdots i_d}) \in \mathbb{R}^{n_1\times n_2\times\cdots\times n_d}$. $\boldsymbol{u}_1 \circ \boldsymbol{u}_2 \circ \cdots \circ \boldsymbol{u}_d$ is called as a best rank-one approximation of $\mathcal{T}$ if*

$$\|\mathcal{T} - \boldsymbol{u}_1 \circ \boldsymbol{u}_2 \circ \cdots \circ \boldsymbol{u}_d\|_F = \min_{\boldsymbol{x}_j\in\mathbb{R}^{n_j}, 1\le j\le d}\{\|\mathcal{T} - \boldsymbol{x}_1 \circ \boldsymbol{x}_2 \circ \cdots \circ \boldsymbol{x}_d\|_F\}. \tag{1}$$

Relating to the spectral norm of a tensor and the best rank-one approximation tensor, the following conclusion is straightforward.

Lemma 2.1. *[8, 16] Let $\mathcal{T} \in \mathbb{R}^{n_1\times n_2\times\cdots\times n_d}$. Suppose that $\boldsymbol{u}_1 \circ \boldsymbol{u}_2 \circ \cdots \circ \boldsymbol{u}_d$ is a best rank-one approximation to $\mathcal{T}$, then*

$$\|\mathcal{T}\|_\sigma = \|\boldsymbol{u}_1 \circ \boldsymbol{u}_2 \circ \cdots \circ \boldsymbol{u}_d\|_F$$

and

$$\begin{aligned}\|\mathcal{T} - \boldsymbol{u}_1 \circ \boldsymbol{u}_2 \circ \cdots \circ \boldsymbol{u}_d\|_F^2 &= \|\mathcal{T}\|_F^2 - \|\boldsymbol{u}_1 \circ \boldsymbol{u}_2 \circ \cdots \circ \boldsymbol{u}_d\|_F^2\\ &= \|\mathcal{T}\|_F^2 - \|\mathcal{T}\|_\sigma^2.\end{aligned}$$

[1] We were not aware [11] in an earlier version of the current paper.

For the sake of convenience, we may write an $n_1 \times n_2 \times n_3$ tensor $\mathcal{T}$ in the following form $(\mathbf{T}_1|\mathbf{T}_2|\cdots|\mathbf{T}_{n_3})$, where $\mathbf{T}_i \in \mathbb{R}^{n_1\times n_2}$, $1 \le i \le n_3$. For example, let $\mathcal{T} = (t_{ijk}) \in \mathbb{R}^{2\times3\times3}$. Then $\mathcal{T}$ is expressed as the following form:

$$\mathcal{T} = \left(\begin{array}{ccc|ccc|ccc} t_{111} & t_{121} & t_{131} & t_{112} & t_{122} & t_{132} & t_{113} & t_{123} & t_{133} \\ t_{211} & t_{221} & t_{231} & t_{212} & t_{222} & t_{232} & t_{213} & t_{223} & t_{233} \end{array}\right).$$

3 Main results

In this section, we provide a new proof to the Conjecture 1.1. Meanwhile, simple examples are given to illustrate the main result.

Let us first propose a lemma.

Lemma 3.1. *Let $\mathcal{W} \in \mathbb{R}^{n_1\times n_2\times\cdots\times n_d}$ be a rank-one tensor. If $\{\mathcal{W}_1, \mathcal{W}_2, \cdots, \mathcal{W}_m\}$ is a general tensor partition of $\mathcal{W}$, then every $\mathcal{W}_i$ $(i = 1, 2, \cdots, m)$ is a rank-one tensor or a zero tensor.*

Proof. Since $\mathcal{W}$ is a rank-one tensor, $\mathcal{W}$ can be written as

$$\mathcal{W} = \mathbf{w}_1 \circ \mathbf{w}_2 \circ \cdots \circ \mathbf{w}_d, \tag{2}$$

where $\mathbf{w}_j \in \mathbb{R}^{n_j}$, $j = 1, 2, \cdots, d$.
According to the definition of the general partition, we know that every $\mathcal{W}_i$ $(i = 1, 2, \cdots, m)$ is a subtensor of $\mathcal{W}$. Without loss of generality, suppose that $\mathcal{W}_i \in \mathbb{R}^{n_{i,1}\times n_{i,2}\times\cdots\times n_{i,d}}$, then it follows from (2) that $\mathcal{W}_i$ can be written as the following form:

$$\mathcal{W}_i = \mathbf{w}_{i,1} \circ \mathbf{w}_{i,2} \circ \cdots \circ \mathbf{w}_{i,d},$$

where every $\mathbf{w}_{i,j} \in \mathbb{R}^{n_{i,j}}$ is a subvector of $\mathbf{w}_j$, $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, d$. This implies that every $\mathcal{W}_i$ $(i = 1, 2, \cdots, m)$ is a rank-one tensor or a zero tensor. □

We are ready to prove the Conjecture 1.1. For the sake of clarity, the Conjecture 1.1 is written as the Theorem 3.1.

