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Research Article

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Arithmetic properties for Andrews' $(48, 6)$- and $(48, 18)$-singular overpartitions

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Abstract: Singular overpartition functions were defined by Andrews. Let $\overline{C}_{k,i}(n)$ denote the number of (k, i)-singular overpartitions of n, which counts the number of overpartitions of n in which no part is divisible by k and only parts $\pm i \pmod k$ may be overlined. A number of congruences modulo 3, 9 and congruences modulo powers of 2 for $\overline{C}_{k,i}(n)$ were discovered by Ahmed and Baruah, Andrews, Chen, Hirschhorn and Sellers, Naika and Gireesh, Shen and Yao for some pairs (k, i). In this paper, we prove some congruences modulo powers of 2 for $\overline{C}_{48,6}(n)$ and $\overline{C}_{48,18}(n)$.

Keywords: congruence, singular overpartition, theta function

MSC: 11P83, 05A17

1 Introduction

In a recent work, Andrews [1] defined combinatorial objects that he called singular overpartitions. Moreover, Andrews proved that these singular overpartitions, which depend on two parameters k and i, can be enumerated by the function $\overline{C}_{k,i}(n)$ which denotes the number of overpartitions of n in which no part is divisible by k and only parts $\equiv \pm i \pmod k$ may be overlined. Andrews also established the generating function for $\overline{C}_{k,i}(n)$. For $k \geq 3$ and $1 \leq i \leq \lfloor \frac{k}{2} \rfloor$, the generating function for $\overline{C}_{k,i}(n)$ is given by

$$\sum_{n=0}^{\infty} \overline{C}_{k,i}(n) q^n = \frac{(-q^i; q^k)_\infty (-q^{k-i}; q^k)_\infty (q^k; q^k)_\infty}{(q; q)_\infty}, \tag{1.1}$$

where

$$(a; q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n).$$

Furthermore, by elementary generating function manipulations, Andrews [1] proved that for all $n \geq 0$,

$$\overline{C}_{3,1}(9n + 3) \equiv \overline{C}_{3,1}(9n + 6) \equiv 0 \pmod 3.$$

Since then, a number of congruences modulo 2, 3, 4, 8, 9, 16, 32 and 64 for $\overline{C}_{k,i}(n)$ have been discovered for some pairs (k, i), see for example, Ahmed and Baruah [2], Chen [3] Chen, Hirschhorn and Sellers [4], Naika and Gireesh [5], Shen [6] and Yao [7].

In this paper, we prove some new congruences modulo powers of 2 for $(48, 6)$- and $(48, 18)$-singular overpartitions.

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2 Congruences modulo 2, 8 and 16 for $\overline{C}_{48,6}(n)$

In this section, we prove some congruences modulo 2, 8 and 16 for $\overline{C}_{48,6}(n)$. It should be noted that congruences for $\overline{C}_{48,6}(n)$ and $\overline{C}_{48,18}(n)$ were not discovered before.

Theorem 2.1. *For $n \geq 0$,*

$$\overline{C}_{48,6}(16n+8) \equiv 0 \pmod 8, \tag{2.1}$$

$$\overline{C}_{48,6}(16n+12) \equiv 0 \pmod 8, \tag{2.2}$$

and

$$\overline{C}_{48,6}(32n+28) \equiv 0 \pmod{16}. \tag{2.3}$$

Theorem 2.2. *For $n, k \geq 0$,*

$$\overline{C}_{48,6}\left(2^{2k+3}n + \frac{31 \times 2^{2k} - 10}{3}\right) \equiv 0 \pmod 2, \tag{2.4}$$

$$\overline{C}_{48,6}\left(2^{2k+4}n + \frac{17 \times 2^{2k+1} - 10}{3}\right) \equiv 0 \pmod 2, \tag{2.5}$$

$$\overline{C}_{48,6}\left(2^{2k+4}n + \frac{23 \times 2^{2k+1} - 10}{3}\right) \equiv 0 \pmod 2, \tag{2.6}$$

$$\overline{C}_{48,6}\left(2^{2k+4}n + \frac{49 \times 2^{2k} - 10}{3}\right) \equiv 0 \pmod 2. \tag{2.7}$$

