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Research Article

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On Partitions and Arf Semigroups

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Abstract: In this study we examine some combinatorial properties of the Arf semigroup. In previous work, the author and Karakaş, Gümüşbaş defined an Arf partition of a positive integer n. Here, we continue this work and give new results on Arf partitions. In particular, we analyze the relation among an Arf partition, its Young dual diagram, and the corresponding rational Young diagram. Additionally, this study contains some results that present the relations between partitions and Arf semigroup polynomials.

Keywords: Arf numerical semigroup, Young diagram, partition, Arf partition.

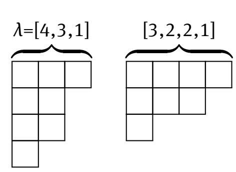
MSC: 20M14, 05A17, 11D07

1 Introduction

A *partition* $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_r]$ of a positive integer n is a non-increasing list of positive integers, $\lambda_r \le \lambda_{r-1} \le \dots \le \lambda_1$, whose sum is n and *length* is r. If $\lambda_i \neq \lambda_{i+1}$, $1 \le i \le r-1$, then λ is called a *strict dominant partition*.

Partitions occur in several branches of physics and mathematics such as representation theory and coding theory, see [12, 13]. Partitions can be visualized with Young diagrams, see [9, 17]. The *Young diagram of a partition* λ consists of a left-justified shape of r columns of boxes with lengths $\lambda_1, \lambda_2, \dots, \lambda_r$. Flipping a Young diagram over its main diagonal (from upper left to lower right) gives the *conjugate diagram*. The *conjugate partition* of λ is the partition corresponding to the conjugate diagram of the Young diagram of λ.

For example, we consider the Young diagram of the partition $\lambda = [4, 3, 1]$. In the Young diagram of λ, we have 4 boxes in the first column, we have 3 boxes in the second column and one box in the third column. Hence, we obtain Young diagrams of λ and the conjugate partition of λ, respectively, as follows:



In a Young diagram, the number of boxes in a column (or a row) is called the *length* of that column (or, respectively, that row). The length of a row is at most the number of columns of the diagram, and there may be more than one row with the same length.

Assume that there are r columns in a Young diagram and there are u_i rows of length i, for each $i = 1, 2, \dots, r$, $u_i \ge 0$. Then we denote such a Young diagram of the form (shape) $Y = 1^{u_1} 2^{u_2} 3^{u_3} \cdots r^{u_r}$ and we have $n = \sum_{j=1}^{r} j u_j$. If there is no row of length j, $1 \le j \le r$, then $u_j = 0$ and we omit j^0 in the presentation of a Young diagram Y.

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If $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_r]$ is the partition corresponding to $Y = 1^{u_1} 2^{u_2} \cdots r^{u_r}$, then

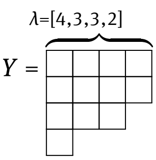
$$\lambda_j = \sum_{i=j}^{r} u_i \,, \ 1 \le j \le r.$$

Note that

$$\lambda_j - \lambda_{j+1} = u_j \ \text{ for each } \ j = 1, \dots, r-1 \ \text{ and } \ \lambda_r = u_r.$$

For example, if $\lambda = [4, 3, 3, 2]$, then $Y = 1^1 3^1 4^2$ and we have the following diagram;

$$Y = 1^1 3^1 4^2 \Rightarrow \begin{array}{l} 4^2 : \text{there are 2 rows with the length of 4,} \\ 3^1 : \text{there is 1 row with the length of 3,} \\ 2^0 : \text{there are no rows with the length of 2,} \\ 1^1 : \text{there is 1 row with the length of 1.} \end{array} \Rightarrow Y =$$

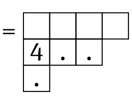


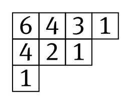
The correspondence $\lambda \to Y_\lambda$ is a bijection between the set of partitions of positive integers and the set of Young diagrams.

The *Young tableau* (plural, “tableaux") of a Young diagram is obtained by placing the numbers $1, \dots, m$ in the m boxes of the diagram. A *standard* Young tableau is a Young tableau in which the numbers form an increasing sequence along each line and along each column.

Given a box of a diagram, the shape formed by the boxes directly to the right of it, the boxes directly below it and the box itself is called the *hook* of that box. The number of boxes in the hook of a box is called the *hook length* of that box. Thus a Young diagram with the hook lengths of boxes in it is a Young tableau. Thus we can identify the Young diagram of a partition with a Young tableau.

$Y = 1^1 3^1 4^1 =$





The *hook set* of a partition λ is the set of hook lengths of the Young diagram of λ. The hook set of a partition encodes information about the other combinatorial objects related to that partition. The most famous is the hook-length formula which gives the degree of the corresponding irreducible representation of the symmetric group and also counts the number of standard Young tableaux that have the shape of that partition, see [9, 13].

We denote the set of positive integers by $\mathbb{N}$ and we put $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The cardinality of any set K will be denoted by $|K|$. For two subsets A, B of $\mathbb{N}_0$, we set

$$A + B = \{u + v : u \in A, v \in B\}, \quad kA = A + A + \cdots + A.$$

A *numerical semigroup* S is a monoid of $\mathbb{N}_0$, and it has a finite complement $G(S) := \mathbb{N}_0 \backslash S$. The elements of $G(S)$ are called *gaps* of S and $g := |G(S)|$ is called the *genus* of S. The largest element of $G(S)$ is called the *Frobenius number* and denoted by $F(S)$. The *conductor* of S is $c := F(S) + 1$. We say that S is generated by $X \subseteq S$, if $S = \{\sum_{i=1}^{m} h_i x_i : m, h_i \in \mathbb{N}_0, x_i \in X, 1 \le i \le m\}$. In this case, X is a *system of generators* of S and we denote S by $\langle X \rangle$. If $X = \{x_1, \dots, x_k\}$, then we write $S = \langle x_1, \dots, x_k \rangle$. Note that a system of generators of a numerical semigroup is a *minimal system of generators* if none of its proper subsets generates the numerical semigroup. Let $\{n_1 < n_2 < \cdots < n_e\}$ be the minimal system of generators of S. Then n_1 is known as *multiplicity*, and e is called the *embedding dimension* of S.

If S is a numerical semigroup, then we assume $S = \{0 = s_0, s_1, \dots, s_r, \longrightarrow\}$, where “$\longrightarrow$" means that all subsequent natural numbers which are bigger then s_r belong to S and r denotes *the number of small elements* of S.