Theorem 3.1. *If $\{\mathcal{T}_1, \mathcal{T}_2, \cdots, \mathcal{T}_m\}$ is a general tensor partition of a tensor $\mathcal{T} \in \mathbb{R}^{n_1\times n_2\times\cdots\times n_d}$, then*

$$\|(\|\mathcal{T}_1\|_\sigma, \|\mathcal{T}_2\|_\sigma, \cdots, \|\mathcal{T}_m\|_\sigma)^T\|_\infty \le \|\mathcal{T}\|_\sigma \le \|(\|\mathcal{T}_1\|_\sigma, \|\mathcal{T}_2\|_\sigma, \cdots, \|\mathcal{T}_m\|_\sigma)^T\|_2, \tag{3}$$

and

$$\|(\|\mathcal{T}_1\|_\star, \|\mathcal{T}_2\|_\star, \cdots, \|\mathcal{T}_m\|_\star)^T\|_2 \le \|\mathcal{T}\|_\star \le \|(\|\mathcal{T}_1\|_\star, \|\mathcal{T}_2\|_\star, \cdots, \|\mathcal{T}_m\|_\star)^T\|_1. \tag{4}$$

Proof. Without loss of generality, we suppose that

$$\|(\|\mathcal{T}_1\|_\sigma, \|\mathcal{T}_2\|_\sigma, \cdots, \|\mathcal{T}_m\|_\sigma)^T\|_\infty = \|\mathcal{T}_1\|_\sigma.$$

Based on the fact that the Frobenious norm of the best rank-one approximation to the subtenor of a tensor is less than or equal to the Frobenious norm of the best rank-one approximation to this tensor, the left hand side of inequality (3) is obviously true. Thus, we only need to prove the right hand side of inequality (3).
Suppose that $\mathcal{W} \in \mathbb{R}^{n_1\times n_2\times\cdots\times n_d}$ is a best rank-one approximation to $\mathcal{T} \in \mathbb{R}^{n_1\times n_2\times\cdots\times n_d}$. By Lemma 2.1, we get

$$\|\mathcal{T}\|_\sigma = \|\mathcal{W}\|_F.$$

Furthermore, suppose that $\{\mathcal{W}_1, \mathcal{W}_2, \cdots, \mathcal{W}_m\}$ is a general tensor partition of $\mathcal{W}$ with the same partition pattern as $\mathcal{T}$, then it follows from the Lemma 2.1 that

$$\begin{aligned} \textstyle\sum_{j=1}^m \|\mathcal{T}_j - \mathcal{W}_j\|_F^2 &= \|\mathcal{T} - \mathcal{W}\|_F^2 \\ &= \|\mathcal{T}\|_F^2 - \|\mathcal{W}\|_F^2 \\ &= \|\mathcal{T}\|_F^2 - \|\mathcal{T}\|_\sigma^2. \end{aligned} \tag{5}$$

Noting that every $\mathcal{W}_j$ ($j = 1, 2, \cdots, m$) is a rank-one tensor or a zero tensor (by Lemma 3.1), we get

$$\begin{aligned}\sum_{j=1}^{m} \|\mathcal{T}_j - \mathcal{W}_j\|_F^2 &\geq \sum_{j=1}^{m}(\|\mathcal{T}_j\|_F^2 - \|\mathcal{T}_j\|_\sigma^2)\\ &= \|\mathcal{T}\|_F^2 - \sum_{j=1}^{m} \|\mathcal{T}_j\|_\sigma^2.\end{aligned} \tag{6}$$

Comparing (5) with (6), we get

$$\|\mathcal{T}\|_\sigma^2 \leq \sum_{j=1}^{m} \|\mathcal{T}_j\|_\sigma^2.$$

This implies that the right hand side of inequality (3) is true.

In what follows we will prove the inequality (4).

Firstly, we prove the right hand side of the inequality (4). As mentioned in [15], the upper bound for the nuclear norm can be obtained through the definition of the nuclear norm.