Theorem 2.3. *For $n, k \geq 0$,*

$$\overline{C}_{48,6}\left(4^{k+2}n + \frac{25 \times 4^{k} - 10}{3}\right) \equiv \begin{cases} 1 \pmod 2, & \text{if } 2n+1 = \dfrac{m(3m-1)}{2} \text{ for some integer } m, \\ 0 \pmod 2, & \text{otherwise.} \end{cases}$$

Proofs of Theorems 2.1–2.3. First, we introduce some notations. Throughout this paper, for any positive integer k, f_k is defined by

$$f_k := (q^k; q^k)_\infty = \prod_{n=1}^{\infty}(1 - q^{kn}),$$

and $S(q)$ and $T(q)$ are the Göllnitz-Gordon functions defined by

$$S(q) = \frac{1}{(q;q^8)_\infty (q^4;q^8)_\infty (q^7;q^8)_\infty}$$

and

$$T(q) = \frac{1}{(q^3;q^8)_\infty (q^4;q^8)_\infty (q^5;q^8)_\infty}.$$

By (1.1), we have

$$\sum_{n=0}^{\infty} \overline{C}_{48,6}(n) q^n = \frac{f_{12}^2 T(-q^6)}{f_1 f_6} \tag{2.8}$$

and

$$\sum_{n=0}^{\infty} \overline{C}_{48,18}(n) q^n = \frac{f_{12}^2 S(-q^6)}{f_1 f_6}. \tag{2.9}$$

It follows from (3.15) and (3.16) in [8] that

$$T(-q^2) = \frac{1}{2q}\left(\frac{f_2^3}{f_1f_4^2} - \frac{f_1}{f_4}\right) \tag{2.10}$$

and

$$S(-q^2) = \frac{1}{2}\left(\frac{f_2^3}{f_1f_4^2} + \frac{f_1}{f_4}\right). \tag{2.11}$$

Substituting (2.10) into (2.8) yields

$$2\sum_{n=0}^{\infty}\overline{C}_{48,6}(n)q^{n+3} = \frac{1}{f_1f_3}\cdot f_6^2 - \frac{f_3}{f_1}\cdot\frac{f_{12}}{f_6}. \tag{2.12}$$

It follows from Lemma 2.6 in [9] that

$$\frac{f_3}{f_1} = \frac{f_4f_6f_{16}f_{24}^2}{f_2^2f_8f_{12}f_{48}} + q\frac{f_6f_8^2f_{48}}{f_2^2f_{16}f_{24}}. \tag{2.13}$$

Xia and Yao [10] proved the following 2-dissection formula for $\frac{1}{f_1f_3}$:

$$\frac{1}{f_1f_3} = \frac{f_8^2f_{12}^5}{f_2^2f_4f_6^4f_{24}^2} + q\frac{f_4^5f_{24}^2}{f_2^4f_6^2f_8^2f_{12}}. \tag{2.14}$$

By the binomial theorem, for any positive integer k and any prime p,

$$f_1^{p^k} \equiv f_p^{p^{k-1}} \pmod{p^k}. \tag{2.15}$$

Substituting (2.14) and (2.13) into (2.12) and employing (2.15), we have

$$2\sum_{n=0}^{\infty}\overline{C}_{48,6}(2n)q^{n+1} = \frac{1}{f_1^4}\cdot\frac{f_2^5f_{12}^2}{f_4^2f_6} - \frac{1}{f_1^2}\cdot\frac{f_4^2f_6f_{24}}{f_8f_{12}} \tag{2.16}$$

and

$$\begin{aligned}2\sum_{n=0}^{\infty}\overline{C}_{48,6}(2n+1)q^{n+2} &= \frac{f_4^2f_6^5}{f_1^2f_2f_3^2f_{12}^2} - \frac{f_2f_8f_{12}^2}{f_1^2f_4f_{24}}\\ &\equiv \frac{f_3^2}{f_1^2}\cdot\frac{f_4^2f_6^3}{f_2f_{12}^2} - \frac{1}{f_1^2}\cdot\frac{f_2f_8f_{12}^2}{f_4f_{24}} \pmod{4}.\end{aligned} \tag{2.17}$$