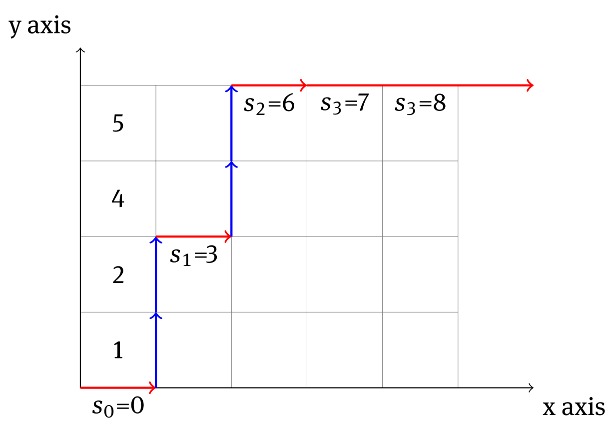
Numerical semigroups have several applications to branches of mathematics such as algebraic geometry, number theory, coding theory. For example, the computation of the minimum distance of algebraic geometric codes involves computations in the Weierstrass semigroup, see [5].

A connection between Young diagrams and numerical semigroups was extended by [7, 15]. A partition λ with no hook lengths divisible by a is called an *a-core partition*. The set of a-cores is infinite but the number of partitions that are both a-cores and b-cores, *simultaneous* (a, b)-cores, is finite. In [7], the authors studied correspondences between numerical sets (subsets of $\mathbb{N}_0$ which have finite complement and contain zero) and partitions of a positive integer. They count the set of simultaneous (a, b)-cores that come from semigroups for a certain pair (a, b). Moreover, some formulas for the number of partitions with a given hook set and some asymptotic results for the number of semigroups are given in [7]. In [4], the authors proved that a numerical semigroup is presented by a unique Dyck path of order given by its genus, and analyzed some properties such as weight, symmetry by means of a square diagram.

Given a numerical semigroup $S \neq \mathbb{N}_0$, we construct a uniquely determined Young diagram and thus a uniquely determined partition as follows. We use the first quadrant of the cartesian xy-plane for the construction by drawing a continuous polygonal path which starts from the origin. Starting with $x = 0$.

- If $x \in S$, then we draw a line segment of unit length to the right.
- If $x \notin S$, then we draw a line segment of unit length up.
- Repeat for $x + 1$.

For any x greater than the Frobenius number of S we draw a line to the right. The lattice lying above the path and below the horizontal line defines the *Young diagram of* S. If the Young diagram of a partition λ and a numerical semigroup S are the same, we say that λ *is the partition of* S. For $S = \{0, 3, 6, \rightarrow\}$ and $G(S) = \{1, 2, 4, 5\}$, we obtain $\lambda = [4, 2]$ and we have the following path.



The association of a Dyck path to a numerical semigroup follows from the association of numerical semigroups with sequences of 0s and 1s and then assigned either up-or-right moves to each, for detail see [17]. There are other papers associating paths in the plane to numerical semigroups and vise versa, for instance [11].

Unless otherwise stated we will make the following assumptions and notations:

- $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_r]$, $r \in \mathbb{N}$, $\sum_{i=1}^{r} \lambda_i = n$ and ${\lambda_1}^r := [\underbrace{\lambda_1, \ldots, \lambda_1}_{r \text{ times}}]$.
- Y_S : The Young diagram corresponding to a numerical semigroup S.
- The *jth column of* Y_S is denoted by G_j, for each $j \geq 0$. The set of hook lengths of boxes which are in the jth column is identified with G_j. The construction of Y_S implies that the jth column G_j corresponds to s_j, $j = 0, 1, \ldots r-1$. For $j \geq r$, s_j is greater than or equal to the Frobenius number of S, then there is no box for s_r at the diagram. We know that $G_0 = G(S)$. Given a box in the G_0, the number of the boxes below shows the number of gaps before the hook length in a given box.

Here note that $\lambda = [2, 2]$ can not represent a numerical semigroup. Otherwise, $1 \in S$ and S must be $\mathbb{N}_0$, but 2, 3 are not in $S = \{0, 1, 4, \rightarrow\}$. We have the following tableau

3	2
2	1

Thus the correspondence $\lambda \longrightarrow S$ is not a bijection between the set of partitions and the set of numerical semigroups. However, the correspondence $S \longrightarrow Y_S$ is a bijection between the set of numerical sets and the set of Young diagrams.

Numerical semigroups have become important because of their applications in algebraic geometry. Valuations of analytically unramified one-dimensional local Noetherian domains are numerical semigroups under certain conditions, and many properties of these rings can be characterized in terms of their associated numerical semigroups, see [2, 3, 18]. Du Val showed geometrically how multiplicity sequences of the blow-ups of a curve can be used to classify singularities. Arf showed the algebraic counterpart of Du Val's results. Arf's aim was to calculate the Arf ring closure of the coordinate ring of a curve and then its value semigroup (which is an Arf numerical semigroup), see [2, 10].

A numerical semigroup S is called an *Arf semigroup* if $x + y - z \in S$, for all $x, y, z \in S$ with $z \leq y \leq x$. This property is equivalent to $2x - y \in S$, for all $x, y \in S$ with $y \leq x$. For example, $\mathbb{N}_0$ and $S = \{0, 7, 14, 21, 24, 27 \rightarrow \}$ are Arf numerical semigroups. The *Arf closure* of a numerical semigroup S is the smallest (with respect to set inclusion) Arf semigroup containing S. There are several equivalent conditions on Arf semigroups, see [3, 8, 10, 14, 18]. In [10, 14], the authors give parametrizations of numerical semigroups with multiplicity up to 5. In [19], an algorithm is given for finding the Arf closure of a numerical set.

Here, we investigate the properties of the Arf partitions of a positive integer using the set of gaps of an Arf numerical semigroup. In Section 2, firstly, we explain the connection between Arf semigroup and Young tableau. Also, we determine partitions of some numerical semigroup families; Proposition 2.7 is about the partitions of Lipman semigroups (see page 347). Let $S_{(k)}$ be a numerical semigroup with the minimal system of generators $< 4, k, k + t, k + t + 2 >$, where $k \equiv 2(mod\ 4)$ and t is an odd integer with $t \geq 7$. Then, we obtain the Arf partition of $S_{(k)}$, more precisely, in Proposition 2.8 we show that this partition is

$$\left[\frac{3k+2t-4}{4}, \frac{3k+2t-4}{4} - 3, \ldots, 3 + \frac{t+1}{2}, \frac{t+1}{2}, \frac{t-1}{2}, \ldots, 2, 1\right].$$

The intersection of two semigroups gives a binary operation over a subset of the set of partitions of positive integers. We denote this operation by $\bigoplus$ and it is detailed in the proof of Theorem 2.9, and we prove that the set of partitions obtained from the sets of gaps of all numerical semigroups is a semigroup with the operation $\bigoplus$. In particular, we obtain that the set of Arf partitions is a semigroup with the operation $\bigoplus$.