It follows from the definition of the nuclear norm that

$$\|\mathcal{T}\|_\star = \max_{\|\mathcal{X}\|_\sigma \leq 1} \{\langle \mathcal{T}, \mathcal{X}\rangle\}.$$

Suppose that $\{\mathcal{X}_1, \mathcal{X}_2, \cdots, \mathcal{X}_m\}$ is a general tensor partition of the arbitrary tensor $\mathcal{X}$ with the same partition pattern as $\mathcal{T}$, then

$$\|\mathcal{X}_j\|_\sigma \leq \|\mathcal{X}\|_\sigma,\ 1 \leq j \leq m,$$

and

$$\begin{aligned}\|\mathcal{T}\|_\star &= \max_{\|\mathcal{X}\|_\sigma \leq 1} \left\{\sum_{j=1}^{m} \langle \mathcal{T}_j, \mathcal{X}_j\rangle\right\}\\ &\leq \sum_{j=1}^{m} \max_{\|\mathcal{X}\|_\sigma \leq 1} \{\langle \mathcal{T}_j, \mathcal{X}_j\rangle\}\\ &\leq \sum_{j=1}^{m} \max_{\|\mathcal{X}_j\|_\sigma \leq 1} \{\langle \mathcal{T}_j, \mathcal{X}_j\rangle\}\\ &= \sum_{j=1}^{m} \|\mathcal{T}_j\|_\star.\end{aligned}$$

Secondly, we prove the left hand side of the inequality (4).

It follows from the right hand side of inequality (3) that

$$\|\mathcal{X}\|_\sigma^2 \leq \sum_{j=1}^{m} \|\mathcal{X}_j\|_\sigma^2.$$

Then, according to the definition of the nuclear norm of a tensor, we get

$$\begin{aligned}\|\mathcal{T}\|_\star &= \max_{\|\mathcal{X}\|_\sigma \leq 1} \{\langle \mathcal{T}, \mathcal{X}\rangle\}\\ &\geq \max_{\sum_{j=1}^{m} \|\mathcal{X}_j\|_\sigma^2 \leq 1} \left\{\sum_{j=1}^{m} \langle \mathcal{T}_j, \mathcal{X}_j\rangle\right\}.\end{aligned} \tag{7}$$

Based on the arbitrariness of the tensor $\mathcal{X}$, we can ensure that all its sub-tensors are non-zero tensors. Let $\|\mathcal{X}_j\|_\sigma = \sigma_j$, then $\sigma_j \neq 0$. Furthermore, let $\mathcal{Y}_j = \frac{1}{\sigma_j}\mathcal{X}_j$, then $\|\mathcal{Y}_j\|_\sigma = 1$ and the inequality (7) can be written as

$$\begin{aligned}\|\mathcal{T}\|_\star &\geq \max_{\sum_{j=1}^{m} \|\mathcal{X}_j\|_\sigma^2 \leq 1} \left\{\sum_{j=1}^{m} \langle \mathcal{T}_j, \mathcal{X}_j\rangle\right\}\\ &= \max_{\sum_{j=1}^{m} \sigma_j^2 \leq 1, \|\mathcal{Y}_j\|_\sigma = 1} \left\{\sum_{j=1}^{m} \langle \mathcal{T}_j, \sigma_j \mathcal{Y}_j\rangle\right\}\\ &= \max_{\sum_{j=1}^{m} \sigma_j^2 \leq 1} \left\{\max_{\|\mathcal{Y}_j\|_\sigma = 1, 1 \leq j \leq m} \left\{\sum_{j=1}^{m} \sigma_j \langle \mathcal{T}_j, \mathcal{Y}_j\rangle\right\}\right\}\\ &= \max_{\sum_{j=1}^{m} \sigma_j^2 \leq 1} \left\{\sum_{j=1}^{m} \sigma_j \max_{\|\mathcal{Y}_j\|_\sigma = 1, 1 \leq j \leq m} \{\langle \mathcal{T}_j, \mathcal{Y}_j\rangle\}\right\}\\ &= \max_{\sum_{j=1}^{m} \sigma_j^2 \leq 1} \left\{\sum_{j=1}^{m} \sigma_j \|\mathcal{T}_j\|_\star\right\}.\end{aligned} \tag{8}$$

By using the Cauchy-Schwarz inequality, we get

$$\sum_{j=1}^{m} \sigma_j \|\mathcal{T}_j\|_\star \le \sqrt{\sum_{j=1}^{m} \sigma_j^2}\sqrt{\sum_{j=1}^{m} \|\mathcal{T}_j\|_\star^2}. \tag{9}$$

Noting the inequality (9), if $\sum_{j=1}^{m} \sigma_j^2 \le 1$, then it holds

$$\sum_{j=1}^{m} \sigma_j \|\mathcal{T}_j\|_\star \le \sqrt{\sum_{j=1}^{m} \|\mathcal{T}_j\|_\star^2}.$$

Thus, we get

$$\max_{\sum_{j=1}^{m} \sigma_j^2 \le 1} \left\{ \sum_{j=1}^{m} \sigma_j \|\mathcal{T}_j\|_\star \right\} = \sqrt{\sum_{j=1}^{m} \|\mathcal{T}_j\|_\star^2}. \tag{10}$$

It follows from (8) and (10) that

$$\sqrt{\sum_{j=1}^{m} \|\mathcal{T}_j\|_\star^2} \le \|\mathcal{T}\|_\star. \quad \square$$

□

At last of this section, we give two simple examples to illustrate the validity of the main result.