From Lemma 2.1 in [11],

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5f_{16}^2} + 2q\frac{f_4^2f_{16}^2}{f_2^5f_8} \tag{2.18}$$

and

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14}f_8^4} + 4q\frac{f_4^2f_8^4}{f_2^{10}}. \tag{2.19}$$

Substituting (2.18) and (2.19) into (2.16), we obtain

$$\sum_{n=0}^{\infty}\overline{C}_{48,6}(4n)q^{n} = 2\cdot\frac{1}{f_1^4}\cdot\frac{1}{f_1f_3}\cdot f_4^4f_6^2 - \frac{1}{f_1^4}\cdot\frac{f_3}{f_1}\cdot\frac{f_2^4f_8^2f_{12}}{f_4^2f_6} \tag{2.20}$$

and

$$2\sum_{n=0}^{\infty}\overline{C}_{48,6}(4n+2)q^{n+1} = \frac{f_2^{12}f_6^2}{f_1^9f_3f_4^4} - \frac{f_2^2f_3f_4^4f_{12}}{f_1^5f_6f_8^2}. \tag{2.21}$$

Substituting (2.19), (2.14) and (2.13) into (2.20) and employing (2.15), we deduce that

$$\sum_{n=0}^{\infty}\overline{C}_{48,6}(8n)q^n \equiv 2\frac{f_2^{17}f_6^5}{f_1^{16}f_3^2f_4^2f_{12}^2} - \frac{f_2^{13}f_8f_{12}^2}{f_1^{12}f_4^3f_{24}} + 4q\frac{f_4^8f_6f_{24}}{f_1^8f_8f_{12}}$$

$$\equiv 2\cdot\frac{1}{f_3^2}\cdot f_2^5f_6 - \frac{1}{f_1^4}\cdot\frac{f_2^9f_8f_{12}^2}{f_4^3f_{24}} + 4qf_4^4f_6^3 \pmod 8 \tag{2.22}$$

and

$$\sum_{n=0}^{\infty}\overline{C}_{48,6}(8n+4)q^n = 2\frac{f_2^{23}f_{12}^2}{f_1^{18}f_4^6f_6} + 8\frac{f_2^5f_4^6f_6^5}{f_1^{12}f_3^2f_{12}^2} - \frac{f_2^{12}f_6f_{24}}{f_1^{12}f_8f_{12}} - 4\frac{f_2f_4^5f_8f_{12}^2}{f_1^8f_{24}}$$

$$\equiv 2\cdot\frac{1}{f_1^2}\cdot\frac{f_2^{15}f_{12}^2}{f_4^6f_6} + 8\frac{f_4^6}{f_2} - \left(\frac{1}{f_1^4}\right)^3\cdot\frac{f_2^{12}f_6f_{24}}{f_8f_{12}} - 4\frac{f_4^5f_8f_{12}^2}{f_2^3f_{24}} \pmod{16}. \tag{2.23}$$

Substituting (2.18) and (2.19) into (2.22) and utilizing (2.15), we see that

$$\sum_{n=0}^{\infty}\overline{C}_{48,6}(16n+8)q^n \equiv 4f_2^4f_3^3 - 4\frac{f_4^5f_6^2}{f_1f_2f_{12}} - 4q\frac{f_1^5f_6^2f_{24}^2}{f_3^4f_{12}}$$

$$\equiv 4f_1F(q) \pmod 8, \tag{2.24}$$

where

$$F(q) = \frac{f_3^3}{f_1}\cdot f_2^4 - \frac{f_2^2f_4^3f_6^2}{f_{12}} - q\frac{f_2^4f_{12}^3}{f_4}. \tag{2.25}$$

Hirschhorn, Garvan and Borwein [12] proved that

$$\frac{f_3^3}{f_1} = \frac{f_4^3f_6^2}{f_2^2f_{12}} + q\frac{f_{12}^3}{f_4}. \tag{2.26}$$

Substituting (2.26) into (2.25), we find that

$$F(q) = 0. \tag{2.27}$$

Congruence (2.1) follows from (2.24) and (2.27).