Let $\lambda = [\lambda_1, \ldots, \lambda_r] \in \mathbb{N}^r$ be a partition of length r. If $\beta_i = \lambda_1 - \lambda_{r+1-i}$, $1 \leq i \leq r$, then β is called the *dual* partition of λ. For more details on the concept of the duality, see [9]. In Section 3, we define the Young dual of a numerical semigroup using the concept of the dual partition. Given a numerical semigroup S, we determine the elements of the numerical set D which is the Young dual of S, and we give conditions for D to be a numerical semigroup and an Arf semigroup, see Propositions 3.4 and 3.5. Let λ be an Arf partition of a natural number n, and r be the length of λ. Then we show that the dual of λ is also a partition of n with the same length, but it may not be an Arf partition. Furthermore, for the rational diagram of a partition λ, defined in Definition 2, we analyze the behavior of the numerical semigroup corresponding to λ. Corollary 3.7 states that for any Arf partition λ, there exists a partition β such that the rational diagram of β can be represented with λ as denominator and another Arf partition.

In Section 4, we give some relations between semigroup polynomials and Arf partitions (Lemma 4.1 and Theorem 4.2), and we achieve the generating functions of semigroups given in Proposition 2.7 and Proposition 2.8.

2 Arf Semigroup and Young Tableau

Let S be a numerical semigroup of genus g and $G(S) = \{b_1, \ldots, b_g\}$. We set $\alpha(S) = (\alpha_1, \ldots \alpha_g)$ with $\alpha_i = b_i - i$, for all $i \leq g$, which is called the *Schubert index* of S. The sum $w(S) = \sum_{i=1}^{g} \alpha_i$ is said to be the *weight* of S. The notion of the weight $w(S)$ indicates the difference between the semigroup $< g + 1, \ldots, 2g + 1 >$ and S.

Lemma 2.1. *Let S be a numerical semigroup and $\lambda = [\lambda_1, \ldots, \lambda_r]$ be the corresponding partition. Then the Schubert index of S is determined by the conjugate partition of $[\lambda_2, \ldots, \lambda_r]$ and $w(S) = \sum_{i=2}^{r} \lambda_i$.*

Proof. The proof follows from definitions. □

For example, if $S = \{0, 3, 6, 8, \longrightarrow\}$, we have the following Young tableau

$$Y_S = \begin{array}{|c|c|c|}\hline 7 & 4 & 1 \\ \hline 5 & 2 & \\ \cline{1-2} 4 & 1 & \\ \cline{1-2} 2 & & \\ \cline{1-1} 1 & & \\ \cline{1-1}\end{array}$$

and the corresponding partition is $\lambda = [5, 3, 1]$, $\alpha(S) = (0, 0, 1, 1, 2)$ and $w(S) = 4$. Note that $[3, 1]$ and the reverse ordering of $[1, 1, 2]$ are conjugate.

For a given numerical semigroup S, we have several related semigroups. For each $i \geq 0$, S_i and $S(i)$ are defined as follows:

$$S_i = \{s \in S : s \geq s_i\}$$

$$S - s_i = \{s - s_i \in \mathbb{N}_0 : s \in S\}$$

$$S(i) = S - S_i = \{z \in N_0 : z + S_i \subseteq S\}.$$

It is obvious that every $S(i)$ is a numerical semigroup, and we obtain a semigroup chain:

$$\cdots \subset S_r \subset S_{r-1} \subset \cdots \subset S_1 \subset S = S(0) \subset S(1) \subset \cdots \subset S(r) = \mathbb{N}_0.$$

For $1 \leq i \leq r$, we define *ith type set* $T(i) := S(i)\backslash S(i-1)$ and $t_i := |T(i)|$. We call $\{t_i\}_{i=1}^{r}$ the *type sequence* of S. The *Lipman semigroup* of S is defined by $L(S) = \bigcup_{k\geq 1}(kS_1 - kS_1)$. We have another finite chain of semigroups obtained by Lipman semigroup of S: $S = L_0 \subseteq L_1 \subseteq L_2 \subseteq \cdots \subseteq L_i \subseteq \cdots$ where $L_i(S) := L(L_{i-1}(S))$ is the *ith Lipman semigroup* of S.

Theorem 2.2 explains that the behavior of the semigroup S over the Young diagram.

Theorem 2.2. *Let $S = \{0 = s_0, s_1, \ldots, s_r, \longrightarrow\}$ be a numerical semigroup, Y_S be the Young diagram of S, and G_i be the hook set of the ith column of Y_S, $S(i) = S - S_i$, for $0 \leq i \leq r$. Let $T(i)$ be the ith type set of S, $1 \leq i \leq r$. Then the following statements hold:*

(1) $G_i = \mathbb{N}_0 \backslash \{s - s_i : s \in S, s \geq s_i\} = G_0 - s_i$ and $|G_i| = s_r - r - (s_i - i)$, $0 \leq i < r$. Moreover, G_i does not contain any element of S, $0 \leq i < r$.

(2) The first hook length of G_i is $\min\{b \in G_0 : b > s_i\} - s_i$, $1 \leq i < r$, the last hook length is $F(S) - s_i$.

(3) $S(i) = \bigcap_{j\geq i}(S - s_j) = \mathbb{N}_0 \backslash \bigcup_{j=i}^{r-1} G_j$.

(4) $x \in T(i)$ if and only if $x \in G_{i-1}$ and $x \notin G_j$, $i - 1 < j < r$.

(5) $x \in T(i)$ if and only if $i = max\{j + 1 : x \in G_0 - s_j, j < r\}$.

Proof. We have $S(i) = \{z \in \mathbb{N}_0 : z \in S - s_j, j \geq i\} = \bigcap_{j\geq i}(S - s_j)$. Then (1), (2) and (3) are clear by using the construction of the diagram Y_S. (4) Since the ith type set is $T(i) = S(i)\backslash S(i-1)$, we obtain

$$\begin{aligned} x \in T(i) &\Leftrightarrow x + s \in S, \text{for all } s \in S,\ s > s_{i-1} \text{ and } x + s_{i-1} \notin S. \\ &\Leftrightarrow x \in G_{i-1} \text{ and } x \notin G_j,\ i - 1 < j < r. \end{aligned}$$

(5) follows from (4). □

Corollary 2.3. *Let S be a numerical semigroup, let λ be the corresponding partition of length r and $n_i := |\{s : s \notin \bigcup_{j\geq i} G_j,\ s \leq F(S) - s_i\}|$, for $0 \leq i \leq r$. Then we have $n_{i-1} - n_i = \lambda_i - \lambda_{i+1} + (1 - t_i)$, for $1 \leq i \leq r$.*

Proof. The proof follows from the definition of the type sequence, Theorem 2.2 and the construction of the diagram Y_S. □

Corollary 2.4. *Let S be a numerical semigroup of genus g, and $\lambda = [\lambda_1, \ldots, \lambda_r]$ be the corresponding partition of length r. Then the following statements hold:*

(1) S is an Arf semigroup if and only if $t_i = \lambda_i - \lambda_{i+1}$, $1 \leq i < r$, $t_r = \lambda_r$.