Example 3.1. *Let*

$$\mathcal{T} = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & -1 & 1 \\ 1 & 0 & 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & -1 & 1 & -1 \end{array}\right) \in \mathbb{R}^{3\times3\times2}.$$

By applying the following general partition,

$$\left(\begin{array}{ccc|ccc} \color{red}{1} & \color{red}{1} & \color{blue}{1} & \color{red}{1} & \color{red}{-1} & \color{blue}{1} \\ \color{magenta}{1} & 0 & \color{blue}{1} & \color{magenta}{1} & 0 & \color{blue}{-1} \\ \color{magenta}{1} & \color{green}{1} & \color{green}{1} & \color{magenta}{-1} & \color{green}{1} & \color{green}{-1} \end{array}\right),$$

the tensor $\mathcal{T}$ is partitioned into five subtensors, each corresponding to one of the five colors. Specifically, let

$$\mathcal{T}_1 = \left(\begin{array}{cc|cc} 1 & 1 & 1 & -1 \end{array}\right) \in \mathbb{R}^{1\times2\times2},$$

$$\mathcal{T}_2 = \left(\begin{array}{c|c} 1 & 1 \\ 1 & -1 \end{array}\right) \in \mathbb{R}^{2\times1\times2},$$

$$\mathcal{T}_3 = \left(\begin{array}{cc|cc} 1 & 1 & 1 & -1 \end{array}\right) \in \mathbb{R}^{1\times2\times2},$$

$$\mathcal{T}_4 = \left(\begin{array}{c|c} 1 & 1 \\ 1 & -1 \end{array}\right) \in \mathbb{R}^{2\times1\times2},$$

and

$$\mathcal{T}_5 = \left(\begin{array}{c|c} 0 & 0 \end{array}\right) \in \mathbb{R}^{1\times1\times2}.$$

Then $\{\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4, \mathcal{T}_5\}$ is a general tensor partition of $\mathcal{T}$.

By a simple computation, we get

$$\|\mathcal{T}_1\|_\sigma = \|\mathcal{T}_2\|_\sigma = \|\mathcal{T}_3\|_\sigma = \|\mathcal{T}_4\|_\sigma = \sqrt{2},$$

$$\|\mathcal{T}_5\|_\sigma = 0,$$

$$\|\mathcal{T}_1\|_\star = \|\mathcal{T}_2\|_\star = \|\mathcal{T}_3\|_\star = \|\mathcal{T}_4\|_\star = 2\sqrt{2},$$

and

$$\|\mathcal{T}_5\|_\star = 0.$$

Then according to the Theorem 3.1, we get

$$\sqrt{2} = \max\{\|\mathcal{T}_1\|_\sigma, \|\mathcal{T}_2\|_\sigma, \cdots, \|\mathcal{T}_5\|_\sigma\} \le \|\mathcal{T}\|_\sigma \le \sqrt{\|\mathcal{T}_1\|_\sigma^2 + \|\mathcal{T}_2\|_\sigma^2 + \cdots + \|\mathcal{T}_5\|_\sigma^2} = 2\sqrt{2}, \tag{11}$$

and

$$4\sqrt{2} = \sqrt{\|\mathcal{T}_1\|_\star^2 + \|\mathcal{T}_2\|_\star^2 + \cdots + \|\mathcal{T}_5\|_\star^2} \le \|\mathcal{T}\|_\star \le \|\mathcal{T}_1\|_\star + \|\mathcal{T}_2\|_\star + \cdots + \|\mathcal{T}_5\|_\star = 8\sqrt{2}. \tag{12}$$

Using the same method above, other upper bounds for the spectral norm and lower bounds for the nuclear norm can be obtained. For the sake of simplicity, we omit the corresponding discussions.