Substituting (2.18) and (2.19) into (2.23) and using (2.15) yields

$$\sum_{n=0}^{\infty}\overline{C}_{48,6}(16n+12)q^n \equiv 4\frac{f_1^{10}f_6^2f_8^2}{f_2^4f_3f_4} + 4\frac{f_2^{30}f_3f_{12}}{f_1^{26}f_4^5f_6}$$

$$\equiv 4\frac{f_2^2f_6^3f_8^2}{f_1^2f_4f_{12}}\left(\frac{f_{12}}{f_3f_6} + f_3\right) \pmod{16}. \tag{2.28}$$

Define

$$\sum_{n=0}^{\infty}b(n)q^n = \frac{f_4}{f_1f_2}. \tag{2.29}$$

Replacing q by $-q$ in (2.29) and employing (2.15) and (2.30), and using the fact that

$$(-q;-q)_\infty = \frac{f_2^3}{f_1f_4}, \tag{2.30}$$

we have

$$\sum_{n=0}^{\infty}(-1)^n b(n)q^n = \frac{f_1f_4^2}{f_2^4} \equiv f_1 \pmod 4. \tag{2.31}$$

Hence,

$$\frac{f_4}{f_1f_2}+f_1\equiv\sum_{n=0}^{\infty}(1+(-1)^n)b(n)q^n=2\sum_{n=0}^{\infty}b(2n)q^{2n} \pmod 4 \tag{2.32}$$

and

$$\frac{f_4}{f_1f_2}-f_1\equiv\sum_{n=0}^{\infty}(1-(-1)^n)b(n)q^n=2\sum_{n=0}^{\infty}b(2n+1)q^{2n+1} \pmod 4. \tag{2.33}$$

In view of (2.15), (2.28) and (2.32),

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{C}_{48,6}(16n+12)q^n &\equiv 8\frac{f_2^2f_6^3f_8^2}{f_1^2f_4f_{12}}\sum_{n=0}^{\infty}b(2n)q^{6n}\\ &\equiv 8\frac{f_6f_8^2}{f_2}\sum_{n=0}^{\infty}b(2n)q^{6n} \pmod{16}.\end{aligned} \tag{2.34}$$

Congruences (2.2) and (2.3) follow from (2.34).

From (3.29) in [10],

$$\frac{f_3^2}{f_1^2}=\frac{f_4^4f_6f_{12}^2}{f_2^5f_8f_{24}}+2q\frac{f_4f_6^2f_8f_{24}}{f_2^4f_{12}}. \tag{2.35}$$

Substituting (2.18) and (2.35) into (2.17) and employing (2.15) yields

$$2\sum_{n=0}^{\infty}\overline{C}_{48,6}(4n+1)q^{n+1}\equiv\frac{f_2^6f_3^4}{f_1^6f_4f_{12}}-\frac{f_4^6f_6^2}{f_1^4f_2f_{12}f_8^2} \pmod 4 \tag{2.36}$$

and

$$\sum_{n=0}^{\infty}\overline{C}_{48,6}(4n+3)q^{n+1}\equiv\frac{f_2^3f_3^5f_4f_{12}}{f_1^5f_6^3}-\frac{f_2f_6^2f_8^2}{f_1^4f_{12}}\equiv\frac{f_3^3}{f_1}\cdot\frac{f_4^2}{f_2}-\frac{f_8^2}{f_2} \pmod 2. \tag{2.37}$$