(2) If S is an Arf semigroup, then $\lambda_i = g - \sum_{j=1}^{i-1} t_j$, $1 \leq i \leq r$, where t_j is the jth type of S.

Proof. (1) S is an Arf semigroup if and only if $S(i) = S - S_i$. Using Corollary 2.3 and Theorem 2.2 (3), we have $n_i + \lambda_i = F(S) - s_i$, for $i \leq r$. Therefore, $t_i = s_i - s_{i-1} - 1 = \lambda_i - \lambda_{i+1}$.

(2) $g = \lambda_1$, $\lambda_2 = g - (s_1 - s_0 - 1)$ and inductively we have $\lambda_i = g - \sum_{j=1}^{i-1}(s_j - s_{j-1} - 1)$, $i \leq r$. □

Proposition 2.5. *If S is an Arf semigroup, then the following statements hold:*
(1) If $Y_S = 1^{u_1}2^{u_2}\cdots r^{u_r}$, *then* $u_i \neq 0$, *for* $1 \leq i \leq r$.
(2) If $Y_S = [\lambda_1, \dots, \lambda_r]$, *then* $\lambda_i \neq \lambda_{i+1}$, $1 \leq i < r$.

Proof. If S is an Arf semigroup, then $g_i - g_{i-1} \leq 2 \leq F(S)$, where $g_i, g_{i-1} \in G_0$ (equivalently, $s_{i+1} - s_i \geq 2$, $1 \leq i \leq c - r$, where c is the conductor of S, $s_i, s_{i-1} \in S$). In fact, if $g_j - g_{j-1} > 2$, for some $g_j < F(S)$, then $g_j - 1$, $g_j - 2 \in S$ and $2(g_j - 1) - (g_j - 2) = g_j \in G_0$. But this is a contradiction. Since $u_i = s_i - s_{i-1} - 1$, $\lambda_i = \sum_{j=i}^{r} u_j$, we obtain $u_i \geq 1$, $1 \leq i \leq r$ and $\lambda_i \neq \lambda_{i+1}$. □

Lemma 2.6. *Let S be a semigroup and* G_i *be the hook set of the ith column of* Y_S, *for* $0 \leq i \leq r$, *and* $S(i) = \{z \in \mathbb{N}_0 : z + S_i \subseteq S\}$. *Then S is an Arf semigroup if and only if* $G_i = \mathbb{N}_0 \backslash S(i)$, *and* $S(i)$ *is Arf,* $0 \leq i \leq r$.

Proof. Using Theorem 2.2, we obtain that the hook set G_i is a subset of the complement of the semigroup $S(i)$, for $0 \leq i \leq r$. For an Arf semigroup S, we have the following equivalent conditions:

$$\text{S Arf} \iff S(i) = S_i - s_i = L_i(S) \iff S(i)(j) = S(i + j),\ 1 \leq i + j \leq r$$

where $L_i(S) = L(L_{i-1}(S))$ is the ith Lipman semigroup of S. Hence, $G_i = G_0 - s_i = \mathbb{N}_0 \backslash S_i - s_i = \mathbb{N}_0 \backslash S(i)$ and $G_{i+j} = G_0 - s_{i+j} = \mathbb{N}_0 \backslash S_{i+j} - s_{i+j} = \mathbb{N}_0 \backslash S(i + j)$, $0 \leq i \leq r$. Thus $S(i)$ is also Arf. □

Hence, the related semigroups with an Arf semigroup S can be obtained over the Young diagram Y_S.

Definition 1. Let λ be a partition of positive integer n. If there exists an Arf semigroup such that the gap set $G(S)$ is the set of hook lengths of the first column of the Young diagram of λ, then λ is called an *Arf partition* of n.

For any positive integer n has at least one Arf partition $\lambda = [n]$ and $S = \{0, n + 1, \rightarrow\}$. Let take $n = 13$. All of the Arf partitions of 13 are [13], [9, 4], [9, 3, 1], [10, 3], [10, 2, 1], [11, 2], [12, 1]. Proposition 2.5 states that an Arf partition is a strict dominant partition.

Determining the Arf partitions of positive integers is equivalent to determining Arf semigroups. Partitions of some semigroup families can be found in Proposition 2.7 and Proposition 2.8.

Proposition 2.7. *Let* $S_{(k)}$ *be a numerical semigroup with the minimal system of generators* $< m, km + 1, km + 2, \dots, km + (m - 1) >$, *where* $m \leq 7$ *is the multiplicity,* $k \in \mathbb{N}$. *Then ith Lipman semigroup is* $L_i(S_{(k)}) =< m, km - im + 1, km - im + 2, \dots, km - im + (m - 1) >$ *and the corresponding partition is*

$$\left[(m - 1)(k - i), (m - 1)(k - i - 1), \dots, (m - 1)\right].$$

Proof. We prove the proposition by induction on i. □

Proposition 2.8. *Let* $S_{(k)}$ *be a numerical semigroup with the minimal system of generators* $< 4, k, k+t, k+t+2 >$, *where* $k \equiv 2 (mod\ 4)$ *and t is an odd integer with* $t \geq 7$. *Then the corresponding Arf partition to* $S_{(k)}$ *is*

$$\left[\frac{3k + 2t - 4}{4}, \frac{3k + 2t - 4}{4} - 3, \dots, 3 + \frac{t + 1}{2}, \frac{t + 1}{2}, \frac{t - 1}{2}, \dots, 2, 1\right]$$

Proof. Using induction method, we prove that the set of gaps of $S_{(k)}$ is

$$G(S_{(k)}) = \{1, 2, 3, 5, 6, 7, 9, \dots, k - 7, k - 5, k - 4, k - 3, k - 1, k + 1, k + 3, \dots, k + t - 2\},$$

and the conductor is $k+t-1$. Hence, the jth part of the partition of $S_{(k)}$ is

$$\lambda_j = \begin{cases} \frac{3(k-2)}{4} + \frac{(t+1)}{2} - 3(j-1), & 1 \le j \le \frac{k-2}{4}, \\ \frac{(t+1)}{2} - j + 1, & 1 + \frac{k-2}{4} \le j \le \frac{(t+1)}{2} - 1. \end{cases}$$

Using Corollary 3.19 in [18] and induction method, we obtain that $S_{(k)}$ is an Arf semigroup. Hence, λ is an Arf partition. □

We remark that the intersection of two numerical semigroups is again a numerical semigroup. A consequence of the closure of this operation can be seen in Theorem 2.9.