Let

$$\mathcal{W} = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{array}\right).$$

Then

$$\mathcal{W} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \circ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \circ \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

is a rank-one tensor and

$$\sqrt{\langle \mathcal{T}, \mathcal{W} \rangle} = 2\sqrt{2}.$$

Thus, the Frobenius norm of the best rank-one approximation to the tensor $\mathcal{T}$ is larger than or equal to

$$\|\mathcal{W}\|_F = 2\sqrt{2}.$$

Then it follows from (11) that

$$\|\mathcal{T}\|_\sigma = 2\sqrt{2}.$$

This implies that for the tensor $\mathcal{T}$ in this simple example, the tensor partition $\{\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3, \mathcal{T}_4, \mathcal{T}_5\}$ is the best choice of all tensor partitions for the estimation of the spectral norm of $\mathcal{T}$. However, we do not know whether the lower bound of the nuclear norm, estimated by (12), is tight, since, there is no an effective way for estimating the nuclear norm [8].

For the sake of completeness, in what follows, we give another example to illustrate that a tight lower bound of the nuclear norm can be obtained by the Theorem 3.1.

Example 3.2. *Let*

$$\mathcal{T} = \left(\begin{array}{cc|cc|cc|cc} 1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 \end{array}\right) \in \mathbb{R}^{4\times2\times4}.$$

Similar to the discussion above, the tensor $\mathcal{T}$ can be partitioned into eight subtensors $\mathcal{T}_i (1 \le i \le 8)$,

$$\mathcal{T} = \left(\begin{array}{cc|cc|cc|cc} 1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & 1 & 1 \end{array}\right) \in \mathbb{R}^{4\times 2\times 4}.$$

where

$$\mathcal{T}_1 = \mathcal{T}_2 = \mathcal{T}_3 = \mathcal{T}_4 = \left(\begin{array}{c|c} 1 & 1 \\ 1 & -1 \end{array}\right) \in \mathbb{R}^{2\times 1\times 2},$$

$$\mathcal{T}_5 = \mathcal{T}_6 = \mathcal{T}_7 = \mathcal{T}_8 = \left(\begin{array}{c|c} -1 & 1 \\ 1 & 1 \end{array}\right) \in \mathbb{R}^{2\times 1\times 2}.$$

Then, by using Theorem 3.1, we get

$$8 = \sqrt{\|\mathcal{T}_1\|_\star^2 + \|\mathcal{T}_2\|_\star^2 + \cdots + \|\mathcal{T}_8\|_\star^2} \le \|\mathcal{T}\|_\star \le \|\mathcal{T}_1\|_\star + \|\mathcal{T}_2\|_\star + \cdots + \|\mathcal{T}_8\|_\star = 16\sqrt{2}. \tag{13}$$

Furthermore, the tensor $\mathcal{T}$ can be decomposed into a sum of two rank-one tensors. That is

$$\mathcal{T} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \circ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \circ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \circ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \circ \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}.$$

Thus, it holds

$$\begin{aligned} \|\mathcal{T}\|_\star &\le \sqrt{1^2 + 1^2 + 1^2 + 1^2}\sqrt{1^2 + 1^2}\sqrt{1^2 + 1^2} \\ &\quad + \sqrt{1^2 + (-1)^2 + 1^2 + (-1)^2}\sqrt{1^2 + 1^2}\sqrt{1^2 + (-1)^2} \\ &= 8. \end{aligned} \tag{14}$$

It follows from (13) and (14) that

$$\|\mathcal{T}\|_\star = 8.$$

This implies that a tight lower bound of the nuclear norm is obtained.

By Theorem 3.1, the following estimates about the spectral norm can also be obtained,

$$\sqrt{2} = \max\{\|\mathcal{T}_1\|_\sigma, \|\mathcal{T}_2\|_\sigma, \cdots, \|\mathcal{T}_8\|_\sigma\} \le \|\mathcal{T}\|_\sigma \le \sqrt{\|\mathcal{T}_1\|_\sigma^2 + \|\mathcal{T}_2\|_\sigma^2 + \cdots + \|\mathcal{T}_8\|_\sigma^2} = 8\sqrt{2}. \tag{15}$$

However, neither the lower bound nor the upper bound given by (15) are tight. Actually, through a series of calculations, we get

$$\|\mathcal{T}\|_\sigma = 4.$$

As discussed above, how to choose a better tensor partition for the estimation of the spectral norm and nuclear norm of a general tensor is no fixed format, and it could be one of the future research.

4 Conclusions

In this paper, by considering the structure of the subtensors of rank-one tensors, we present a new proof to the conjecture proposed by Li [15]. The proof is different and simper than the method for proving the main results relating to the bounds for the spectral norm and nuclear norm in [11]. As discussed in [15], we believe these inequalities will have great potential in various applications. In the future, we will find more applications of these inequalities.

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