Substituting (2.26) into (2.37) and using (2.15), we deduce that

$$\sum_{n=0}^{\infty}\overline{C}_{48,6}(4n+3)q^{n+1}\equiv\frac{f_4^5f_6^2}{f_2^3f_{12}}-\frac{f_8^2}{f_2}+q\frac{f_4f_{12}^3}{f_2}\equiv q\frac{f_4f_{12}^3}{f_2} \pmod 2,$$

which implies that for $n\geq 0$,

$$\overline{C}_{48,6}(8n+7)\equiv 0 \pmod 2. \tag{2.38}$$

By (2.15) and (2.36),

$$2\sum_{n=0}^{\infty}\overline{C}_{48,6}(4n+1)q^{n+1}\equiv\frac{1}{f_1^2}\cdot\frac{f_4f_6^2}{f_{12}}-\frac{f_4^2f_6^2}{f_2^3f_{12}} \pmod 4. \tag{2.39}$$

Substituting (2.18) into (2.39) and employing (2.15), we have

$$2\sum_{n=0}^{\infty}\overline{C}_{48,6}(8n+5)q^{n+1}\equiv\frac{f_2f_3^2f_4^5}{f_1^5f_6f_8^2}-\frac{f_2^2f_3^2}{f_1^3f_6}\equiv\frac{f_3^2}{f_6}\left(\frac{f_4}{f_1f_2}-f_1\right) \pmod 4. \tag{2.40}$$

Thanks to (2.33) and (2.40),

$$\sum_{n=0}^{\infty}\overline{C}_{48,6}(8n+5)q^{n}\equiv\frac{f_3^2}{f_6}\sum_{n=0}^{\infty}b(2n+1)q^{2n}$$

$$\equiv \sum_{n=0}^{\infty} b(2n+1)q^{2n} \pmod 2 \qquad \text{(by (2.15))}.$$

Therefore, for $n \geq 0$,

$$\overline{C}_{48,6}(16n+13) \equiv 0 \pmod 2 \tag{2.41}$$

and

$$\overline{C}_{48,6}(16n+5) \equiv b(2n+1) \pmod 2. \tag{2.42}$$

It is well-known that

$$f_1 = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}}. \tag{2.43}$$

Combining (2.43) and (2.31) yields

$$b(n) \equiv \begin{cases} 1 \pmod 2, & \text{if } n = \dfrac{m(3m-1)}{2} \text{ for some integer } m, \\ 0 \pmod 2, & \text{otherwise.} \end{cases} \tag{2.44}$$

Thanks to (2.15) and (2.21),

$$2\sum_{n=0}^{\infty} \overline{C}_{48,6}(4n+2)q^{n+1} \equiv \frac{f_3^3}{f_1} - \frac{f_3 f_{12}}{f_1 f_6} \pmod 4. \tag{2.45}$$

Based on (2.15), (2.12) and (2.45), we see that for $n \geq 0$,

$$\overline{C}_{48,6}(4n+10) \equiv C_{48,6}(n) \pmod 2. \tag{2.46}$$

By (2.46) and mathematical induction, for $n,\ k \geq 0$,

$$\overline{C}_{48,6}\left(4^k n + \frac{10(4^k-1)}{3}\right) \equiv \overline{C}_{48,6}(n) \pmod 2. \tag{2.47}$$

Replacing n by $8n+7$, $16n+8$, $16n+12$, $16n+13$ in (2.47) and using (2.38), (2.1), (2.2), (2.41) respectively, we obtain the congruences stated in Theorem 2.2.

It follows from (2.42) and (2.44) that

$$\overline{C}_{48,6}(16n+5) \equiv \begin{cases} 1 \pmod 2, & \text{if } 2n+1 = \dfrac{m(3m-1)}{2} \text{ for some integer } m, \\ 0 \pmod 2, & \text{otherwise.} \end{cases} \tag{2.48}$$

Theorem 2.3 follows form (2.47) and (2.48). This completes the proof.

3 Congruences modulo 2, 8 and 16 for $\overline{C}_{48,18}(n)$

We prove several congruences modulo 2, 8 and 16 for $\overline{C}_{48,18}(n)$ in this section.