Theorem 2.9. *Let P denote the set of partitions obtained from the set of numerical semigroups. Then P is a semigroup.*

Proof. Let S and T be numerical semigroups corresponding to partitions λ and β, respectively. Since $S \bigcap T = T \bigcap S$, we may assume that $F(S) \ge F(T)$. Now, we use the following notations: $\lambda = [\lambda_1, \dots, \lambda_r] = 1^{u_1} 2^{u_2} \cdots k^{u_k}$, $\beta = [\beta_1, \cdots, \beta_f] = 1^{v_1} 2^{v_2} \cdots h^{v_h}$, $u_i \ge 0, v_j \ge 0$, $1 \le i \le k$, $1 \le j \le h$. Let $M = \{s \in S \bigcap T : s < F(S)\} = \{s_{i_1}, s_{i_2}, \cdots s_{i_l}\}$. The effect of the intersection of two semigroups can be explained as follows: let $\lambda_1 \ge \beta_1$. If there is an element $b = s_d \in S \backslash T$, then the corresponding column of the diagram of S must be deleted and $b = \sum_{i=1}^{d} u_i + d$. Since u_{d+1} is the number of consecutive gap numbers which are between s_d and s_{d+1}, the number u_{d+1} must be added to u_d. Thus the previous column has $u_{d+1}+1$ more boxes for gaps which are between s_d and s_{d+1}. Hence, we obtain $u_{d+1}+1+u_d$ consecutive gap numbers in $S \bigcap T$. Denote $\alpha_j = |\{b \in S \backslash T : s_{i_j} < b < s_{i_{j+1}}\}|$, $j \le l$, and let p_j denote the number of the consecutive elements of $S \bigcap T$ which are greater than or equal to s_{i_j}. If t_j denotes the length of the jth gap block of $S \bigcap T$, then we obtain $t_j = (t_{j-1} + p_j)$ and this number repeats $m_j = \sum_{z=i_j+1}^{i_{j+1}} u_z + \alpha_j$ times. If $\lambda_1 < \beta_1$, the proof follows from the similar argument.

Then the intersection of two semigroups gives a binary operation which is denoted by $\bigoplus$ in P: $\alpha \bigoplus \beta = \gamma$, where $\gamma = t_1^{m_1} t_2^{m_2} \cdots t_l^{m_l}$, $t_j = (t_{j-1} + p_j)$ and $m_j = \sum_{z=i_j+1}^{i_{j+1}} u_z + \alpha_j$, Associativity is clear as a property of intersection, [0] is unit which represents $\mathbb{N}_0$. □

Example 2.10. Let $S = \{0, 4, 7, 8, 11, 12, 14, 15, 16, 18, \rightarrow\}$ and $T = \{0, 3, 6, 7, 9, 10, 12, \rightarrow\}$. Then we have $S \cap T = \{0, 7, 12, 14, 15, 16, 18, \rightarrow\}$. $Y_S = 1^3 2^2 4^2 6^1 9^1 = [9, 6, 4, 4, 2, 2, 1, 1, 1]$, $Y_T = 1^2 2^2 4^1 6^1 = [6, 4, 2, 2, 1, 1]$. By using the proof of the Theorem 2.9, we obtain the following integers

$$\begin{aligned} &\alpha_1 = |\{4\}|, && p_1 = 1, && m_1 = u_1 + u_2 + 1 = 3 + 2 + 1 = 6, \\ &\alpha_2 = |\{8, 11\}|, && p_2 = 1, && m_2 = u_3 + u_4 + u_5 + 2 = 2 + 2 = 4, \\ &\alpha_3 = 0, && p_3 = 1, && m_3 = u_6 + 0 = 1 + 0 = 1, \\ &\alpha_4 = 0, && p_4 = 3, && m_4 = u_7 + u_8 + u_9 + 0 = 1 + 0 = 1, \end{aligned}$$

and

$$\begin{aligned} &t_1 = (0+1)^{u_1+u_2+1} && t_4 = (3+3)^{u_7+u_8+u_9+0} \\ &t_2 = (1+1)^{u_3+u_4+u_5+2} && t_5 = (4+2)^{u_{10}+0} = 0 \\ &t_3 = (2+1)^{u_6+0} && t_6 = (5+1)^{u_{11}+0} = 0. \end{aligned}$$

Hence we get

$$Y_{S \cap T} = 1^{u_1+u_2+1}(1+1)^{u_3+u_4+u_5+2}(2+1)^{u_6+0}(3+3)^{u_7+u_8+u_9+0} = 1^6 2^4 3^1 6^1 = [12, 6, 2, 1, 1, 1].$$

$$1^3 2^2 4^2 6^1 9^1 \bigoplus 1^2 2^2 4^1 6^1 = 1^6 2^4 3^1 6^1.$$

$Y_S =$

17	13	10	9	6	5	3	2	1
13	9	6	5	2	1			
10	6	3	2					
9	5	2	1					
6	2							
5	1							
3								
2								
1								

$Y_T =$

11	8	5	4	2	1
8	5	2	1		
5	2				
4	1				
2					
1					

$Y_{S \cap T} =$

17	10	5	3	2	1
13	6	1			
11	4				
10	3				
9	2				
8	1				
6					
5					
4					
3					
2					
1					

Corollary 2.11. *Let A be the set of Arf partitions. Then A is a semigroup with the operation $\bigoplus$.*

Proof. Because of the fact that the intersection of two Arf semigroups is an Arf semigroup, the proof follows from Proposition 2.5 and Theorem 2.9. □

With the same notation as in the proof of Theorem 2.9, we have that $\alpha \bigoplus \beta = \gamma = 1^{m_1} 2^{m_2} \cdots l^{m_l}$, where $\alpha, \beta \in A$, $m_j = \sum_{z=1}^{i_{j+1}-i_j} u_{i_j+z} + \alpha_j$ and $m_j \neq 0$, for $1 \le j \le l$.

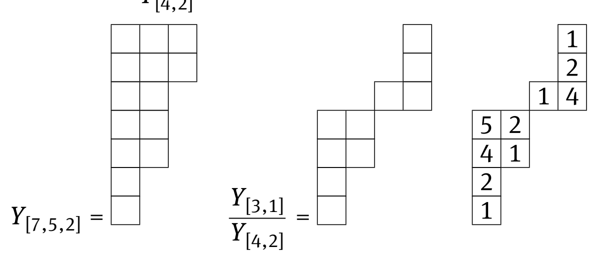
3 The Young dual of an Arf semigroup

Definition 2. For a strict dominant partition $\lambda = [\lambda_1, \ldots, \lambda_r]$, the ratio of Young diagrams of the partitions $[-v_{r+1}, -v_r, \ldots, -v_{k+1}]$ and $[v_1, \ldots, v_k]$ is called the *rational diagram* of λ, where $v = [\lambda_1, \ldots, \lambda_r, 0] - \mathbf{r}^{\mathbf{r+1}} = [v_1, \ldots, v_k, v_{k+1}, \ldots, v_{r+1}]$, the separation in two blocks corresponding to the values $v_i \geq 0$, or $v_j < 0$, $1 \le i \le k$, $1 + k \le j \le r + 1$. The rational diagram of λ is denoted by $\frac{Y_{[-v_{r+1}, \ldots, -v_{k+1}]}}{Y_{[v_1, \ldots, v_k]}}$.

Schubert calculus uses the Young diagrams for polynomials. The rational diagram is used for the calculation of rational Schubert polynomials, see [1].

Note that the block $[-v_r, \ldots, -v_{k+1}]$ of the rational diagram which corresponds to the values $v_j < 0$ gives an inverted diagram. Therefore, we consider the reverse ordering for calculating the hook lengths of the block $[-v_{r+1}, \ldots, -v_{k+1}]$.

Example 3.1. If we take $\lambda = [7, 5, 2]$, then we have $v = [7, 5, 2, 0] - [3, 3, 3, 3] = [4, 2, -1, -3]$. Thus the rational diagram corresponding to λ is $\frac{Y_{[3,1]}}{Y_{[4,2]}}$,



where the last tableau is the rational diagram of $[7, 5, 2]$ containing the hook lengths of boxes. Here, $G_1 = \{1, 2, 4, 5\}$ and $G_2 = \{1, 2, 4\}$ are the gap sets of two numerical semigroups $H_1 = \{0, 3, 6, \rightarrow\}$ and $H_2 = \{0, 3, 5 \rightarrow\}$, respectively.

The concept of the dual of a numerical semigroup has been viewed in [3]. Now, we define a new duality concept for a numerical semigroup. The main motivation comes from Schubert calculus (see duality theorem) and it arises naturally by using partitions.

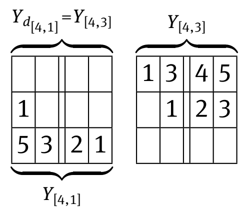
Definition 3. Let S be a numerical semigroup and $\lambda = [\lambda_1, \ldots, \lambda_r] \in \mathbb{N}^r$ be the corresponding partition.

(1) The partition $[\lambda_1 - 0, \lambda_1 - \lambda_r, \ldots, \lambda_1 - \lambda_1]$ is called the *dual partition* of S, and denoted by d_λ.

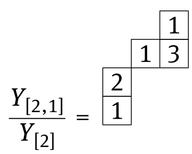
(2) The set of hook lengths of d_λ is called the *Young dual* of S.

Since S is a numerical semigroup, $1 \in G(S)$ and $\lambda_1 - \lambda_2 > 1$. Note that s_r corresponds to the part λ_{r+1}. Then we may assume $\lambda_{r+1} = 0$, we need this for compatibility of the construction of the dual partition of a numerical semigroup.

Example 3.2. Now we consider $\lambda = [4, 1]$. The duality relation between $Y_{[4,1]}$ and $Y_{[4,3]}$ can be seen by the following diagrams



where the double line determines the rational diagram. Hence, we have two hook sets, $G = \{1, 2, 3, 5\}$ and $T = \{1, 3, 4, 5\}$. We obtain G as the hook set of $Y_{[4,1]}$ and T as the hook set of the diagram $Y_{[4,3]}$. We note that $Y_{[4,1]}$ and $Y_{[4,3]}$ are dual to each other. But T is not the set of gaps of a numerical semigroup. The rational diagram corresponding to λ and hook sets can be seen as follows:



Therefore, we have two semigroups $\{0, 3, 4, 5 \longrightarrow\}$ and $\{0, 2, 4, 5 \longrightarrow\}$. Both are Arf semigroups.

Proposition 3.3. *Let S be a numerical semigroup and λ be the corresponding partition. Let c be the conductor of S and $G(S) = \{b_1, b_2, \ldots, b_{\lambda_1}\}$ (resp., $G(S_{d_\lambda}) = \{\bar{b}_1, \ldots, \bar{b}_{\lambda_1}\}$). Then the following statements hold:*

(1) If s_i (resp., $\bar{s}_i$) denotes the ith element of S (resp., S_{d_λ}), then we have

$$c = b_i + \bar{b}_{\lambda_1 - i + 1}, \quad 1 \le i \le \lambda_1,$$
$$c = s_i + \bar{s}_{r-i}, \qquad 1 \le i \le r.$$

(2) $c^2 = \sum_{i=1}^{\lambda_1} b_i + \bar{b}_{\lambda_1 - i+1} + \sum_{i=1}^{r} s_i + \bar{s}_{r-i}$.

Proof. Definition of the dual partition gives (1), using (1) and $c = \lambda_1 + r$, we obtain (2). □

In general, the set S_{d_λ} is a numerical set but it may not be a semigroup. For $S = \{0, 6, 8, \rightarrow\}$, the partition of S is $\lambda = [6, 1]$, then $d_\lambda = [6, 5]$ and $S_{d_\lambda} = \{0, 2, 8, \rightarrow\}$ is not a semigroup.

Proposition 3.4. *Let S be a numerical semigroup and D be its Young dual. If the conductor of S is $c = s_r$, then the following statements hold:*

(1) $D = (d_i) = \begin{cases} s_r - s_{r-i}, & i \le r, \\ s_r + i - r, & i > r. \end{cases}$

(2) For any $i, j \le r$, if there exists $k \le r$ such that $s_m = s_{r-i} + s_{r-j}$ with $m = r + s_{r-k}$ or $m = r - k$, then D is a numerical semigroup.

Proof. (1) is clear by the definition. To prove (2), we consider the following cases: let $i, j \le r$ and set $d := d_i + d_j$. Then we have,

$$d = s_r - (s_{r-i} + s_{r-j} - s_r) = s_r - (s_m - s_r).$$

If $m = r + s_{r-k}$, then $d = s_r - (s_r + s_{r-k} - s_r) = s_r - s_{r-k} \in D$. If $m = r - k$, then $d = s_r + (s_r - s_{r-i} - s_{r-j}) = s_r + (s_r - s_{r-k}) \in D$.

Take $i \le r < j$. Then $d_i = s_r - s_{r-i}$, $d_j = s_r + (j-r)$ and $d = s_r + \underbrace{(s_r - s_{r-i}) + (j-r)}_{u>0} \in D$. Define $t = r + u$, then we get $d = d_t = s_r + u = s_{r+u}$. Now, take $j, i > r$. Then $i + j - 2r > 0$ and we obtain $d = s_r + (i-r) + s_r + (j-r) = 2s_r + (i + j - 2r)$. □

Proposition 3.5. *Let S be a numerical semigroup and let D be the Young dual of S. If* $2s_{r-i} - s_{r-j} \in S$, *for* $r \ge i \ge j$, *and D is a numerical semigroup, then D is an Arf semigroup.*

Proof. If $r \ge i \ge j$, then $s_{r-i} \le s_{r-j} \le s_r$. Assume that there exists $m > 0$ such that $2s_{r-i} - s_{r-j} = s_m$. Then we have

$$s_r - s_m = s_r - (2s_{r-i} - s_{r-j}) = 2s_r + (s_{r-j} - 2s_{r-i} - s_r)$$
$$s_r - s_m = 2[s_r - s_{r-i}] - [s_r - s_{r-j}] = 2d_i - d_j.$$

In this case, if $m \le r$, then $m = r - t$, $t \le r$ and $d := 2d_i - d_j = s_r - s_{r-t} = d_t \in D$. If $m > r$, then $d = s_r - s_m < 0$. If $u := s_{r-j} - 2s_{r-i} \ge 0$, then $s_r + (s_{r-j} - 2s_{r-i}) = s_r + u$ and $d = 2[s_r - s_{r-i}] - [s_r - s_{r-j}] = s_{r+u} \in D$.