Theorem 3.1. *For $n \geq 0$,*

$$\overline{C}_{48,18}(16n+11) \equiv 0 \pmod 8, \tag{3.1}$$

$$\overline{C}_{48,18}(16n+15) \equiv 0 \pmod 8, \tag{3.2}$$

$$\overline{C}_{48,18}(32n+15) \equiv 0 \pmod{16}. \tag{3.3}$$

Theorem 3.2. *For* $n,\ k \geq 0$,

$$\overline{C}_{48,18}\left(2^{2k+3}n + \frac{7 \times 2^{2k} - 1}{3}\right) \equiv 0 \pmod 2, \tag{3.4}$$

$$\overline{C}_{48,18}\left(2^{2k+4}n + \frac{25 \times 2^{2k} - 1}{3}\right) \equiv 0 \pmod 2, \tag{3.5}$$

$$\overline{C}_{48,18}\left(2^{2k+4}n + \frac{17 \times 2^{2k+1} - 1}{3}\right) \equiv 0 \pmod 2, \tag{3.6}$$

$$\overline{C}_{48,18}\left(2^{2k+4}n + \frac{23 \times 2^{2k+1} - 1}{3}\right) \equiv 0 \pmod 2. \tag{3.7}$$

Theorem 3.3. *For* $n,\ k \geq 0$,

$$\overline{C}_{48,18}\left(4^{k+2}n + \frac{4^k - 1}{3}\right) \equiv \begin{cases} 1 \pmod 2, & \text{if } 2n = \dfrac{m(3m-1)}{2} \text{ for some integer } m, \\ 0 \pmod 2, & \text{otherwise.} \end{cases} \tag{3.8}$$

Proofs of Theorems 3.1–3.3. In view of (2.9) and (2.11),

$$2\sum_{n=0}^{\infty} \overline{C}_{48,18}(n)q^n = \frac{1}{f_1 f_3} \cdot f_6^2 + \frac{f_3}{f_1} \cdot \frac{f_{12}}{f_6}. \tag{3.9}$$

Substituting (2.14) and (2.13) into (3.9) yields

$$2\sum_{n=0}^{\infty} \overline{C}_{48,18}(n)q^n = \frac{f_8^2 f_{12}^5}{f_2^2 f_4 f_6^2 f_{24}^2} + \frac{f_4 f_{16} f_{24}^2}{f_2^2 f_8 f_{48}} + q\frac{f_4^5 f_{24}^2}{f_2^4 f_8^2 f_{12}} + q\frac{f_8^2 f_{12} f_{48}}{f_2^2 f_{16} f_{24}}.$$

Therefore,

$$2\sum_{n=0}^{\infty} \overline{C}_{48,18}(2n)q^n = \frac{f_4^2 f_6^5}{f_1^2 f_2 f_3^2 f_{12}^2} + \frac{f_2 f_8 f_{12}^2}{f_1^2 f_4 f_{24}} \tag{3.10}$$

and

$$2\sum_{n=0}^{\infty} \overline{C}_{48,18}(2n+1)q^n = \frac{1}{f_1^4} \cdot \frac{f_2^5 f_{12}^2}{f_4^2 f_6} + \frac{1}{f_1^2} \cdot \frac{f_4^2 f_6 f_{24}}{f_8 f_{12}}. \tag{3.11}$$

Substituting (2.18) and (2.19) into (3.11) yields

$$2\sum_{n=0}^{\infty} \overline{C}_{48,18}(4n+1)q^n = \frac{f_2^2 f_3 f_4^4 f_{12}}{f_1^5 f_6 f_8^2} + \frac{f_2^{12} f_6^2}{f_1^9 f_3 f_4^4} \tag{3.12}$$

and

$$\sum_{n=0}^{\infty} \overline{C}_{48,18}(4n+3)q^n = \frac{1}{f_1^4} \cdot \frac{f_3}{f_1} \cdot \frac{f_2^4 f_8^2 f_{12}}{f_4^2 f_6} + 2 \cdot \frac{1}{f_1 f_3} \cdot \frac{1}{f_1^4} \cdot f_4^4 f_6^2. \tag{3.13}$$