If $j < r < i$, then $d = 2[s_r + i - r] - [s_r - s_{r-j}] = s_r + 2(i-r) + s_{r-j} > s_r$ and $d \in D$. If $i > j > r$, then $d = s_r + (i-j) > s_r$ and $d \in D$. □

Corollary 3.6. *Let* $\lambda = [\lambda_1, \dots, \lambda_r]$ *be an Arf partition of a positive integer n. Then the following statements hold:*

(1) The dual of λ *is a partition of n and its length is r.*

(2) If $v = [\lambda_1, \dots, \lambda_r, 0] - \mathbf{r}^{r+1}$, *then* $v_p := [v_1, \dots, v_k]$ *is an Arf partition, where* v_i *is the ith part of v,* $v_i \ge 0$, $i \le k \le r+1$.

Proof. Using Proposition 3.4 and the definition of Arf partition, one can obtain the Corollary 3.6 (1). For (2), it is enough to see that the partition $v_p = [v_1, \dots, v_k]$ presents a semigroup containing the semigroup of λ. □

Corollary 3.7. *Let* $\lambda = [\lambda_1, \dots, \lambda_r]$ *be an Arf partition. Then there exists a partition* β *such that the rational diagram of* β *can be represented with* λ *as denominator and another Arf partition.*

Proof. Define $\beta = [\lambda_1, \dots, \lambda_r, 0] + [\mathbf{r+1}]^r$. Then the corresponding rational diagram to β is $\frac{Y_{[r]}}{Y_\lambda}$. Here Y_λ corresponds to the Arf semigroup of λ and $Y_{[r]}$ is the diagram of the semigroup $\{0, r+1, \longrightarrow\}$ which is also Arf. □

4 Arf semigroup polynomial

For a numerical semigroup S, we have

$$\frac{1}{1-x} = \sum_{s \ge 0} x^s = \sum_{s \in S} x^s + \sum_{s \in \mathbb{N}_0 \setminus S} x^s.$$

$H_S(x) = \sum_{s \in S} x^s$ is called the *generating function* associated to S and $P_S(x) = (1-x)\sum_{s \in S} x^s$ its *semigroup polynomial*. Here, $H_S(x)$ is not a polynomial but $P_S(x)$ is. On the other hand, we have that

$$P_S(x) = (1-x)H_S(x) = 1 + (x-1)\sum_{s \in \mathbb{N}_0 \setminus S} x^s.$$

There are several papers dealing with the polynomial $P_S(x)$, see [6] and [16]. We can associate semigroup polynomials with a partition of a natural number n. For $G_0 = \mathbb{N}_0 \setminus S$, we have a partition whose hook set is G_0. For any hook number j which occurs in the first column, we form a polynomial involving a sum of powers x^j. Adding a column to the left of the diagram of λ means the multiplication of the polynomial of λ by x. We can illustrate this association in the following table.

partition	*tableau*	*polynomial*
$\lambda = [0]$	–	
$\lambda = [1]$	1	x
$\lambda = [2]$	2 1	$x + x^2$
$\lambda = [1, 1]$	2 1	$x^2 = x.x$
$\lambda = [2, 1]$	3 1 1	$x + x.x^2$
$\lambda = [3, 1]$	4 1 2 1	$x + x^2 + x.x^3$
$\lambda = [2, 2, 1]$	4 3 1 2 1	$x[x + x^3]$

Lemma 4.1. *Let $Y_S = 1^{u_1} 2^{u_2} \cdots r^{u_r}$ be the Young diagram of a given semigroup S and let define $S(x) = \sum_{s \in \mathbb{N}_0 \setminus S} x^s$. Then we have*

$$S(x) = \sum_{j=1}^{r} \sum_{i=1}^{u_j} x^{i+s_{j-1}}.$$

Proof. The complement of the hook set of the first column of Y_S is

$$\{0, u_1 + 1, u_1 + u_2 + 2, u_1 + u_2 + u_3 + 3, \dots, u_1 + u_2 + \cdots + u_r + r, \longrightarrow\}.$$

Hence, we obtain

$$S(x) = \sum_{i=1}^{u_1} x^i + x^{u_1+1} \sum_{i=1}^{u_2} x^i + x^{u_1+u_2+2} \sum_{i=1}^{u_3} x^i + \dots + x^{u_1+\dots+u_{r-1}+r-1} \sum_{i=1}^{u_r} x^i$$

by rearranging, we have $S(x) = \sum_{i=1}^{u_1} x^i + x^{s_1} \sum_{i=1}^{u_2} x^i + \dots + x^{s_{r-1}} \sum_{i=1}^{u_r} x^i$. □

Theorem 4.2. *Let S be an Arf semigroup with type sequence $\{t_i\}_{i=1}^r$, and $S(k) = S - S_k$, $0 \le k \le r$. If $S(x) = \sum_{s \in \mathbb{N}_0 \setminus S} x^s$, then the following statements hold:*

(1) $S(x) = \sum_{j=1}^{r} \sum_{i=1}^{t_j} x^{i+s_{j-1}}$.

(2) $S(0)(x) = S(x)$ *and for* $k \ge 1$, *we have*

$$S(k)(x) = \sum_{i=1}^{t_{k+1}} x^i + \sum_{i=1}^{r-1-k-1} \sum_{j=1}^{t_{k+i+1}} x^{j+s_{k+i-1}-\sum_{i=1}^{k} s_i}.$$

(3) $S(k-1)(x) = \sum_{i=1}^{t_k} x^i + x^{t_k+1} S(k)(x)$ *for* $k \ge 1$.

(4) The semigroup polynomial of S is $P_S(x) = 1 + (x-1)S(x)$.

Proof. Since S is an Arf semigroup, we have $t_i = u_i$ by Corollary 2.4. Using Lemma 4.1, Theorem 2.2 and the definition of the semigroup $S(k)$, we obtain (1)-(4). □

Corollary 4.3 follows from Theorem 4.2.