Substituting (2.19), (2.14) and (2.13) into (3.13) and employing (2.15), we have

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{C}_{48,18}(8n+3)q^n &= \frac{f_2^{13} f_8 f_{12}^2}{f_1^{12} f_4^3 f_{24}} + 2\frac{f_2^{17} f_6^5}{f_1^{16} f_3^2 f_4^2 f_{12}^2} + 4q\frac{f_4^8 f_6 f_{24}}{f_1^8 f_8 f_{12}} + 8q\frac{f_2^{11} f_4^2 f_{12}^2}{f_1^{14} f_6} \\ &\equiv \frac{1}{f_1^4} \cdot \frac{f_2^9 f_8 f_{12}^2}{f_4^3 f_{24}} + 2 \cdot \frac{1}{f_3^2} \cdot f_2^5 f_6 + 4q\frac{f_4^8 f_6 f_{24}}{f_8^2 f_{12}} \pmod 8 \end{aligned} \tag{3.14}$$

and

$$\sum_{n=0}^{\infty}\overline{C}_{48,18}(8n+7)q^n = \frac{f_2^{12}f_6f_{24}}{f_1^{12}f_8f_{12}} + 4\frac{f_2f_4^5f_8f_{12}^2}{f_1^8f_{24}} + 2\frac{f_2^{23}f_{12}^2}{f_1^{18}f_4^6f_6} + 8\frac{f_2^5f_4^6f_6^5}{f_1^{12}f_3^2f_{12}^2}$$
$$\equiv \left(\frac{1}{f_1^4}\right)^3\cdot\frac{f_2^{12}f_6f_{24}}{f_8f_{12}} + 4\frac{f_2f_4^3f_8f_{12}^2}{f_{24}}$$
$$+ 2\cdot\frac{1}{f_1^2}\cdot\frac{f_2^{15}f_{12}^2}{f_4^6f_6} + 8\frac{f_4^6f_6^4}{f_2f_{12}^2} \pmod{16}. \tag{3.15}$$

Substituting (2.18) and (2.19) into (3.14) yields

$$\sum_{n=0}^{\infty}\overline{C}_{48,18}(16n+11)q^n \equiv 4\frac{f_4^5f_6^2}{f_1f_2f_{12}} + 4\frac{f_2^8f_3f_{12}}{f_4^2f_6} + 4q\frac{f_1^5f_6^2f_{24}^2}{f_3^4f_{12}}$$
$$\equiv 4f_1F(q) \pmod 8, \qquad \text{(by (2.15))} \tag{3.16}$$

where $F(q)$ is defined by (2.25). Congruence (3.1) follows from (2.27) and (3.16).

Substituting (2.18) and (2.19) into (3.15) yields

$$\sum_{n=0}^{\infty}\overline{C}_{48,18}(16n+15)q^n \equiv 4\frac{f_1^{10}f_6^2f_8^2}{f_2^4f_3f_4} - 4\frac{f_2^{30}f_3f_{12}}{f_1^{26}f_4^5f_6}$$
$$\equiv 4\frac{f_2^2f_8^2f_{12}}{f_1^2f_4f_6}\left(\frac{f_{12}}{f_3f_6} - f_3\right) \qquad \text{(by (2.15))}$$
$$\equiv 8\frac{f_2^2f_8^2f_{12}}{f_1^2f_4f_6}\sum_{n=0}^{\infty}b(2n+1)q^{6n+3} \pmod{16} \quad \text{(by (2.33))}$$
$$\equiv 8\frac{f_8^2f_{12}}{f_2f_6}\sum_{n=0}^{\infty}b(2n+1)q^{6n+3} \pmod{16} \quad \text{(by (2.15))}. \tag{3.17}$$

Congruences (3.2) and (3.3) follow from (3.17).