Corollary 4.3. *Let S be an Arf semigroup and c be the conductor of S, $S(x) = \sum_{s \in \mathbb{N}_0 \setminus S} x^s$. Then the following statements hold:*

(1) *If* $s_1 = 2$, *then* $S(x) = \sum_{j=0}^{\frac{c}{2}-1} x^{2j+1}$.

(2) *If* $S =< 3, c+1, c+2 >$ *and* $c \equiv 0(mod\ 3)$, *then* $S(x) = \sum_{j=0}^{\frac{c}{3}-1} x^{3j}(x+x^2)$.

(3) *If* $S =< 3, c, c+2 >$ *and* $c \equiv 2(mod\ 3)$, *then* $S(x) = \sum_{j=0}^{\frac{c-2}{3}-1} x^{3j}(x+x^2) + x^{c-1}$.

In Proposition 2.7 and Proposition 2.8, we obtained partitions of some semigroup families. The generating functions associated to these families are given in Corollary 4.4 and Corollary 4.5.

Corollary 4.4. *Let* $S_{(k)} =< 4, k, k+t, k+t+2 >$ *and* t *be an odd integer with* $t \geq 7$, $k \equiv 2(mod\ 4)$. *Then* $S_{(k)}(x) = \frac{x^{k+t}+x^{k+t-2}+x^k-x-x^2-x^3}{x^4-1}$ *and the generating function associated to* $S_{(k)}$ *is*

$$H_{S_{(k)}}(x) = \frac{x^{k+t}}{(1-x^2)} + \frac{x^k+1}{(1-x^4)}.$$

Proof. Let $k = 4v+2$, $t = 7+2a$, $a \geq 0$. Proposition 2.8 states the partition of $S_{(k)}$ and we get $Y_S = 1^3 2^3 \cdots v^3 (v+1) \cdots (v+a+4)$. Since $S_{(k)}$ is an Arf semigroup, the type sequence is $\{\underbrace{3,\ldots,3}_{v \text{ times}}, \underbrace{1,\ldots,1}_{a+4 \text{ times}}\}$. Therefore,

$$\begin{aligned} S_{(k)}(x) &= (\sum_{l=1}^{3} x^l)(1+x^4+x^8+\ldots+x^{4(v-1)}) + x^{4v+1}(1+x^2+x^4+\ldots+x^{2a+6}) \\ &= \frac{(x^{4v}-1)(x^3+x^2+x)+x^{4v+1}(x^{2a+8}-1)(x^2+1)}{x^4-1} = \frac{(-x-x^2-x^3+x^{k+t}+x^k+x^{k+t+2})}{x^4-1}. \end{aligned}$$

Hence, $P_{S_{(k)}}(x) = 1 + (x-1)\sum_{s\in\mathbb{N}_0\setminus S_{(k)}} x^s = \frac{x^{k+t}+x^k+x^{k+t+2}+1}{(x+1)(x^2+1)}$ and the generating function associated to $S_{(k)}$ is

$$H_{S_{(k)}}(x) = \frac{P_{S_{(k)}}(x)}{(1-x)} = \frac{x^{k+t}+x^{k+t+2}+x^k+1}{(1-x^4)} = \frac{x^{k+t}}{(1-x^2)} + \frac{x^k+1}{(1-x^4)}.$$

□

Corollary 4.5. *Let* $S_{(k)}$ *be a numerical semigroup with the minimal system of generators* $< m, km+1, km+2, \ldots, km+(m-1) >$, *and* $L_i(S_{(k)})$ *be the ith Lipman semigroup of* $S_{(k)}$, *where* $m \leq 7$ *is the multiplicity of* $S_{(k)}$, $0 \leq i < k$, $k \in \mathbb{N}$. *Then* $L_i(S_{(k)})(x) = \frac{(x^{m(k-i)}-1)(x^m-x)}{(x^m-1)(x-1)}$, *for* $0 \leq i \leq k-1$, *and the generating function associated to* $L_i(S_{(k)})$ *is*

$$H_{L_i(S_{(k)})}(x) = \frac{x^{(k-i)m}}{(1-x)} - \frac{1}{(x^m-1)}.$$

Proof. We see that $S_{(k)}$ is an Arf semigroup and $L_i(S_{(k)})$ is also Arf, $0 \leq i < k$. Proposition 2.7 states the partition of the semigroup $L_i(S_{(k)})$, and we get $Y_{L_i(S_{(k)})} = 1^{m-1}2^{m-1}\cdots(k-i)^{m-1}$. Using Corollary 2.4, we obtain the type sequence of $L_i(S_{(k)})$, $\{t_j\}_{j=1}^{k-i} = \{m-1,\ldots,m-1\}$. By Theorem 4.2, we get

$$L_i(S_{(k)})(x) = \sum_{l=1}^{m-1} x^l + x^m \sum_{l=1}^{m-1} x^l + \ldots + x^{(k-i-1)m} \sum_{l=1}^{m-1} x^l = \frac{(x^{m(k-i)}-1)(x^m-x)}{(x^m-1)(x-1)}.$$

Therefore, $P_{L_i(S_{(k)})}(x) = \frac{x^{m(k+1-i)}-x^{(k-i)m+1}+x-1}{x^m-1}$, the generating function associated to $L_i(S_{(k)})$ is

$$H_{L_i(S_{(k)})}(x) = \frac{x^{(k-i)m}(x^m-1)+x-1}{(x^m-1)(1-x)} = \frac{x^{(k-i)m}}{(1-x)} - \frac{1}{(x^m-1)}.$$

□

Corollary 4.6. *Let S be an Arf semigroup with type sequence* $\{t_i\}_{i=1}^{r}$ *and D be its Young dual. Then*

$$D(x) = \sum_{j=1}^{r} \sum_{i=1}^{t_{r-j+1}} x^{i+s_r-s_{r+1-j}}.$$

Proof. If $\lambda = [\lambda_1, \dots, \lambda_r]$ is the partition of S, then $t_i = \lambda_i - \lambda_{i+1}$, $1 \leq i < r$ and $t_r = \lambda_r$. The Young dual of S is $D = \{0, s_r - s_{r-1}, s_r - s_{r-2}, \dots, s_r - s_1, s_r, \longrightarrow\}$ and the dual partition is $d_\lambda = [\lambda_1, \lambda_1 - \lambda_r, \dots, \lambda_1 - \lambda_2]$. Hence, $v_1 = \lambda_r = t_r$, $v_i = (\lambda_1 - \lambda_{r-i+2}) - (\lambda_1 - \lambda_{r-i+1}) = \lambda_{r-i+1} - \lambda_{r-i+2} = t_{r-i+1}$, $2 \leq i \leq r$. In other words, the sequence $v_1, \dots, v_r$ is the reverse ordering of $t_1, \dots, t_r$. By Theorem 4.2, we obtain $D(x) = \sum_{j=1}^{r} \sum_{i=1}^{t_{r-j+1}} x^{i+s_r-s_{r+1-j}}$. □

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