Based on (2.15) and (3.10),

$$2\sum_{n=0}^{\infty}\overline{C}_{48,18}(2n)q^n \equiv \frac{f_3^2}{f_1^2}\cdot\frac{f_4^2}{f_2f_6} + \frac{1}{f_1^2}\cdot\frac{f_2f_8f_{12}^2}{f_4f_{24}} \pmod 4. \tag{3.18}$$

Substituting (2.18) and (2.35) into (3.18) and employing (2.15), we deduce

$$2\sum_{n=0}^{\infty}\overline{C}_{48,18}(4n)q^n \equiv \frac{f_2^6f_6^2}{f_1^6f_4f_{12}} + \frac{f_4^6f_6^2}{f_1^4f_2f_8^2f_{12}} \equiv \frac{1}{f_1^2}\cdot\frac{f_4f_6^2}{f_{12}} + \frac{f_2f_6^2}{f_{12}} \pmod 4 \tag{3.19}$$

and

$$\sum_{n=0}^{\infty}\overline{C}_{48,18}(4n+2)q^n \equiv \frac{f_2^3f_3f_4f_{12}}{f_1^5f_6} + \frac{f_2f_6^2f_8^2}{f_1^4} \equiv \frac{f_2^3f_3^3}{f_1} + \frac{f_8^2}{f_2} \pmod 2. \tag{3.20}$$

Substituting (2.26) into (3.20) and employing (2.15), we deduce that

$$\sum_{n=0}^{\infty}\overline{C}_{48,18}(4n+2)q^n \equiv \frac{f_2f_4^3f_6^2}{f_{12}} + \frac{f_8^2}{f_2} + q\frac{f_2^3f_{12}^3}{f_4} \equiv qf_2f_{12}^3 \pmod 2. \tag{3.21}$$

It follows from (3.21) that for $n \geq 0$,

$$\overline{C}_{48,18}(8n+2) \equiv 0 \pmod 2. \tag{3.22}$$

Substituting (2.18) into (3.19) and utilizing (2.15) and (2.32), we see that, modulo 4,

$$2\sum_{n=0}^{\infty}\overline{C}_{48,18}(8n)q^n \equiv \frac{f_2f_3^2f_4^5}{f_1^5f_6f_8^2}+\frac{f_1f_3^2}{f_6} \equiv \frac{f_3^2f_4}{f_1f_2f_6}+f_1\frac{f_3^2}{f_6}$$

$$=\frac{f_3^2}{f_6}\left(\frac{f_4}{f_1f_2}+f_1\right)=2\frac{f_3^2}{f_6}\sum_{n=0}^{\infty}b(2n)q^{2n}\equiv 2\sum_{n=0}^{\infty}b(2n)q^{2n},$$

which implies that for $n \geq 0$,

$$\overline{C}_{48,18}(16n+8)\equiv 0 \pmod 2 \tag{3.23}$$

and

$$\overline{C}_{48,18}(16n)\equiv b(2n) \pmod 2. \tag{3.24}$$

By (2.15) and (3.12),

$$2\sum_{n=0}^{\infty}\overline{C}_{48,18}(4n+1)q^n \equiv \frac{f_3f_{12}}{f_1f_6}+\frac{f_6^2}{f_1f_3} \pmod 4. \tag{3.25}$$

Thanks to (3.9) and (3.25), we see that for $n \geq 0$,

$$C_{48,18}(4n+1)\equiv C_{48,18}(n) \pmod 2. \tag{3.26}$$

By (3.26) and mathematical induction, we deduce that for $n, k \geq 0$,

$$C_{48,18}\left(4^kn+\frac{4^k-1}{3}\right)\equiv C_{48,18}(n) \pmod 2. \tag{3.27}$$

Replacing n by $8n+2$, $16n+8$, $16n+11$, $16n+15$ in (3.27) and using (3.22), (3.23), (3.1) and (3.2) respectively, we arrive at the congruences stated in Theorem 3.2. It follows from (2.44) and (3.24) that

$$C_{48,18}(16n)\equiv \begin{cases} 1 \pmod 2, & \text{if } 2n=\dfrac{m(3m-1)}{2} \text{ for some integer } m,\\ 0 \pmod 2, & \text{otherwise.}\end{cases} \tag{3.28}$$

Theorem 3.3 follows from (3.27) and (3.28). This completes the proof.

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